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Optimal inequalities for a Toader-type mean by quadratic and contraharmonic means

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Abstract

In this paper, we present the best possible parameters α_i, β_i ($i = 1, 2, 3$) and $\alpha_4, \beta_4 \in (1/2, 1)$ such that the double inequalities

$$\alpha_1 Q(a,b) + (1-\alpha_1)C(a,b) < T_{Q,C}(a,b) < \beta_1 Q(a,b) + (1-\beta_1)C(a,b),$$
$$Q^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b) < T_{Q,C}(a,b) < Q^{\beta_2}(a,b)C^{1-\beta_2}(a,b),$$
$$\frac{Q(a,b)C(a,b)}{\alpha_3 Q(a,b) + (1-\alpha_3)C(a,b)} < T_{Q,C}(a,b) < \frac{Q(a,b)C(a,b)}{\beta_3 Q(a,b) + (1-\beta_3)C(a,b)},$$
$$C\left(\sqrt{\alpha_4 a^2 + (1-\alpha_4)b^2}, \sqrt{(1-\alpha_4)a^2 + \alpha_4 b^2}\right) < T_{Q,C}(a,b) < C\left(\sqrt{\beta_4 a^2 + (1-\beta_4)b^2}, \sqrt{(1-\beta_4)a^2 + \beta_4 b^2}\right)$$

hold for all $a, b > 0$ with $a \neq b$, where $Q(a,b)$, $C(a,b)$, and $T(a,b)$ are the quadratic, contraharmonic, and Toader means, respectively, and $T_{Q,C}(a,b) = T[Q(a,b), C(a,b)]$. As consequences, we provide new bounds for the complete elliptic integral of the second kind.

Keywords: Toader mean, elliptic integral, quadratic mean, contraharmonic mean.

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1. Introduction

For $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$, the pth power mean $M_p(a,b)$ [7, 9, 17, 18, 22, 32, 35, 37, 38], pth Lehmer mean $L_p(a,b)$[27, 34], harmonic mean $H(a,b)$, geometric mean $G(a,b)$, arithmetic mean $A(a,b)$, Toader mean $T(a,b)$ [10, 14, 16, 28], centroidal mean $\overline{C}(a,b)$[6, 36], quadratic mean $Q(a,b)$[19],

contraharmonic mean $C(a,b)$[5, 13] are, respectively, defined by

$$M_p(a,b) = \begin{cases} [(a^p + b^p)/2]^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \quad L_p(a,b) = \frac{a^{p+1} + b^{p+1}}{a^p + b^p},$$

$$H(a,b) = \frac{2ab}{a+b}, \quad G(a,b) = \sqrt{ab}, \quad A(a,b) = \frac{a+b}{2},$$

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta, \quad \overline{C}(a,b) = \frac{2(a^2 + ab + b^2)}{3(a+b)},$$

$$Q(a,b) = \sqrt{\frac{a^2 + b^2}{2}}, \quad C(a,b) = \frac{a^2 + b^2}{a+b}. \tag{1.1}$$

It is well-known that $M_p(a,b)$ and $L_p(a,b)$ are continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, the following inequality chain

$$H(a,b) = M_{-1}(a,b) = L_{-1}(a,b) < G(a,b) = M_0(a,b) = L_{-1/2}(a,b)$$
$$< A(a,b) = M_1(a,b) = L_0(a,b) < T(a,b) < \overline{C}(a,b) \tag{1.2}$$
$$< Q(a,b) = M_2(a,b) < C(a,b) = L_1(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

The Toader mean $T(a,b)$ has been well known in the mathematical literature for many years (see [20, 21, 24]), which is related to the complete elliptic integral of the second kind $\mathcal{E}(r) = \int_0^{\pi/2}(1 - r^2\sin^2\theta)^{1/2}d\theta(r \in (0,1))$[12, 15, 25, 30, 31, 33, 39, 40] and it can be rewritten as

$$T(a,b) = \begin{cases} 2a\mathcal{E}(\sqrt{1 - (b/a)^2})/\pi, & a \geqslant b, \\ 2b\mathcal{E}(\sqrt{1 - (a/b)^2})/\pi, & a < b. \end{cases} \tag{1.3}$$

Let $r \in (0,1)$ and $r' = \sqrt{1 - r^2}$, then the complete elliptic integral of the first kind is given by $\mathcal{K}(r) = \int_0^{\pi/2}(1 - r^2\sin^2\theta)^{-1/2}d\theta$. We clearly see that $\mathcal{K}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2, +\infty)$ and $\mathcal{E}(r)$ is strictly decreasing from $(0,1)$ onto $(1, \pi/2)$. Moreover, $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the following Landen identities and derivatives formulas (see [2, Appendix E, p.474-475])

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \quad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E}(r) - r'^2\mathcal{K}(r)}{1+r},$$

$$\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{rr'^2}, \quad \frac{d\mathcal{E}(r)}{dr} = \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r},$$

$$\frac{d[\mathcal{E}(r) - r'^2\mathcal{K}(r)]}{dr} = r\mathcal{K}(r), \quad \frac{d[\mathcal{K}(r) - \mathcal{E}(r)]}{dr} = \frac{r\mathcal{E}(r)}{r'^2}.$$

The special values $\mathcal{K}(\sqrt{2}/2)$ and $\mathcal{E}(\sqrt{2}/2)$ will be used later, which can be expressed as (see [4, Theorem 1.7])

$$\mathcal{K}\left(\sqrt{2}/2\right) = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}} = 1.854\cdots, \quad \mathcal{E}\left(\sqrt{2}/2\right) = \frac{4\Gamma^2(3/4) + \Gamma^2(1/4)}{8\sqrt{\pi}} = 1.350\cdots,$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the classical Euler's gamma function.

The special bivariate mean $T_{X,Y}(a,b)$ derived from Toader mean for any bivariate means $X(a,b)$ and $Y(a,b)$ of positive numbers a, b is given by

$$T_{X,Y}(a,b) = T[X(a,b), Y(a,b)], \tag{1.4}$$

which is called a Toader-type mean. We denote the pairs of means $\{X, Y\}$ the generating means of the Toader-type mean defined in (1.4).

Recently, the Toader mean has been the subject of intensive research. Vuorinen [29] conjectured that the inequality

$$T(a,b) > M_{3/2}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen [26], and Barnard et al. [3], respectively.

Alzer and Qiu [1] presented a best possible upper power mean bound of the Toader mean as follows

$$T(a,b) < M_{\log 2/(\log \pi - \log 2)}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Chu and Wang [11] prove the double inequality

$$L_p(a,b) < T(a,b) < L_q(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only $p \leqslant 0$ and $q \geqslant 1/4$.

Very recently, optimal bounds for $T_{A,C}(a,b)$ and $T_{A,Q}(a,b)$ by several convex combinations of their generating means were established. Li et al. [23] presented the best possible parameters α_i and β_i with $i = 1, 2, 3, 4$ such that the double inequalities

$$\alpha_1 A(a,b) + (1-\alpha_1)C(a,b) < T_{A,C}(a,b) < \beta_1 A(a,b) + (1-\beta_1)C(a,b),$$
$$A^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b) < T_{A,C}(a,b) < A^{\beta_2}(a,b)C^{1-\beta_2}(a,b),$$
$$\frac{\alpha_3}{A(a,b)} + \frac{1-\alpha_3}{C(a,b)} < \frac{1}{T_{A,C}(a,b)} < \frac{\beta_3}{A(a,b)} + \frac{1-\beta_3}{C(a,b)},$$
$$C\left(\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a\right) < T_{A,C}(a,b) < C\left(\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a\right)$$

hold for all $a, b > 0$ with $a \neq b$.

In [8], the authors found the best possible parameters $\lambda_1, \mu_1, \lambda_2, \mu_2 \in (1/2, 1)$ such that the double inequalities

$$Q(a,b;\lambda_1) < T_{A,Q}(a,b) < Q(a,b;\mu_1),$$
$$C(a,b;\lambda_2) < T_{A,Q}(a,b) < C(a,b;\mu_2)$$

hold for all $a, b > 0$ with $a \neq b$, where $M(a,b;p) = M[pa + (1-p)b, pb + (1-p)a]$ is the one-parameter mean of a and b. Besides, another expression of optimal bounds for $T_{A,Q}(a,b)$ was given by Zhao et al. [41]. Explicitly, they proved the double inequality

$$[\alpha(r)A^r(a,b) + (1-\alpha(r))Q^r(a,b)]^{1/r} < T_{A,Q}(a,b) < [\beta(r)A^r(a,b) + (1-\beta(r))Q^r(a,b)]^{1/r}$$

holds for all $r \leqslant 1$ and $a, b > 0$ with $a \neq b$ if and only if $\alpha(r) \geqslant 1/2$ and $\beta(r) \leqslant \lambda(r)$, where $\lambda(r)$ is defined by $\lambda(r) = [1 - (2\mathcal{E}(\sqrt{2}/2)/\pi)^r]/[1 - (\sqrt{2}/2)^r]$ $(r \neq 0)$ and $\lambda(0) = \log[2\mathcal{E}(\sqrt{2}/2)/\pi]/\log(\sqrt{2}/2)$.

From (1.1) and (1.2) together with the properties of a mean, we clearly see that

$$Q(a,b) < T_{Q,C}(a,b) < C(a,b). \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$.

Motivated by inequality (1.5) and the results of [8], it is natural to ask what are the best possible parameters α_i, β_i $(i = 1, 2, 3)$ and $\alpha_4, \beta_4 \in (1/2, 1)$ such that the double inequalities

$$\alpha_1 Q(a,b) + (1-\alpha_1)C(a,b) < T_{Q,C}(a,b) < \beta_1 Q(a,b) + (1-\beta_1)C(a,b),$$
$$Q^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b) < T_{Q,C}(a,b) < Q^{\beta_2}(a,b)C^{1-\beta_2}(a,b),$$
$$\frac{Q(a,b)C(a,b)}{\alpha_3 Q(a,b) + (1-\alpha_3)C(a,b)} < T_{Q,C}(a,b) < \frac{Q(a,b)C(a,b)}{\beta_3 Q(a,b) + (1-\beta_3)C(a,b)},$$
$$C\left(\sqrt{\alpha_4 a^2 + (1-\alpha_4)b^2}, \sqrt{(1-\alpha_4)a^2 + \alpha_4 b^2}\right) < T_{Q,C}(a,b) < C\left(\sqrt{\beta_4 a^2 + (1-\beta_4)b^2}, \sqrt{(1-\beta_4)a^2 + \beta_4 b^2}\right)$$

hold for all $a, b > 0$ with $a \neq b$? The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove the desired theorem, we need several lemmas which we present in this section.

Lemma 2.1 ([2, Theorem 1.25]). *For* $-\infty < a < b < \infty$, *let* $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$, *and be differentiable on* (a, b), *let* $g'(x) \neq 0$ *on* (a, b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a, b), *then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad and \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2.

(1) $[\mathcal{K}(r) - \mathcal{E}(r)]/r^2$ *is strictly increasing on* $(0, 1)$;

(2) $[\mathcal{K}(r) - \mathcal{E}(r)]/\log(1/r')$ *is strictly decreasing from* $(0, 1)$ *onto* $(1, \pi/2)$;

(3) $r'^c \mathcal{E}(r)$ *is strictly increasing on* $(0, 1)$ *if and only if* $c \leqslant -1/2$.

Proof. Parts (1)-(3) follow from [2, Exercise 3.43 (11) and (15), Theorem 3.21 (8)]. $\qquad\square$

Lemma 2.3. *Let* $\delta_1 = 2[\pi - 2\mathcal{E}(\sqrt{2}/2)]/(2 - \sqrt{2})\pi = 0.4785 \cdots$ *and*

$$f(r) = \frac{1 - 2\mathcal{E}(r)/\pi}{1 - \sqrt{1 - r^2}},$$

then $f(r)$ *is strictly decreasing from* $(0, \sqrt{2}/2)$ *onto* $(\delta_1, 1/2)$.

Proof. Let $f_1(r) = 1 - 2\mathcal{E}(r)/\pi$ and $f_2(r) = 1 - \sqrt{1 - r^2}$. Then we clearly see that $f_1(0) = f_2(0) = 0$ and $f(r) = f_1(r)/f_2(r)$.

Taking the derivative of $f_1(r)$ and $f_2(r)$ yields

$$\frac{f_1'(r)}{f_2'(r)} = \frac{2}{\pi} \cdot \frac{\mathcal{K}(r) - \mathcal{E}(r)}{\log(1/r')} \cdot \frac{r' \log(1/r')}{1 - r'^2}. \tag{2.1}$$

An easy calculation yields

$$\frac{d[r' \log(1/r')/(1 - r'^2)]}{dr'} = \frac{(1 - r'^2) + (1 + r'^2)\log r'}{(1 - r'^2)^2} > 0 \tag{2.2}$$

for $r' \in (0, 1)$.

It follows from (2.2) together with the monotonicity of $r' = \sqrt{1 - r^2}$ that $r' \log(1/r')/(1 - r'^2)$ is strictly decreasing on $(0, 1)$ with respect to r. This conjunction with (2.1) and Lemma 2.2 (2) implies that $f_1'(r)/f_2'(r)$ is strictly decreasing on $(0, 1)$. Therefore, Lemma 2.3 follows from Lemma 2.1 and the limiting values $f(0^+) = 1/2$ and $f(\frac{\sqrt{2}}{2}^-) = \delta_1$.

$\qquad\square$

Lemma 2.4. *Let* $\delta_2 = 2\log[\pi/(2\mathcal{E}(\sqrt{2}/2))]/\log 2 = 0.435698 \cdots$ *and*

$$g(r) = \frac{2[\log(2/\pi) + \log \mathcal{E}(r)]}{\log(1 - r^2)},$$

then $g(r)$ *is strictly decreasing from* $(0, \sqrt{2}/2)$ *onto* $(\delta_2, 1/2)$.

Proof. Let $g_1(r) = 2[\log(2/\pi) + \log \mathcal{E}(r)]$ and $g_2(r) = \log(1 - r^2)$, then it is easy to see that $g_1(0) = g_2(0) = 0$, $g(r) = g_1(r)/g_2(r)$ and

$$\frac{g_1'(r)}{g_2'(r)} = \frac{g_{11}(r)}{g_{22}(r)}, \tag{2.3}$$

where $g_{11}(r) = (1 - r^2)[\mathcal{K}(r) - \mathcal{E}(r)]$ and $g_{22}(r) = r^2 \mathcal{E}(r)$.

Observe that $g_{11}(0) = g_{22}(0) = 0$. Taking the derivative of $g_{11}(r)$ and $g_{22}(r)$ yields

$$\frac{g_{11}'(r)}{g_{22}'(r)} = \frac{3\mathcal{E}(r) - 2\mathcal{K}(r)}{3\mathcal{E}(r) - \mathcal{K}(r)} = 1 - \frac{1}{3\mathcal{E}(r)/\mathcal{K}(r) - 1}. \tag{2.4}$$

It follows from (2.4) and the monotonicity of $\mathcal{E}(r)/\mathcal{K}(r)$ that $g_{11}'(r)/g_{22}'(r)$ is strictly decreasing on $(0,1)$. This conjunction with (2.3) and Lemma 2.1 implies that $g(r)$ is strictly decreasing on $(0,1)$.

The proof of Lemma 2.4 is completed from the limit values $g(0^+) = 1/2$ and $g(\frac{\sqrt{2}}{2}^-) = \delta_2$. $\square$

Lemma 2.5. *Let* $\delta_3 = [\sqrt{2}\mathcal{E}(\sqrt{2}/2) - \pi/2]/(\sqrt{2} - 1)\mathcal{E}(\sqrt{2}/2) = 0.606488\cdots$ *and*

$$h(r) = \frac{\pi/[2\mathcal{E}(r)] - 1/\sqrt{1 - r^2}}{1 - 1/\sqrt{1 - r^2}},$$

then $h(r)$ *is strictly increasing from* $(0, \sqrt{2}/2)$ *onto* $(1/2, \delta_3)$.

Proof. Let $h_1(r) = \pi/[2\mathcal{E}(r)] - 1/\sqrt{1 - r^2}$ and $h_2(r) = 1 - 1/\sqrt{1 - r^2}$, then we clearly see that $h_1(0) = h_2(0) = 0, h(r) = h_1(r)/h_2(r)$ and

$$\frac{h_1'(r)}{h_2'(r)} = 1 - \frac{\pi}{2} \cdot \frac{r'[\mathcal{K}(r) - \mathcal{E}(r)]}{1 - r'^2} \cdot \frac{1}{[r'^{-1}\mathcal{E}(r)]^2}. \tag{2.5}$$

Lemma 2.2 (3) and (2.1) together with the proof of Lemma 2.3 lead to the conclusion that $1/[r'^{-1}\mathcal{E}(r)]^2$ and $r'[\mathcal{K}(r) - \mathcal{E}(r)]/(1 - r'^2)$ are strictly decreasing on $(0,1)$. This conjunction with (2.5) implies that $h_1'(r)/h_2'(r)$ is strictly increasing on $(0,1)$.

Therefore, Lemma 2.5 follows immediately from Lemma 2.1 and the limiting values $h(0^+) = 1/2$ and $h(\frac{\sqrt{2}}{2}^-) = \delta_3$. $\square$

Lemma 2.6. *The function* $\varphi(r) = [1 - r^2/4 - 2\mathcal{E}(r)/\pi]/r^4$ *is strictly increasing from* $(0, \sqrt{2}/2)$ *onto* $(3/64, 7/2 - 8\mathcal{E}(\sqrt{2}/2)/\pi)$.

Proof. Let $\varphi_1(r) = 1 - r^2/4 - 2\mathcal{E}(r)/\pi$ and $\varphi_2(r) = r^4$, then $\varphi_1(0) = \varphi_2(0) = 0, \varphi(r) = \varphi_1(r)/\varphi_2(r)$, and

$$\frac{\varphi_1'(r)}{\varphi_2'(r)} = \frac{\mathcal{K}(r) - \mathcal{E}(r) - \pi/4}{2\pi r^4}. \tag{2.6}$$

Taking the derivate of (2.6) yields

$$\frac{d[\varphi_1'(r)/\varphi_2'(r)]}{dr} = \frac{\mu(r)}{2\pi r^5(1 - r^2)}, \tag{2.7}$$

where

$$\mu(r) = (4 - 3r^2)\mathcal{E}(r) + (1 - r^2)[\pi - 4\mathcal{K}(r)].$$

An easy calculation leads to

$$\mu(\sqrt{2}/2) = 1.23926\cdots, \quad \mu'(r) = r[7\mathcal{K}(r) - 9\mathcal{E}(r) - 2\pi] < r[7\mathcal{K}(\sqrt{2}/2) - 9\mathcal{E}(\sqrt{2}/2) - 2\pi] < -5r \tag{2.8}$$

for $r \in (0, \sqrt{2}/2)$. It follows from (2.8) that $\mu(r) > 0$ for $r \in (0, \sqrt{2}/2)$. This conjunction with (2.7) implies that $\varphi_1'(r)/\varphi_2'(r)$ is strictly increasing on $(0, \sqrt{2}/2)$.

Therefore, Lemma 2.6 follows from Lemma 2.1 together with the limit values $\varphi(0^+) = 3/64$ and $\varphi(\frac{\sqrt{2}}{2}^-) = 7/2 - 8\mathcal{E}(\sqrt{2}/2)/\pi$. $\square$

Notice that $7/2 - 8\mathcal{E}(\sqrt{2}/2)/\pi = 0.0606\cdots < 1/8$, then from Lemma 2.6 we can get the following corollary directly.

Corollary 2.7. *The double inequality*

$$1 - \frac{r^2}{4} - \frac{r^4}{8} < \frac{2}{\pi}\mathcal{E}(r) < 1 - \frac{r^2}{4} - \frac{3r^4}{64}$$

holds for $r \in (0, \sqrt{2}/2)$.

Lemma 2.8. *Let* $\delta_4 = \left[\pi \sqrt{2\mathcal{E}^2(\sqrt{2}/2) - \pi^2/4} \right] / \mathcal{E}^2(\sqrt{2}/2) = 1.87157 \cdots$, $\lambda \in (0, 2]$ *and*

$$\Phi_\lambda(r) = \frac{\sqrt{1 + \lambda r r'} - \sqrt{1 - \lambda r r'}}{\lambda r},$$

then the following statements are true:

(1) $\Phi_{\sqrt{2}}(r) < 1 - r^2/4 - r^4/4$ *holds for* $r \in (0, \sqrt{2}/2)$;

(2) $\Phi_{\delta_4}(r) > 1 - r^2/4$ *holds for* $r \in (0, 17/25)$;

(3) $2\mathcal{E}(r)/\pi + r/2$ *is strictly increasing on* $(0, \sqrt{2}/2)$;

(4) $\Phi_{\delta_4}(r) + r/2$ *is strictly decreasing on* $(17/25, \sqrt{2}/2)$.

Proof.

(1). We first claim that

$$\sqrt{1 - 2r^2 + 2r^4} > 1 - r^2 + \frac{r^4}{2} + \frac{r^6}{2} \tag{2.9}$$

for $r \in (0, \sqrt{2}/2)$. Indeed, it follows easily from

$$1 - 2r^2 + 2r^4 - \left(1 - r^2 + \frac{r^4}{2} + \frac{r^6}{2}\right)^2 = \frac{r^8}{4}(1 - r^2)(r^2 + 3) > 0$$

for $r \in (0, \sqrt{2}/2)$. From (2.9), we clearly see that

$$\Phi_{\sqrt{2}}^2(r) - \left(1 - \frac{r^2}{4} - \frac{r^4}{4}\right)^2 = \frac{1 - \sqrt{1 - 2r^2 + 2r^4}}{r^2} - \left(1 - \frac{r^2}{4} - \frac{r^4}{4}\right)^2$$

$$< 1 - \frac{r^2}{2} - \frac{r^4}{2} - \left(1 - \frac{r^2}{4} - \frac{r^4}{4}\right)^2 = -\frac{r^4(r^2 + 1)}{16} < 0.$$

This completes the proof of Lemma 2.8 (1).

(2). In order to prove that $\Phi_{\delta_4}(r) > 1 - r^2/4$ for $r \in (0, 17/25)$, by squaring both sides of the inequality and simplifying, it suffices to show that

$$\sqrt{1 - \delta_4^2 r^2 (1 - r^2)} < 1 - \frac{\delta_4^2}{2} r^2 \left(1 - \frac{r^2}{4}\right)^2 \tag{2.10}$$

holds for $r \in (0, 17/25)$.

We consider the difference of both sides squares of (2.10) as follows

$$1 - \delta_4^2 r^2 (1 - r^2) - \left[1 - \frac{\delta_4^2}{2} r^2 \left(1 - \frac{r^2}{4}\right)^2\right]^2 = -\frac{\delta_4^2 r^4}{1024} \eta(r), \tag{2.11}$$

where

$$\eta(r) = \delta_4^2 r^8 - 16\delta_4^2 r^6 + 96\delta_4^2 r^4 - 256\delta_4^2 r^2 - 64\delta_4^2 r^2 + 256\delta_4^2 - 512.$$

An easy calculation yields

$$\eta(17/25) = 6.9982\cdots, \quad \eta'(r) = -8r[\delta_4^2(4-r^2)^3 + 16] < 0. \tag{2.12}$$

It follows from (2.12) that $\eta(r) > 0$ for $r \in (0, 17/25)$. This conjunction with (2.11) completes the proof of Lemma 2.8 (2).

(3). It suffices to determine the sign of the derivate of $2\mathcal{E}(r)/\pi + r/2$. An easy computation yields

$$\frac{d[2\mathcal{E}(r)/\pi + r/2]}{dr} = \frac{1}{2} - \frac{2r}{\pi} \cdot \frac{\mathcal{K}(r) - \mathcal{E}(r)}{r^2}. \tag{2.13}$$

It follows from (2.13) and Lemma 2.2 (1) that $d[2\mathcal{E}(r)/\pi + r/2]/dr$ is strictly decreasing on $(0,1)$. The monotonicity of (2.13) leads to the conclusion that

$$\frac{d[2\mathcal{E}(r)/\pi + r/2]}{dr} > \frac{1}{2} - \frac{2\sqrt{2}}{\pi}\left[\mathcal{K}(\sqrt{2}/2) - \mathcal{E}(\sqrt{2}/2)\right] = 0.046753\cdots.$$

This completes the proof.

(4). Let $\omega_1(r) = \sqrt{1 + \delta_4 rr'}$ and $\omega_2(r) = \sqrt{1 - \delta_4 rr'}$, then $\omega_1(r) > \omega_2(r) > 0$ for $r \in (0, \sqrt{2}/2)$. By easy computations, one has

$$\begin{aligned}
\frac{d[\omega_1(r) - \omega_2(r)]}{dr} &= \frac{\delta_4(1-2r^2)}{2\sqrt{1-r^2}}\left[\frac{1}{\omega_1(r)} + \frac{1}{\omega_2(r)}\right] > 0, \\
\frac{d[1/\omega_1(r) + 1/\omega_2(r)]}{dr} &= \frac{\delta_4(1-2r^2)}{2\sqrt{1-r^2}}\left[\frac{1}{\omega_2^3(r)} - \frac{1}{\omega_1^3(r)}\right] > 0
\end{aligned} \tag{2.14}$$

for $r \in (0, \sqrt{2}/2)$.

Moreover, it is easy to see that $(1 - 2r^2)/\sqrt{1-r^2} = 2\sqrt{1-r^2} - 1/\sqrt{1-r^2}$ is strictly decreasing on $(0, \sqrt{2}/2)$. This conjunction with (2.14) implies that

$$\begin{aligned}
\frac{d[\Phi_{\delta_4}(r) + r/2]}{dr} &= -\frac{\omega_1(r) - \omega_2(r)}{\delta_4 r^2} + \frac{1-2r^2}{2\sqrt{1-r^2}}\left[\frac{1}{\omega_1(r)} + \frac{1}{\omega_2(r)}\right] + \frac{1}{2} \\
&< -\frac{\omega_1(17/25) - \omega_2(17/25)}{\delta_4 r^2} \\
&\quad + \frac{1 - 2\times(17/25)^2}{2\sqrt{1-(17/25)^2}}\left[\frac{1}{\omega_1(\sqrt{2}/2)} + \frac{1}{\omega_2(\sqrt{2}/2)}\right] + \frac{1}{2} \\
&< -\frac{3}{5r^2} + \frac{1}{4r} + \frac{1}{2} \\
&< \frac{1}{r^2}\left[-\frac{3}{5} + \frac{1}{4}\times\frac{\sqrt{2}}{2} + \frac{1}{2}\times\left(\frac{\sqrt{2}}{2}\right)^2\right] = -\frac{14 - 5\sqrt{2}}{40r^2} < 0
\end{aligned} \tag{2.15}$$

for $r \in (17/25, \sqrt{2}/2)$. Therefore, Lemma 2.8 (4) follows directly from (2.15). $\square$

3. Main results

Theorem 3.1. *The double inequality*

$$\alpha_1 Q(a,b) + (1 - \alpha_1)C(a,b) < T_{Q,C}(a,b) < \beta_1 Q(a,b) + (1 - \beta_1)C(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \geqslant 1/2$ *and* $\beta_1 \leqslant \delta_1 = 2[\pi - 2\mathcal{E}(\sqrt{2}/2)]/(2-\sqrt{2})\pi = 0.4785\cdots$.

Proof. Since $Q(a,b)$, $C(a,b)$, and $T(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we may assume that $a > b > 0$. Let $r = (a-b)/\sqrt{2(a^2+b^2)} \in (0, \sqrt{2}/2)$, then (1.1) and (1.3) lead to

$$Q(a,b) = \frac{A(a,b)}{\sqrt{1-r^2}}, \quad C(a,b) = \frac{A(a,b)}{1-r^2}, \quad T_{Q,C}(a,b) = \frac{2A(a,b)}{\pi(1-r^2)}\mathcal{E}(r). \tag{3.1}$$

It follows from (3.1) that

$$\frac{C(a,b) - T_{Q,C}(a,b)}{C(a,b) - Q(a,b)} = \frac{1 - 2\mathcal{E}(r)/\pi}{1 - \sqrt{1-r^2}} = f(r), \tag{3.2}$$

where $f(r)$ is defined as in Lemma 2.3. Therefore, Theorem 3.1 follows easily from Lemma 2.3 and (3.2). $\square$

Theorem 3.2. *The double inequality*

$$Q^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b) < T_{Q,C}(a,b) < Q^{\beta_2}(a,b)C^{1-\beta_2}(a,b)$$

holds for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha_2 \geqslant 1/2$ *and* $\beta_2 \leqslant \delta_2 = 2\log[\pi/(2\mathcal{E}(\sqrt{2}/2))]/\log 2 = 0.435698\cdots$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/\sqrt{2(a^2+b^2)} \in (0, \sqrt{2}/2)$, then from (3.1) we clearly see that

$$\frac{\log C(a,b) - \log T_{Q,C}(a,b)}{\log C(a,b) - \log Q(a,b)} = g(r), \tag{3.3}$$

where $g(r)$ is defined as in Lemma 2.4.

Therefore, Theorem 3.2 follows directly from (3.3) and Lemma 2.4. $\square$

Theorem 3.3. *The double inequality*

$$\frac{Q(a,b)C(a,b)}{\alpha_3 Q(a,b) + (1-\alpha_3)C(a,b)} < T_{Q,C}(a,b) < \frac{Q(a,b)C(a,b)}{\beta_3 Q(a,b) + (1-\beta_3)C(a,b)}$$

holds for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha_3 \leqslant 1/2$ *and* $\beta_3 \geqslant \delta_3 = [\sqrt{2}\mathcal{E}(\sqrt{2}/2) - \pi/2]/(\sqrt{2}-1)\mathcal{E}(\sqrt{2}/2) = 0.606488\cdots$.

Proof. In order to prove the double inequality in Theorem 3.3, it suffices to find α_3 and β_3 such that

$$\alpha_3 < \frac{\frac{1}{T_{Q,C}(a,b)} - \frac{1}{Q(a,b)}}{\frac{1}{C(a,b)} - \frac{1}{Q(a,b)}} < \beta_3 \tag{3.4}$$

holds for all $a,b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/\sqrt{2(a^2+b^2)} \in (0, \sqrt{2}/2)$, then (3.1) leads to

$$\frac{\frac{1}{T_{Q,C}(a,b)} - \frac{1}{Q(a,b)}}{\frac{1}{C(a,b)} - \frac{1}{Q(a,b)}} = h(r), \tag{3.5}$$

where $h(r)$ is defined as in Lemma 2.5. Therefore, Theorem 3.3 follows directly from (3.4), (3.5), and Lemma 2.5. $\square$

Theorem 3.4. *Let $\alpha_4, \beta_4 \in (1/2, 1)$, then the double inequality*

$$C\left(\sqrt{\alpha_4 a^2 + (1-\alpha_4)b^2}, \sqrt{(1-\alpha_4)a^2 + \alpha_4 b^2}\right) < T_{Q,C}(a,b)$$

$$< C\left(\sqrt{\beta_4 a^2 + (1-\beta_4)b^2}, \sqrt{(1-\beta_4)a^2 + \beta_4 b^2}\right)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leqslant (\sqrt{2}+2)/4$ and $\beta_4 \geqslant (\delta_4 + 2)/4 = 0.967894\cdots$, where δ_4 is defined as in Lemma 2.8.

Proof. Since $T(a,b)$ and $C(a,b)$ are symmetric and homogeneous of degree one, we assume that $a > b > 0$. Let $r = (a-b)/\sqrt{2(a^2+b^2)} \in (0, \sqrt{2}/2)$, then (1.1), (3.1) lead to

$$T_{Q,C}(a,b) - C\left(\sqrt{pa^2 + (1-p)b^2}, \sqrt{pb^2 + (1-p)a^2}\right)$$
$$= C(a,b)\left[\frac{2}{\pi}\mathcal{E}(r) - \frac{\sqrt{1+(4p-2)rr'} - \sqrt{1-(4p-2)rr'}}{(4p-2)r}\right] = C(a,b)\left[\frac{2}{\pi}\mathcal{E}(r) - \Phi_{4p-2}(r)\right], \quad (3.6)$$

where $\Phi_\lambda(r)$ is defined as in Lemma 2.8.

It is easy to be verified that $C\left(\sqrt{pa^2 + (1-p)b^2}, \sqrt{pb^2 + (1-p)a^2}\right)$ is continuous and strictly increasing on $[1/2, 1]$ with respect to p for fixed $a, b > 0$ with $a \neq b$.

We divide the proof into three cases.

Case 1. $p_1 = (\sqrt{2}+2)/4$. Then it follows from 2.7 and Lemma 2.8 (1) that

$$\frac{2}{\pi}\mathcal{E}(r) - \Phi_{4p_1-2}(r) = \frac{2}{\pi}\mathcal{E}(r) - \Phi_{\sqrt{2}}(r) > 1 - \frac{r^4}{4} - \frac{r^4}{8} - \left(1 - \frac{r^4}{4} - \frac{r^4}{4}\right) = \frac{r^4}{8} > 0 \quad (3.7)$$

for $r \in (0, \sqrt{2}/2)$. Therefore, $T_{Q,C}(a,b) > C\left(\sqrt{p_1 a^2 + (1-p_1)b^2}, \sqrt{(1-p_1)a^2 + p_1 b^2}\right)$ follows from (3.6) and (3.7).

Case 2. $p_2 = (\delta_4 + 2)/4$. Then from Corollary 2.7 and Lemma 2.8 (2) we clearly see that

$$\frac{2}{\pi}\mathcal{E}(r) - \Phi_{4p_2-2}(r) = \frac{2}{\pi}\mathcal{E}(r) - \Phi_{\delta_4}(r) < 1 - \frac{r^4}{4} - \frac{3r^4}{64} - \left(1 - \frac{r^4}{4}\right) = -\frac{3r^4}{64} < 0 \quad (3.8)$$

for $r \in (0, 17/25)$. Furthermore, it follows from Lemma 2.8 (3) and (4) that $2\mathcal{E}(r)/\pi - \Phi_{\delta_4}(r) = 2\mathcal{E}(r)/\pi + r/2 - [\Phi_{\delta_4}(r) + r/2]$ is strictly increasing on $(17/25, \sqrt{2}/2)$. This implies

$$\frac{2}{\pi}\mathcal{E}(r) - \Phi_{4p_2-2}(r) < \frac{2}{\pi}\mathcal{E}(\sqrt{2}/2) - \Phi_{4p_2-2}(\sqrt{2}/2) = 0 \quad (3.9)$$

for $r \in (17/25, \sqrt{2}/2)$.

Therefore, $T_{Q,C}(a,b) < C\left(\sqrt{p_2 a^2 + (1-p_2)b^2}, \sqrt{(1-p_2)a^2 + p_2 b^2}\right)$ follows from (3.6), (3.8), and (3.9).

Case 3. $(\sqrt{2}+2)/4 < p_3 < (\delta_4 + 2)/4$. On the one hand, if $r \to 0$, then making use of Taylor series yields

$$\frac{2}{\pi}\mathcal{E}(r) - \Phi_{4p_3-2}(r) = -2\left[\left(p_3 - \frac{2-\sqrt{2}}{4}\right)\left(p_3 - \frac{\sqrt{2}+2}{4}\right)\right]r^2 + o(r^4). \quad (3.10)$$

Equations (3.6) and (3.10) lead to the conclusion that there exists small enough $\tau_1 \in (0, \sqrt{2}/2)$ such that $T_{Q,C}(a,b) < C\left(\sqrt{p_3 a^2 + (1-p_3)b^2}, \sqrt{(1-p_3)a^2 + p_3 b^2}\right)$ for all $a > b > 0$ with $(a-b)/\sqrt{2(a^2+b^2)} \in (0, \tau_1)$.

On the other hand, it is easy to see that

$$\frac{2}{\pi}\mathcal{E}(\sqrt{2}/2) - \Phi_{4p_3-2}(\sqrt{2}/2) = \frac{2}{\pi}\mathcal{E}(\sqrt{2}/2) + \frac{\sqrt{1-p_3} - \sqrt{p_3}}{2p_3 - 1}$$

is strictly decreasing on $(1/2, 1)$ with respect to p_3. This implies that

$$\frac{2}{\pi}\mathcal{E}(\sqrt{2}/2) - \Phi_{4p_3-2}(\sqrt{2}/2) < \frac{2}{\pi}\mathcal{E}(\sqrt{2}/2) - \Phi_{\delta_4}(\sqrt{2}/2) = 0. \tag{3.11}$$

Equations (3.6) and (3.11) lead to the conclusion that there exists small enough $\tau_2 \in (0, \sqrt{2}/2)$ such that $T_{Q,C}(a,b) > C\left(\sqrt{p_3 a^2 + (1-p_3)b^2}, \sqrt{(1-p_3)a^2 + p_3 b^2}\right)$ for all $a > b > 0$ with $(a-b)/\sqrt{2(a^2+b^2)} \in (\sqrt{2}/2 - \tau_2, \sqrt{2}/2)$. □

The following corollary follows from Theorems 3.1, 3.2, 3.3, and 3.4 immediately.

Corollary 3.5. *Let* $r' = \sqrt{1-r^2}$ *and*

$$m(r) = \max\left\{\frac{1+r'}{2}, \sqrt{r'}, \frac{2r'}{1+r'}, \frac{\sqrt{1+\sqrt{2}rr'} - \sqrt{1-\sqrt{2}rr'}}{\sqrt{2}r}\right\},$$
$$M(r) = \min\left\{1 - \delta_1 + \delta_1 r', r'^{\delta_2}, \frac{r'}{1-\delta_3+\delta_3 r'}, \frac{\sqrt{1+\delta_4 rr'} - \sqrt{1-\delta_4 rr'}}{\delta_4 r}\right\},$$

where $\delta_1, \delta_2, \delta_3,$ *and* δ_4 *are defined as in Lemmas 2.3, 2.4, 2.5, 2.8, respectively. Then the double inequality*

$$\frac{\pi}{2}m(r) < \mathcal{E}(r) < \frac{\pi}{2}M(r)$$

holds for all $r \in (0, \sqrt{2}/2)$.

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