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The fuzzy C-delta integral on time scales

Xuexiao You[a,b], Dafang Zhao[b,c], Jian Cheng[b], Tongxing Li[d,e,*]

[a]College of Computer and Information, Hohai University, Nanjing, Jiangsu 210098, P. R. China.

[b]School of Mathematics and Statistics, Hubei Normal University, Huangshi, Hubei 435002, P. R. China.

[c]College of Science, Hohai University, Nanjing, Jiangsu 210098, P. R. China.

[d]LinDa Institute of Shandong Provincial Key Laboratory of Network Based Intelligent Computing, Linyi University, Linyi, Shandong 276005, P. R. China.

[e]School of Information Science and Engineering, Linyi University, Linyi, Shandong 276005, P. R. China.

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Abstract

In this paper, we introduce and study the C-delta integral of interval-valued functions and fuzzy-valued functions on time scales. Also, some basic properties of the fuzzy C-delta integral are proved. Finally, we give two necessary and sufficient conditions of integrability.

Keywords: C-Delta integral, fuzzy-valued function, time scale.

2010 MSC: 26A42, 26E50, 26E70.

1. Introduction

It is well known that the Henstock-Kurzweil integral integrates highly oscillating functions and encompasses Newton, Riemann, and Lebesgue integrals [22, 28]. As an important branch in the Henstock-Kurzweil integration theory, the theory of fuzzy Henstock-Kurzweil integral has been studied extensively [7, 12, 15, 16, 24, 29, 31, 32]. In 1986, Bruckner et al. [9] considered the function

$$F(x) = \begin{cases} x \sin \frac{1}{x^2}, & \text{if } 0 < x \leqslant 1, \\ 0, & \text{if } x = 0. \end{cases} \tag{1.1}$$

It is a primitive for the Henstock integral, but it is neither a Lebesgue primitive, a differentiable function, nor a sum of a Lebesgue primitive and a differentiable function. The natural question is: is there a minimal

*Corresponding author
Email addresses: youxuexiao@126.com (Xuexiao You), dafangzhao@163.com (Dafang Zhao), jianchenghs@163.com (Jian Cheng), litongx2007@163.com (Tongxing Li)

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integral including the Lebesgue integral and derivatives? To solve this question, Bongiorno [4] provided a minimal constructive integration process of Riemann type, i.e., C-integral, which includes the Lebesgue integral and also integrates the derivatives of differentiable functions. The theory of C-integration has developed rather intensively in the past few years; see, for instance, the papers [5, 6, 8, 11, 13, 20, 21, 26, 30, 33, 34, 36, 38] and the references cited there.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. The theory of time scales was born in 1988 with the Ph.D. thesis of Hilger [18]. The aim of this theory is to unify various definitions and results from the theories of discrete and continuous dynamical systems, and to extend such theories to more general classes of dynamical systems. It has been extensively studied on various aspects by several authors; see, e.g., [1–3, 17, 23, 25, 27, 35, 37]. To the best of our knowledge, the C-delta integral of fuzzy-valued functions has not received attention in the literature of time scales. The main goal of this paper is to generalize the results above by constructing the C-delta integral of fuzzy-valued functions on time scales.

The paper is organized as follows. Section 2 contains basic concepts of fuzzy sets, time scales, and C-integral. In Section 3, we give the definition of C-delta integral of interval-valued functions, and discuss some of its basic properties. In Section 4, the definition of fuzzy C-delta integral is introduced, and two necessary and sufficient conditions of integrability are presented. We end with Section 5 of conclusions and future work.

2. Preliminaries

In this section, we recall some basic definitions, notation, properties, and results on fuzzy sets and the time scale calculus, which are used throughout the paper. Let us denote by $\mathbb{R}_{\mathcal{I}}$ the set of all nonempty compact intervals of $\mathbb{R}$, that is, $\mathbb{R}_{\mathcal{I}} = \{[\underline{u}, \overline{u}] \mid \underline{u}, \overline{u} \in \mathbb{R} \text{ and } \underline{u} < \overline{u}\}$. $\underline{u}$ and $\overline{u}$ are called the lower and the upper branches of $[\underline{u}, \overline{u}]$, respectively. The usual interval operations, i.e., Minkowski addition and scalar multiplication, are defined by

$$[\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}]$$

and

$$\lambda[\underline{u}, \overline{u}] = \begin{cases} [\lambda\underline{u}, \lambda\overline{u}], & \text{if } \lambda > 0, \\ \{0\}, & \text{if } \lambda = 0, \\ [\lambda\overline{u}, \lambda\underline{u}], & \text{if } \lambda < 0. \end{cases}$$

The distance between intervals $[\underline{u}, \overline{u}]$ and $[\underline{v}, \overline{v}]$ is defined by

$$d([\underline{u}, \overline{u}], [\underline{v}, \overline{v}]) = \max\left\{|\underline{u} - \underline{v}|, |\overline{u} - \overline{v}|\right\}.$$

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of the real axis. Assume $u : \mathbb{R} \to [0, 1]$ satisfies the following properties:

(1) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;

(2) u is a convex fuzzy set, i.e., for all $x_1, x_2 \in \mathbb{R}$, $\lambda \in (0, 1)$, we have

$$u(\lambda x_1 + (1 - \lambda)x_2) \geqslant \min\{u(x_1), u(x_2)\};$$

(3) u is upper semi-continuous;

(4) $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of the set A.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. For $0 < \alpha \leqslant 1$, denote $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geqslant \alpha\}$. From the conditions (1)-(4), it follows that the α-level set $[u]^\alpha$ is a nonempty compact interval for all $\alpha \in [0, 1]$. We write $[u]^\alpha = \left[\underline{u^\alpha}, \overline{u^\alpha}\right]$. As a distance between fuzzy numbers we use the Hausdorff metric defined by

$$D(u,v) = \sup_{\alpha\in[0,1]} d\big([u]^\alpha,[v]^\alpha\big) = \sup_{\alpha\in[0,1]} \max\left\{|\underline{u^\alpha} - \underline{v^\alpha}|, |\overline{u^\alpha} - \overline{v^\alpha}|\right\}$$

for $u, v \in \mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space. The following properties are well-known:

(1) $D(u \oplus w, v \oplus w) = D(u,v)$;

(2) $D(\lambda \odot u, \lambda \odot v) = |\lambda| D(u,v)$;

(3) $D(u \oplus v, w \oplus e) \leqslant D(u,w) + D(v,e)$ for all $u, v, w, e, \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$.

Let $\mathbb{T}$ be a time scale, i.e., a nonempty closed subset of $\mathbb{R}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. The open and half-open intervals are defined in a similar way. For $t \in \mathbb{T}$ we define the forward jump operator σ by $\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$, where $\inf \emptyset = \sup \mathbb{T}$, while the backward jump operator ρ is defined by $\rho(t) = \sup\{s < t : s \in \mathbb{T}\}$, where $\sup \emptyset = \inf \mathbb{T}$.

If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$, then we say that t is left-scattered. If $\sigma(t) = t$ and $t < \sup \mathbb{T}$, then we say that t is right-dense, while if $\rho(t) = t$ and $t > \inf \mathbb{T}$, then we say that t is left-dense. A point $t \in \mathbb{T}$ is dense if it is right-dense and left-dense at the same time; isolated if it is right-scattered and left-scattered at the same time. The forward graininess function $\mu : \mathbb{T} \to [0,\infty)$ and the backward graininess function $\eta : \mathbb{T} \to [0,\infty)$ are defined by $\mu(t) = \sigma(t) - t$ and $\eta(t) = t - \rho(t)$ for all $t \in \mathbb{T}$, respectively. If $\sup \mathbb{T}$ is finite and left-scattered, then we define $\mathbb{T}^k = \mathbb{T} \backslash \{\sup \mathbb{T}\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$. If $\inf \mathbb{T}$ is finite and right-scattered, then $\mathbb{T}_k = \mathbb{T} \backslash \{\inf \mathbb{T}\}$; otherwise, $\mathbb{T}_k = \mathbb{T}$. We set $\mathbb{T}^k_k = \mathbb{T}^k \cap \mathbb{T}_k$.

Throughout this paper, all considered intervals will be intervals in $\mathbb{T}$. A partition $\mathcal{D}$ of $[a,b]_{\mathbb{T}}$ is a finite collection of interval-point pairs $\{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$, where

$$\{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$$

and $\xi_i \in [a,b]_{\mathbb{T}}$ for $i = 1, 2, \ldots, n$. By $\Delta t_i = t_i - t_{i-1}$ we denote the length of the ith subinterval in the partition $\mathcal{D}$. $\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))$ is a Δ-gauge for $[a,b]_{\mathbb{T}}$ provided that $\delta_L(\xi) > 0$ on $(a,b]_{\mathbb{T}}$, $\delta_R(\xi) > 0$ on $[a,b)_{\mathbb{T}}$, $\delta_L(a) \geqslant 0, \delta_R(b) \geqslant 0$, and $\delta_R(\xi) \geqslant \mu(\xi)$ for all $\xi \in [a,b)_{\mathbb{T}}$. Let $\delta^1(\xi)$ and $\delta^2(\xi)$ be Δ-gauges for $[a,b]_{\mathbb{T}}$ such that $0 < \delta^1_L(\xi) < \delta^2_L(\xi)$ for all $\xi \in (a,b]_{\mathbb{T}}$ and $0 < \delta^1_R(\xi) < \delta^2_R(\xi)$ for all $\xi \in [a,b)_{\mathbb{T}}$. Then $\delta^1(\xi)$ is finer than $\delta^2(\xi)$ and we write $\delta^1(\xi) < \delta^2(\xi)$. We say that $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ is

(1) a partial partition of $[a,b]_{\mathbb{T}}$ if $\bigcup_{i=1}^n [t_{i-1}, t_i]_{\mathbb{T}} \subset [a,b]_{\mathbb{T}}$;

(2) a partition of $[a,b]_{\mathbb{T}}$ if $\bigcup_{i=1}^n [t_{i-1}, t_i]_{\mathbb{T}} = [a,b]_{\mathbb{T}}$;

(3) a δ-fine McShane partition of $[a,b]_{\mathbb{T}}$ if $[t_{i-1}, t_i]_{\mathbb{T}} \subset (\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i))_{\mathbb{T}}$ and $\xi_i \in [a,b]_{\mathbb{T}}$ for all $i = 1, 2, \ldots, n$;

(4) a δ-fine C-partition of $[a,b]_{\mathbb{T}}$ if it is a δ-fine McShane partition of $[a,b]_{\mathbb{T}}$ satisfying the condition

$$\sum_{i=1}^n \mathrm{dist}(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}}) < \frac{1}{\varepsilon}$$

for the given arbitrary $\varepsilon > 0$, where $\mathrm{dist}(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})$ denotes the distance of ξ_i from $[t_{i-1}, t_i]_{\mathbb{T}}$.

Given a δ-fine C-partition (McShane partition) $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ of $[a,b]_{\mathbb{T}}$, we write

$$S(f, \mathcal{D}, \delta) = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1})$$

for integral sums over $\mathcal{D}$, whenever $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ or $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$.

Definition 2.1. A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is called McShane delta integrable on $[a, b]_{\mathbb{T}}$ if there exists an $A \in \mathbb{R}$ such that for each $\epsilon > 0$ there exists a Δ-gauge, δ, for $[a, b]_{\mathbb{T}}$ such that

$$\left|S(f, \mathcal{D}, \delta), A\right| < \epsilon$$

for each δ-fine McShane partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^{n}$ of $[a, b]_{\mathbb{T}}$. In this case, A is called the McShane delta integral of f on $[a, b]_{\mathbb{T}}$ and is denoted by $A = (M)\int_a^b f(t)\Delta t$.

Definition 2.2. A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is called C-delta integrable on $[a, b]_{\mathbb{T}}$ if there exists an $A \in \mathbb{R}$ such that for each $\epsilon > 0$ there exists a Δ-gauge, δ, for $[a, b]_{\mathbb{T}}$ such that

$$\left|S(f, \mathcal{D}, \delta), A\right| < \epsilon$$

for each δ-fine C-partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^{n}$ of $[a, b]_{\mathbb{T}}$. In this case, A is called the C-delta integral of f on $[a, b]_{\mathbb{T}}$ and is denoted by $A = (C)\int_a^b f(t)\Delta t$. The collection of all functions that are C-delta integrable on $[a, b]_{\mathbb{T}}$ will be denoted by $\mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$.

Lemma 2.3 ([19]). *If $u \in \mathbb{R}_{\mathcal{F}}$, then*

(1) $[u]^{\alpha}$ *is a closed interval,* $\alpha \in [0, 1]$*;*
(2) $[u]^{\alpha_1} \supset [u]^{\alpha_2}$ *whenever* $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$*;*
(3) *for any α_n converging increasingly to* $\alpha \in (0, 1]$, $\bigcap_{n=1}^{\infty}[u]^{\alpha_n} = [u]^{\alpha}$.

Conversely, if $\{\tilde{A}^{\alpha} : \alpha \in [0, 1]\}$ *is a family of subsets of* $\mathbb{R}$ *satisfying* (1)-(3)*, then there exists a* $u \in \mathbb{R}_{\mathcal{F}}$ *such that* $[u]^{\alpha} = \tilde{A}^{\alpha}$ *for* $\alpha \in (0, 1]$ *and*

$$[u]^0 = \overline{\bigcup_{0<\alpha\leqslant 1} \tilde{A}^{\alpha}} \subset \tilde{A}^0.$$

Lemma 2.4 ([14]). *If $u \in \mathbb{R}_{\mathcal{F}}$, then*

(1) $\underline{u^{\alpha}}$ *is a bounded nondecreasing function on* $[0, 1]$*;*
(2) $\overline{u^{\alpha}}$ *is a bounded nonincreasing function on* $[0, 1]$*;*
(3) $\underline{u^1} \leqslant \overline{u^1}$*;*
(4) *for* $c \in (0, 1]$, $\lim_{\alpha\to c^-} \underline{u^{\alpha}} = \underline{u^c}$, $\lim_{\alpha\to c^-} \overline{u^{\alpha}} = \overline{u^c}$*;*
(5) $\lim_{\alpha\to 0^+} \underline{u^{\alpha}} = \underline{u^0}$, $\lim_{\alpha\to 0^+} \overline{u^{\alpha}} = \overline{u^0}$.

Conversely, if $\underline{u^{\alpha}}$ *and* $\overline{u^{\alpha}}$ *satisfy* (1)-(5)*, then there exists a* $v \in \mathbb{R}_{\mathcal{F}}$ *such that* $[v]^{\alpha} = \left[\underline{v^{\alpha}}, \overline{v^{\alpha}}\right] = \left[\underline{u^{\alpha}}, \overline{u^{\alpha}}\right]$.

3. C-Delta integral of interval-valued functions

Definition 3.1. An interval-valued function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{I}}$ is called interval C-delta integrable (IC Δ-integrable) on $[a, b]_{\mathbb{T}}$ if there exists an $A \in \mathbb{R}_{\mathcal{I}}$ such that for each $\epsilon > 0$ there exists a Δ-gauge, δ, for $[a, b]_{\mathbb{T}}$ such that

$$d\big(S(f, \mathcal{D}, \delta), A\big) < \epsilon$$

for each δ-fine C-partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^{n}$ of $[a, b]_{\mathbb{T}}$. In this case, A is called the IC Δ-integral of f on $[a, b]_{\mathbb{T}}$ and is denoted by $A = (IC)\int_a^b f(t)\Delta t$. The collection of all functions that are IC Δ-integrable on $[a, b]_{\mathbb{T}}$ will be denoted by $\mathcal{IC}_{(\Delta,\ [a,b]_{\mathbb{T}})}$.

For the IC Δ-integral, we have the following properties.

Theorem 3.2. *An interval-valued function* $f(t) \in \mathcal{IC}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ *if and only if* $\overline{f(t)}, \underline{f(t)} \in \mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ *and*

$$(IC)\int_a^b f(t)\Delta t = \left[(C)\int_a^b \underline{f(t)}\Delta t, (C)\int_a^b \overline{f(t)}\Delta t\right].$$

Proof. Let $f(t) \in \mathcal{IC}_{(\Delta,\ [a,b]_{\mathbb{T}})}$. Then there exists an $A \in \mathbb{R}_{\mathcal{I}}$ such that for each $\epsilon > 0$ there exists a Δ-gauge, δ, for $[a,b]_{\mathbb{T}}$ such that

$$d\big(S(f,\mathcal{D},\delta),A\big) < \epsilon$$

for each δ-fine C-partition $\mathcal{D} = \{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^{n}$ of $[a,b]_{\mathbb{T}}$. It follows that

$$\max\Big(\big|\underline{S(f,\mathcal{D},\delta)}-\underline{A}\big|,\big|\overline{S(f,\mathcal{D},\delta)}-\overline{A}\big|\Big) < \epsilon.$$

Then we have

$$\big|\underline{S(f,\mathcal{D},\delta)}-\underline{A}\big| < \epsilon, \quad \big|\overline{S(f,\mathcal{D},\delta)}-\overline{A}\big| < \epsilon.$$

By Definition 2.2, $\overline{f(t)}$, $\underline{f(t)} \in \mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ and

$$(IC)\int_a^b f(t)\Delta t = \left[(C)\int_a^b \underline{f(t)}\Delta t,\ (C)\int_a^b \overline{f(t)}\Delta t\right].$$

Conversely, if $\underline{f(t)} \in \mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$, then there exists an $\underline{A} \in \mathbb{R}$ such that for each $\epsilon > 0$ there exists a Δ-gauge, δ_1, for $[a,b]_{\mathbb{T}}$ such that

$$\big|\underline{S(f,\mathcal{D}_1,\delta_1)}-\underline{A}\big| < \epsilon$$

for each δ_1-fine C-partition $\mathcal{D}_1$ of $[a,b]_{\mathbb{T}}$. Similarly, there exists a Δ-gauge δ_2 such that

$$\big|\overline{S(f,\mathcal{D}_2,\delta_2)}-\overline{A}\big| < \epsilon$$

for each δ_2-fine C-partition $\mathcal{D}_2$ of $[a,b]_{\mathbb{T}}$.

Let $\delta_2 = \min\{\delta_1,\delta_2\}$ and $A = [\underline{A},\overline{A}]$. Then

$$d\big(S(f,\mathcal{D},\delta),A\big) < \epsilon$$

for each δ-fine C-partition $\mathcal{D}$ of $[a,b]_{\mathbb{T}}$ and the proof is complete. □

The following Theorems 3.3 and 3.4 are obvious, because their proofs are similar to those of [27].

Theorem 3.3. *If* $f(t), g(t) \in \mathcal{IC}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ *and* $\alpha,\beta \in \mathbb{R}$, *then* $\alpha f(t)+\beta g(t) \in \mathcal{IC}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ *and*

$$(IC)\int_a^b (\alpha f(t)+\beta g(t))\Delta t = \alpha(IC)\int_a^b f(t)\Delta t + \beta(IC)\int_a^b g(t)\Delta t.$$

Theorem 3.4. *Let* $a<c<b$. *If* $f(t) \in \mathcal{IC}_{(\Delta,\ [a,c]_{\mathbb{T}})}$ *and* $f(t) \in \mathcal{IC}_{(\Delta,\ [c,b]_{\mathbb{T}})}$, *then so it is on* $[a,b]_{\mathbb{T}}$ *and*

$$(IC)\int_a^b f(t)\Delta t = (IC)\int_a^c f(t)\Delta t + (IC)\int_c^b f(t)\Delta t.$$

4. C-Delta integral of fuzzy-valued functions

Definition 4.1. A fuzzy-valued function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$ is called fuzzy C-delta integrable (FC Δ-integrable) on $[a,b]_{\mathbb{T}}$ if there exists a fuzzy number $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$ such that for each $\epsilon > 0$ there exists a Δ-gauge, δ, for $[a,b]_{\mathbb{T}}$ such that

$$D\big(S(f,\mathcal{D},\delta),\tilde{A}\big) < \epsilon$$

for each δ-fine C-partition $\mathcal{D} = \{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^{n}$ of $[a,b]_{\mathbb{T}}$. In this case, $\tilde{A}$ is called the FC Δ-integral of f on $[a,b]_{\mathbb{T}}$ and is denoted by $\tilde{A} = (FC)\int_a^b f(t)\Delta t$. The collection of all functions that are FC Δ-integrable on $[a,b]_{\mathbb{T}}$ will be denoted by $\mathcal{FC}_{(\Delta,\ [a,b]_{\mathbb{T}})}$.

Remark 4.2. It is clear that if f is a real-valued function, then Definition 4.1 yields the definition of C-delta integral introduced by [13].

For the FC Δ-integral, we have the following properties.

Theorem 4.3. *If $f(t) \in \mathcal{FC}_{(\Delta,\ [a,c]_{\mathbb{T}})}$, then the integral of $f(t)$ is determined uniquely.*

Theorem 4.4. *If $f(t), g(t) \in \mathcal{FC}_{(\Delta,\ [a,c]_{\mathbb{T}})}$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f(t) + \beta g(t) \in \mathcal{FC}_{(\Delta,\ [a,c]_{\mathbb{T}})}$ and*

$$(FC)\int_a^b (\alpha f(t) + \beta g(t))\Delta t = \alpha (FC)\int_a^b f(t)\Delta t + \beta (FC)\int_a^b g(t)\Delta t.$$

Theorem 4.5 (Cauchy-Bolzano condition)**.** *A function $f(t) \in \mathcal{FC}_{(\Delta,\ [a,c]_{\mathbb{T}})}$ if and only if for each $\epsilon > 0$ there exists a Δ-gauge, δ, for $[a,b]_{\mathbb{T}}$ such that*

$$D\big(S(f, \mathcal{D}_1, \delta), S(f, \mathcal{D}_2, \delta)\big) < \epsilon$$

for each pair of δ-fine C-partitions $\mathcal{D}_1, \mathcal{D}_2$ of $[a,b]_{\mathbb{T}}$.

Theorem 4.6. *Let $a < c < b$. If $f(t) \in \mathcal{FC}_{(\Delta,\ [a,c]_{\mathbb{T}})}$ and $f(t) \in \mathcal{FC}_{(\Delta,\ [c,b]_{\mathbb{T}})}$, then so it is on $[a,b]_{\mathbb{T}}$ and*

$$(FC)\int_a^b f(t)\Delta t = (FC)\int_a^c f(t)\Delta t + (FC)\int_c^b f(t)\Delta t.$$

Theorem 4.7. *If $f(t) \in \mathcal{FC}_{(\Delta,\ [a,b]_{\mathbb{T}})}$, then $f(t) \in \mathcal{FC}_{(\Delta,\ [c,d]_{\mathbb{T}})}$ for any $[c,d]_{\mathbb{T}} \subset [a,b]_{\mathbb{T}}$.*

Proof. We only prove that Theorem 4.5 holds, the others are obvious.

(Necessity). Suppose that $f(t) \in \mathcal{FC}_{(\Delta,\ [a,c]_{\mathbb{T}})}$ and $\epsilon > 0$. Then, there exists a Δ-gauge, δ, for $[a,b]_{\mathbb{T}}$ such that

$$D\left(S(f, \mathcal{D}, \delta), (FC)\int_a^b f(t)\Delta t\right) < \frac{\epsilon}{2}$$

for each δ-fine C-partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ of $[a,b]_{\mathbb{T}}$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two δ-fine C-partitions of $[a,b]_{\mathbb{T}}$. Then,

$$D\big(S(f, \mathcal{D}_1, \delta), S(f, \mathcal{D}_2, \delta)\big) \leqslant D\left(S(f, \mathcal{D}_1, \delta), (FC)\int_a^b f(t)\Delta t\right) + D\left(S(f, \mathcal{D}_2, \delta), (FC)\int_a^b f(t)\Delta t\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(Sufficiency). For each $n \in \mathbb{N}$, choose a Δ_n-gauge, δ_n, for $[a,b]_{\mathbb{T}}$ such that for any two δ_n-fine C-partitions $\mathcal{D}_1$ and $\mathcal{D}_2$ of $[a,b]_{\mathbb{T}}$ we have

$$D\big(S(f, \mathcal{D}_1, \delta_n), S(f, \mathcal{D}_2, \delta_n)\big) < \frac{1}{n}.$$

Replacing δ_n with $\bigcap_{j=1}^n \delta_j = \delta_n$, we may assume that $\delta_{n+1} \subset \delta_n$. For each n, fix a δ_n-fine C-partitions $\mathcal{D}_n$. Note that for $j > n$, since $\delta_j \subset \delta_n$, $\mathcal{D}_j$ is a δ_n-fine C-partitions of $[a,b]_{\mathbb{T}}$. Thus,

$$D\big(S(f, \mathcal{D}_n, \delta_n), S(f, \mathcal{D}_j, \delta_n)\big) < \frac{1}{n},$$

which implies that the sequence $\{S(f, \mathcal{D}_n, \delta_n)\}$ is a Cauchy sequence, and hence converges. Let $\tilde{A}$ be the limit of this sequence. It follows from the previous inequality that

$$D\big(S(f, \mathcal{D}_n, \delta_n), \tilde{A}\big) < \frac{1}{n}.$$

It remains to show that $\tilde{A}$ satisfies Definition 4.1. Fix $\epsilon > 0$ and choose $N > 2/\epsilon$. Let $\mathcal{D}$ be a δ_N-fine C-partitions of $[a,b]_{\mathbb{T}}$. Then,

$$D\big(S(f, \mathcal{D}, \delta_N), \tilde{A}\big) \leqslant D\big(S(f, \mathcal{D}_1, \delta_N), S(f, \mathcal{D}, \delta_N)\big) + D\big(S(f, \mathcal{D}, \delta_N), \tilde{A}\big) < \frac{1}{N} + \frac{1}{N} < \epsilon.$$

It follows now that $f(t) \in \mathcal{FC}_{(\Delta,\ [a,c]_{\mathbb{T}})}$. The proof is complete. □

Definition 4.8 ([27])**.** We say that a subset E of a time scale $\mathbb{T}$ has delta measure zero provided that E contains no right-scattered points and E has Lebesgue measure zero. We say that a property A holds delta almost everywhere (delta a.e.) on $\mathbb{T}$ provided that there is a subset E of $\mathbb{T}$ such that the property A holds for all $t \in E$ and E has delta measure zero.

Theorem 4.9. *If* $f(t) = g(t)$ *holds delta a.e. on* $[a, b]_{\mathbb{T}}$ *and* $f(t) \in \mathcal{FC}_{(\Delta,\ [a,b]_{\mathbb{T}})}$, *then* $g(t) \in \mathcal{FC}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ *and*

$$(FC)\int_a^b f(t)\Delta t = (FC)\int_a^b g(t)\Delta t.$$

Proof. Let $\tilde{A}$ denote the integral value of $f(t)$ on $[a, b]_{\mathbb{T}}$. Given $\epsilon > 0$ there exists a Δ-gauge, δ, for $[a, b]_{\mathbb{T}}$ such that

$$D\big(S(f, \mathcal{D}, \delta), \tilde{A}\big) < \epsilon$$

for each δ-fine C-partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^{n}$ of $[a, b]_{\mathbb{T}}$.

Set $E = \sum_{j=1}^{\infty} E_j$, where

$$E_j = \{t : j-1 < D(f(t), g(t)) \leqslant j,\ j = 1, 2, \dots,\ t \in [a, b]_{\mathbb{T}}\}.$$

For each j, there is an F_j which is the union of a countable number of open intervals with the total length less than $\epsilon \cdot 2^{-j} \cdot j^{-1}$ and such that $E_j \subset F_j$. Then define

$$\delta(\xi) = \begin{cases} (\delta_L^0(\xi), \delta_R^0(\xi)), & \text{if } \xi \in [a, b]_{\mathbb{T}} \backslash E, \\ (\delta_L^1(\xi), \delta_R^1(\xi)), & \text{such that } (\xi - \delta_L^1(\xi), \xi + \delta_R^1(\xi))_{\mathbb{T}} \subset F_j,\ \xi \in E_j. \end{cases}$$

Then, for any δ-fine C-partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^{n}$ of $[a, b]_{\mathbb{T}}$, we have

$$\begin{aligned} D\big(S(g, \mathcal{D}, \delta),\ \tilde{A}\big) &= D\Big(\sum_{\xi_i \in [a,b]_{\mathbb{T}}} g(\xi_i)(t_i - t_{i-1}),\ \tilde{A}\Big) \\ &= D\Big(\sum_{\xi_i \in E} g(\xi_i)(t_i - t_{i-1}) + \sum_{\xi_i \in [a,b]_{\mathbb{T}} \backslash E} g(\xi_i)(t_i - t_{i-1}),\ \tilde{A}\Big) \\ &= D\Big(\sum_{\xi_i \in E} g(\xi_i)(t_i - t_{i-1}) + \sum_{\xi_i \in [a,b]_{\mathbb{T}} \backslash E} f(\xi_i)(t_i - t_{i-1}) + \sum_{\xi_i \in E} f(\xi_i)(t_i - t_{i-1}), \\ &\qquad \tilde{A} + \sum_{\xi_i \in E} f(\xi_i)(t_i - t_{i-1})\Big) \\ &\leqslant D\Big(\sum_{\xi_i \in [a,b]_{\mathbb{T}}} f(\xi_i)(t_i - t_{i-1}),\ \tilde{A}\Big) + D\Big(\sum_{\xi_i \in E} g(\xi_i)(t_i - t_{i-1}),\ \sum_{\xi_i \in E} f(\xi_i)(t_i - t_{i-1})\Big) \\ &\leqslant \epsilon + \sum_{j=1}^{\infty} \sum_{\xi_i \in E_j} D\big(g(\xi_i), f(\xi_i)\big)(t_i - t_{i-1}) \leqslant 2\epsilon. \end{aligned}$$

The proof is complete. □

Theorem 4.10 (Dominated convergence theorem)**.** *Assume*

(1) $\lim_{n\to\infty} f_n(t) = f(t)$ *holds delta a.e. on* $[a, b]_{\mathbb{T}}$;

(2) $g(t) \leqslant f_n \leqslant h(t)$ *holds delta a.e. on* $[a, b]_{\mathbb{T}}$ *and* $f_n, g, h \in \mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$.

Then $f(t) \in \mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ *and*

$$\lim_{n\to\infty} (C)\int_a^b f_n(t)\Delta t = (C)\int_a^b f(t)\Delta t.$$

Proof. By hypotheses, the function $\phi(t) = h(t) - g(t)$ is McShane delta integrable on $[a, b]_{\mathbb{T}}$ and $|f_n(t) - f_m(t)| \leqslant \phi(t)$ on $[a, b]_{\mathbb{T}}$ for all n and m. By the dominated convergence theorem for McShane integral (see [35]), $f(t) - f_1(t)$ is McShane delta integrable on $[a, b]_{\mathbb{T}}$ and

$$\lim_{n\to\infty} (C)\int_a^b (f_n(t) - f_1(t))\Delta t = (C)\int_a^b (f(t) - f_1(t))\Delta t.$$

In particular, the sequence $\left\{(C)\int_a^b f_n(t)\Delta t\right\}$ converges. Let $\epsilon > 0$. Since the function $\Phi(x) = \{\int_a^x \phi(t)\Delta t\}$ is absolutely continuous on $[a, b]_{\mathbb{T}}$ (see [10]), there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |\Phi(t_i) - \Phi(t_{i-1})| < \epsilon$$

whenever $t_{i-1} \leqslant t_i$ and $\{[t_{i-1}, t_i)_{\mathbb{T}}\}_{i=1}^n$ is a finite collection of non-overlapping intervals in $[a, b]_{\mathbb{T}}$ satisfying

$$\sum_{i=1}^{n} (t_i - t_{i-1}) < \delta.$$

By Egorov's theorem, there exists an open set G with $m(G) < \delta$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ uniformly for $t \in [a, b]_{\mathbb{T}}\backslash G$. Choose a positive integer N such that

$$\left|(C)\int_a^b f_n(t)\Delta t - (C)\int_a^b f_m(t)\Delta t\right| < \epsilon \quad \text{and} \quad |f_n - f_m| < \epsilon$$

for all $m, n > N$ and for all $t \in [a, b]_{\mathbb{T}}\backslash G$. Let $\delta_\Phi(\xi) = (\delta_L(\xi), \delta_R(\xi))$ be a Δ-gauge for $[a, b]_{\mathbb{T}}$ such that

$$\left|S(\Phi, \mathcal{D}, \delta_\Phi) - (M)\int_a^b \Phi(t)\Delta t\right| < \epsilon \quad \text{and} \quad \left|S(f_n, \mathcal{D}, \delta_\Phi) - (C)\int_a^b f_n(t)\Delta t\right| < \epsilon$$

for $1 \leqslant n \leqslant N$ whenever $\mathcal{D}$ is a δ_Φ-fine C-partition of $[a, b]_{\mathbb{T}}$. Define a Δ-gauge, δ, for $[a, b]_{\mathbb{T}}$ by

$$\delta(\xi) = \begin{cases} \delta_\Phi(\xi), & \text{if } \xi \in [a, b]_{\mathbb{T}}\backslash G, \\ \min\{\delta_\Phi(\xi), \rho(\xi, G)\}, & \text{if } \xi \in G, \end{cases}$$

where $\rho(\xi, G) = \inf\{|\xi - \xi'| : \xi' \in G\}$. Suppose that $\mathcal{D}$ is a δ-fine C-partition of $[a, b]_{\mathbb{T}}$ and fix $n > N$. Then

$$\begin{aligned}
\left|S(f_n, \mathcal{D}, \delta) - (C)\int_a^b f_n(t)\Delta t\right| &\leqslant |S(f_n, \mathcal{D}, \delta) - S(f_N, \mathcal{D}, \delta)| + \left|S(f_N, \mathcal{D}, \delta) - \int_a^b f_N(t)\Delta t\right| \\
&\quad + \left|(C)\int_a^b f_N(t)\Delta t - (C)\int_a^b f_n(t)\Delta t\right| \\
&\leqslant \left|\sum_{\xi_i\in[a,b]_{\mathbb{T}}\backslash G} f_n(\xi_i)(t_i - t_{i-1}) - \sum_{\xi_i\in[a,b]_{\mathbb{T}}\backslash G} f_N(\xi_i)(t_i - t_{i-1})\right| \\
&\quad + \left|\sum_{\xi_i\in G} f_n(\xi_i)(t_i - t_{i-1}) - \sum_{\xi_i\in G} f_N(\xi_i)(t_i - t_{i-1})\right| + \epsilon + \epsilon \\
&\leqslant (b - a + 2)\epsilon + \sum_{\xi_i\in G} |f_n(\xi_i) - f_N(\xi_i)|(t_i - t_{i-1}) \\
&\leqslant (b - a + 2)\epsilon + \left|\sum_{\xi_i\in G} \phi(\xi_i)(t_i - t_{i-1}) - (M)\int_G \phi(t)\Delta t\right| + \left|(M)\int_G \phi(t)\Delta t\right| \\
&\leqslant (b - a + 4)\epsilon.
\end{aligned}$$

It follows that $\left\{(C)\int_a^b f_n(t)\Delta t\right\}$ is a Cauchy sequence. Consequently, we have $f(t) \in \mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ and

$$\lim_{n\to\infty}(C)\int_a^b f_n(t)\Delta t = (C)\int_a^b f(t)\Delta t.$$

The proof is complete. □

Now, we have the necessary machinery to prove the following theorem.

Theorem 4.11. *Let* $f(t)$ *be a fuzzy-valued function. Then* $f(t) \in \mathcal{FC}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ *if and only if* $\underline{u^\alpha}, \overline{u^\alpha} \in \mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ *for any* $\alpha \in [0,1]$ *uniformly, i.e., where* Δ*-gauge in Definition 4.1 is independent of* $\alpha \in [0,1]$.

Proof.

(Necessity). Let $\tilde{A}$ denote the integral value of $f(t)$ on $[a,b]_{\mathbb{T}}$. Given $\epsilon > 0$ there exists a Δ-gauge, δ, for $[a,b]_{\mathbb{T}}$ such that

$$D\big(S(f,\mathcal{D},\delta),\tilde{A}\big) < \epsilon$$

for each δ-fine C-partition $\mathcal{D} = \{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^n$ of $[a,b]_{\mathbb{T}}$. Then

$$\begin{aligned}
&\sup_{\alpha\in[0,1]}\max\left\{\left|\underline{\left[\sum_{i=1}^n f(\xi_i)(t_i-t_{i-1})\right]^\alpha} - \underline{\tilde{A}^\alpha}\right|, \left|\overline{\left[\sum_{i=1}^n f(\xi_i)(t_i-t_{i-1})\right]^\alpha} - \overline{\tilde{A}^\alpha}\right|\right\}\\
&= \sup_{\alpha\in[0,1]}\max\left\{\left|\sum_{i=1}^n \underline{f(\xi_i)^\alpha}(t_i-t_{i-1}) - \underline{\tilde{A}^\alpha}\right|, \left|\sum_{i=1}^n \overline{f(\xi_i)^\alpha}(t_i-t_{i-1}) - \overline{\tilde{A}^\alpha}\right|\right\} < \epsilon.
\end{aligned}$$

Hence, for any $\alpha \in [0,1]$ and for any δ-fine C-partition $\mathcal{D} = \{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^n$ of $[a,b]_{\mathbb{T}}$, we have

$$\left|\sum_{i=1}^n \underline{f(\xi_i)^\alpha}(t_i-t_{i-1}) - \underline{\tilde{A}^\alpha}\right| < \epsilon, \quad \left|\sum_{i=1}^n \overline{f(\xi_i)^\alpha}(t_i-t_{i-1}) - \overline{\tilde{A}^\alpha}\right| < \epsilon.$$

This implies that $\underline{u^\alpha}, \overline{u^\alpha} \in \mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ for any $\alpha \in [0,1]$ uniformly.

(Sufficiency). Since $\underline{u^\alpha}, \overline{u^\alpha} \in \mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ for any $\alpha \in [0,1]$ uniformly, given $\epsilon > 0$ there exists a Δ-gauge, δ, for $[a,b]_{\mathbb{T}}$ such that

$$\left|\sum_{i=1}^n \underline{f(\xi_i)^\alpha}(t_i-t_{i-1}) - \underline{\tilde{A}^\alpha}\right| < \epsilon, \quad \left|\sum_{i=1}^n \overline{f(\xi_i)^\alpha}(t_i-t_{i-1}) - \overline{\tilde{A}^\alpha}\right| < \epsilon$$

for any δ-fine C-partition $\mathcal{D} = \{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^n$ of $[a,b]_{\mathbb{T}}$ and for any $\alpha \in [0,1]$, where $\underline{\tilde{A}^\alpha}$ and $\overline{\tilde{A}^\alpha}$ are the integral values of $\underline{f(\xi_i)^\alpha}$ and $\overline{f(\xi_i)^\alpha}$, respectively.

We can prove that the class of closed intervals $\left\{\left[\underline{\tilde{A}^\alpha}, \overline{\tilde{A}^\alpha}\right], \alpha \in [0,1]\right\}$ determines a fuzzy number. In fact, $\left[\underline{\tilde{A}^\alpha}, \overline{\tilde{A}^\alpha}\right]$ satisfies all conditions of Lemma 2.3.

(1). Since $\underline{f(t)^\alpha} \leqslant \overline{f(t)^\alpha}$, $\alpha \in [0,1]$, we have $\underline{\tilde{A}^\alpha} \leqslant \overline{\tilde{A}^\alpha}$, i.e., $\left[\underline{\tilde{A}^\alpha}, \overline{\tilde{A}^\alpha}\right]$ is a closed interval, $\alpha \in [0,1]$.

(2). Since $\underline{f(t)^\alpha}$ is a nondecreasing function on $[0,1]$ and $\overline{f(t)^\alpha}$ is a nonincreasing function on $[0,1]$, for any $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$, we get

$$(FC)\int_a^b \underline{f(t)^{\alpha_1}}\Delta t \leqslant (FC)\int_a^b \underline{f(t)^{\alpha_2}}\Delta t \leqslant (FC)\int_a^b \overline{f(t)^{\alpha_2}}\Delta t \leqslant (FC)\int_a^b \overline{f(t)^{\alpha_1}}\Delta t,$$

which yields $\left[\underline{\tilde{A}^{\alpha_1}}, \overline{\tilde{A}^{\alpha_1}}\right] \supset \left[\underline{\tilde{A}^{\alpha_2}}, \overline{\tilde{A}^{\alpha_2}}\right]$.

(3). For any α_n converging increasingly to $\alpha \in (0,1]$, $\bigcap_{n=1}^{\infty}\left[f(t)\right]^{\alpha_n} = \left[f(t)\right]^{\alpha}$, i.e.,

$$\bigcap_{n=1}^{\infty}\left[\underline{f(t)^{\alpha_n}}, \overline{f(t)^{\alpha_n}}\right] = \left[\underline{f(t)^{\alpha}}, \overline{f(t)^{\alpha}}\right].$$

That is,

$$\lim_{n\to\infty} \underline{f(t)^{\alpha_n}} = \underline{f(t)^{\alpha}}, \quad \lim_{n\to\infty} \overline{f(t)^{\alpha_n}} = \overline{f(t)^{\alpha}}.$$

We also have

$$\underline{f(t)^{0}} \leqslant \underline{f(t)^{\alpha_n}} \leqslant \underline{f(t)^{1}}, \quad \overline{f(t)^{1}} \leqslant \overline{f(t)^{\alpha_n}} \leqslant \overline{f(t)^{0}}.$$

Thanks to Theorem 4.10, we infer that $\underline{f(t)^{\alpha}}, \overline{f(t)^{\alpha}} \in \mathcal{C}_{(\Delta,\ [a,b]_{\mathbb{T}})}$ and

$$\lim_{n\to\infty} (C)\int_a^b \underline{f(t)^{\alpha_n}}\Delta t = (C)\int_a^b \underline{f(t)^{\alpha}}\Delta t, \quad \lim_{n\to\infty} (C)\int_a^b \overline{f(t)^{\alpha_n}}\Delta t = (C)\int_a^b \overline{f(t)^{\alpha}}\Delta t.$$

Consequently, we obtain

$$\bigcap_{n=1}^{\infty}\left[\underline{\tilde{A}^{\alpha_n}}, \overline{\tilde{A}^{\alpha_n}}\right] = \left[\underline{\tilde{A}^{\alpha}}, \overline{\tilde{A}^{\alpha}}\right].$$

Define $\tilde{A}$ as a fuzzy number which is determined by the closed intervals class $\left\{\left[\underline{\tilde{A}^{\alpha}}, \overline{\tilde{A}^{\alpha}}\right], \alpha \in [0,1]\right\}$. Then, for any δ-fine C-partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^{n}$ of $[a,b]_{\mathbb{T}}$, we have

$$D\big(S(f,\mathcal{D},\delta), \tilde{A}\big) < \epsilon.$$

The proof is complete. □

5. Conclusions

This paper investigated the C-delta integral of interval-valued functions and fuzzy-valued functions on time scales. we gave generalizations of some results on the C-delta integral on time scales. The next steps in the research direction proposed here is to study the characterizations of fuzzy C-delta integrable functions.

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