Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 172–188
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Solutions of p-Kirchhoff type problems with critical nonlinearity in $\mathbb{R}^N$

Check for updates

Yueqiang Song[a], Shaoyun Shi[b,*]

[a]Scientific Research Department, Changchun Normal University, Changchun 130032, Jilin, P. R. China.

[b]School of Mathematics & State Key Laboratory of Automotive Simulation and Control, Jilin University, Changchun 130012, Jilin, P. R. China.

Communicated by V. K. Le

Abstract

In this paper, we are interested in the existence of weak solutions for the fractional p-Laplacian equation with critical nonlinearity in $\mathbb{R}^N$. By using fractional version of concentration compactness principle together with variational method, we obtain the existence and multiplicity of solutions for the above problem.

Keywords: Fractional p-Laplacian equation, critical nonlinearity, variational method, critical points.

2010 MSC: 35J60, 47G20.

1. Introduction

In this paper, we study the existence of weak solutions for the fractional p-Laplacian equation with critical nonlinearity in $\mathbb{R}^N$:

$$\begin{cases} \varepsilon^{ps}\left[a+b[u]_{s,p}^{\theta-1}\right](-\Delta)_p^s u(x)+V(x)|u|^{p-2}u=|u|^{p_s^*-2}u+h(x,u), \; x\in\mathbb{R}^N, \\ u(x)\to 0, \quad \text{as } |x|\to\infty, \end{cases} \tag{1.1}$$

where $a, b > 0$, $\theta \in [1, N/(N-sp))$, $V(x)$ is a nonnegative potential, $N > sp$ with $s \in (0,1)$, $[u]_{s,p}$ will be given later, $p_s^* = Np/(N-ps)$ is the fractional critical exponent, and $(-\Delta)_p^s$ is the fractional p-Laplacian operator which (up to normalization factors) may be defined for any $x \in \mathbb{R}^N$ as

$$(-\Delta)_p^s u(x) := \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy$$

along any $u \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$. For more details about the fractional p-Laplacian operator, we refer to [10, 36, 40].

*Corresponding author
Email addresses: songyq16@mails.jlu.edu.cn (Yueqiang Song), shisy@mail.jlu.edu.cn (Shaoyun Shi)

doi: 10.22436/jnsa.011.02.01

Received: 2017-06-10 *Revised:* 2017-11-22 *Accepted:* 2017-11-24

The study on semilinear elliptic equation involving critical exponent begins from the seminal paper by Brézis and Nirenberg [6]. After that many authors were dedicated to investigating all kinds of elliptic equations with critical growth in bounded domain or in the whole space, we refer to [17–19, 22, 26].

When $p = 2$ and $\theta = 1$, the problem (1.1) arises in the study of the nonlinear fractional Schrödinger equation

$$i\frac{\partial\varphi}{\partial t} = (-\Delta)^s\varphi + W(x)\varphi - g(x, |\varphi|)\varphi, \quad (t, x) \in \mathbb{R}\times\mathbb{R}^N.$$

If we look for standing wave solutions with the form $\varphi(t, x) = u(x)e^{-iEt}$, then we obtain that u satisfies

$$(-\Delta)^s u + V(x)u = \widetilde{f}(x, u),\ x \in \mathbb{R}^N,$$

with $V(x) = W(x) - E$ and $\widetilde{f}(x, u) = g(x, |u|)u$ for a suitable $E > 0$ (see [15, 16]).

In the literature there are many papers on the existence of solutions for fractional Laplacian equations, we refer the reader to [20, 29]. In [12, 35], the authors investigated the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N \tag{1.2}$$

and established some existence theorems on one or infinitely many weak solutions. In [13], the authors used the concentration compactness principle to show that (1.2) has at least two nontrivial radial solutions without assuming the classical Ambrosetti-Rabinowitz condition. In [34], the authors studied the nonlinear fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N$$

and obtained existence and multiplicity results using a perturbed variational method. Recently, some contributions on the existence of solutions for critical fractional Laplacian equations in bounded domain are given in [4], where the effects of lower order perturbations are considered. A Brézis-Nirenberg type result for non-local fractional Laplacian in bounded domain with homogeneous Dirichlet boundary datum is given in [33] by variational techniques, see also [32] for further results. Nonexistence results for nonlocal equations involving critical and supercritical nonlinearities can be found in [31]. A multiplicity result for fractional Laplacian problems in $\mathbb{R}^N$ is obtained in [3] by using the mountain pass theorem and the direct method in variational methods, where one of two superlinear nonlinearities could be critical or even supercritical. It is worth mentioning that the interest in nonlocal fractional problems goes beyond the mathematical curiosity. Indeed, this type of operators arises in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, see for example [1, 7, 10] and the references therein. The literature on non-local operators and their applications is quite large, here we just quote a few, see [23–25, 42] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the readers to [10].

When $p \neq 2$ and $\theta = 1$, there are also some interesting results obtained. In [14], the authors studied the fractional p-Laplacian equation

$$\begin{cases} (-\Delta)_p^s u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega \end{cases}$$

and existence and multiplicity results were established using Morse theory. In [38], the authors investigated a Kirchhoff type problem driven by a non-local integro-differential operator of elliptic type

$$\begin{cases} M\left(\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x-y)dxdy\right) L_K^p u = f(x, u), & \text{in } \ \Omega, \\ u = 0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

the authors obtained two existence theorems on nontrivial weak solutions. In [9], the authors studied a nonlocal equation involving the fractional p-Laplacian

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x, u) + \lambda h \quad \text{in } \mathbb{R}^n.$$

When the nonlinearity f is assumed to have exponential growth, by using a fixed point method, the authors established an existence result on weak solutions. In [38], the authors investigated the existence of solutions for Kirchhoff type problem involving the fractional p-Laplacian via variational methods, where the nonlinearity is subcritical and the Kirchhoff function is non-degenerate. By using the mountain pass theorem and Ekeland's variational principle, the authors in [39] studied the multiplicity of solutions to a nonhomogeneous Kirchhoff type problem driven by the fractional p-Laplacian, where the nonlinearity is convex-concave and the Kirchhoff function is degenerate. Using the same methods as in [39], Pucci et al. in [28] obtained the existence of multiple solutions for the nonhomogeneous fractional p-Laplacian equations of Schrödinger-Kirchhoff type in the whole space. Indeed, the fractional Kirchhoff problems have been extensively studied in recent years, for instance, we also refer to [27] about non-degenerate Kirchhoff type problems and to [2, 29] about degenerate Kirchhoff type problems for the recent advances in this direction.

Motivated by the above and the idea of [11], the aim of this paper is to study the existence and multiplicity of semiclassical solutions for fractional p-Laplacian equation. However, to our best knowledge, there is no result in the literature on problem (1.1). Therefore, in the present paper we are interested in the existence and multiplicity of solutions for problem (1.1) involving the fractional p-Laplacian in $\mathbb{R}^N$. There is no doubt that we encounter serious difficulties because of the lack of compactness and of the nonlocal nature of the p-fractional Laplacian. It is worthwhile to remark that in the arguments developed in [11], one of the key points is to prove the $(PS)_c$ condition. Here we use the fractional version of Lions' second concentration compactness principle and concentration compactness principle at infinity to prove that the $(PS)_c$ condition holds which is different from methods used in [11]. As far as we known, this is the first time that the fractional version of Lions' concentration compactness principle and variational methods are combined to get multiple solutions for perturbed fractional Schrödinger equations with critical nonlinearity (1.1).

We make the following assumptions on $V(x)$ and $h(x, u)$ throughout this paper:

(V) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $V(x_0) = \min V = 0$ and there is $\tau_0 > 0$ such that the set $V^{\tau_0} = \{x \in \mathbb{R}^N : V(x) < \tau_0\}$ has finite Lebesgue measure;

(H) (h_1) $h \in C(\mathbb{R}^N \times [0, +\infty), \mathbb{R})$ and $h(x, t) = o(|t|^{p-1})$ uniformly in x as $t \to 0$;

(h_2) there are $C_0 > 0$ and $1 < q < p_s^*$ such that $|h(x, t)| \leqslant C_0(1 + t^q)$;

(h_3) there are $l_0 > 0$, $\theta p < \nu < p_s^*$, and $\theta p < \mu < p_s^*$ such that $H(x, t) \geqslant l_0|t|^\nu$ and $\mu H(x, t) \leqslant h(x, t)t$ for all (x, t), where $H(x, t) = \int_0^t h(x, s)ds$.

Our main result is the following.

Theorem 1.1. *Let* (V) *and* (H) *be satisfied. Thus*

(1) *for any* $\kappa > 0$ *there is* $\mathcal{E}_\kappa > 0$ *such that if* $\varepsilon \leqslant \mathcal{E}_\kappa$, *problem* (1.1) *has at least one solution* u_ε *satisfying*

$$\frac{\mu - p}{p} \int_{\mathbb{R}^N} H(x, u_\varepsilon)dx + \frac{s}{N} \int_{\mathbb{R}^N} |u_\varepsilon|^{p_s^*} dx \leqslant \kappa\varepsilon^N, \tag{1.3}$$

$$\left(\frac{1}{p} - \frac{1}{\mu}\right) \left[\iint_{\mathbb{R}^{2N}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^p}{|x - y|^{N+sp}} dxdy + \lambda \int_{\mathbb{R}^N} V(x)|u_\varepsilon|^p dx\right] \leqslant \kappa\lambda^{N-ps}. \tag{1.4}$$

Moreover, $u_\varepsilon \to 0$ *as* $\varepsilon \to 0$.

(2) *Assume additionally that* $h(x, t)$ *is odd in* t, *for any* $m \in \mathbb{N}$ *and* $\kappa > 0$ *there is* $\mathcal{E}_{m\kappa} > 0$ *such that if* $\varepsilon \leqslant \mathcal{E}_{m\kappa}$, *problem* (1.1) *has at least* m *pairs of solutions* $u_{\varepsilon,i}$, $u_{\varepsilon,-i}$, $i = 1, 2, \cdots, m$ *which satisfy the estimates* (1.3) *and* (1.4). *Moreover,* $u_{\varepsilon,i} \to 0$ *as* $\varepsilon \to 0$, $i = 1, 2, \cdots, m$.

2. Main results

We set $\lambda = \varepsilon^{-ps}$ and rewrite (1.1) in the following form

$$\begin{cases} \left[a + b[u]_{s,p}^{\theta-1}\right] (-\Delta)_p^s u(x) + \lambda V(x)|u|^{p-2}u = \lambda |u|^{p_s^*-2}u + \lambda h(x,u), \ x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{2.1}$$

where $(-\Delta)_p^s u$ is the fractional p-Laplacian operator.

We recall some results related to the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$, for more details, see [6]. Define the Gagliardo seminorm by

$$[u]_{s,p} = \left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}},$$

where $u : \mathbb{R}^N \to \mathbb{R}$ is a measurable function. Now, the fractional Sobolev space is given by

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : u \text{ is measurable and } [u]_{s,p} < \infty\}$$

with the norm

$$\|u\|_{s,p} = \left([u]_{s,p}^p + \|u\|_p^p\right)^{\frac{1}{p}},$$

where

$$\|u\|_p := \left(\int_{\mathbb{R}^N} |u(x)|^p dx \right)^{\frac{1}{p}}.$$

We recall the Sobolev embedding theorem.

Lemma 2.1 ([10])**.** *Let $s \in (0,1)$ and $p \in [0,+\infty)$ be such that $sp < N$. Then there exists a positive constant $C = C(N,p,s)$ such that*

$$\|u\|_{L^{p_s^*}}^p \leqslant C \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy,$$

where $p_s^* = \frac{Np}{N-sp}$ is the so-called fractional critical exponent. Consequently, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_s^*]$. Moreover the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$ is compact for $q \in [p, p_s^*)$.

In view of the presence of the potential $V(x)$, we consider the fractional Sobolev space

$$E := \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^p dx < \infty \right\},$$

with the norm

$$\|u\|_E := \left([u]_{s,p}^p + \|V(x)^{1/p}u\|_p^p \right)^{\frac{1}{p}}.$$

From condition (V), Lemma 2.1, and Hölder inequality, it follows from that the following embedding

$$E \hookrightarrow L^q(\mathbb{R}^N), \quad p \leqslant q \leqslant p_s^*,$$

is continuous. Moreover, the following compactness result holds. It was proved in [8] in the case $p = 2$. For the general case, the proof is similar.

Lemma 2.2. *Suppose that (V) holds. Then $E \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $q \in [p, p_s^*)$.*

From above facts, for any $\xi \in [p, p_s^*]$, there is $\mu_\xi > 0$ independent of λ such that if $\lambda \geqslant 1$,

$$\|u\|_\xi \leqslant \mu_\xi \|u\|_E \leqslant \mu_\xi \|u\|_\lambda, \tag{2.2}$$

where

$$\|u\|_\lambda := \left[\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy + \lambda \int_{\mathbb{R}^N} V(x)|u|^p dx\right]^{\frac{1}{p}}.$$

In the following , we denote $\|\cdot\|_s$ is the norm in $L^s(\mathbb{R}^N)$ $(p \leqslant s \leqslant p_s^*)$, and $\|u\|_E$ is the norm in E. Note that the norm $\|\cdot\|_E$ is equivalent to the $\|\cdot\|_\lambda$ for each $\lambda > 0$.

The energy functional $J_\lambda : E \to \mathbb{R}$ associated with problem (2.1) is well defined as

$$J_\lambda(u) := \frac{a}{p}[u]_{s,p}^p + \frac{b}{\theta p}[u]_{s,p}^{\theta p} + \frac{\lambda}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \frac{\lambda}{p_s^*}\int_{\mathbb{R}^N} |u|^{p_s^*} dx - \lambda \int_{\mathbb{R}^N} H(x,u)dx.$$

Thus, it is easy to check that as arguments [30, 37] $J_\lambda \in C^1(E, \mathbb{R})$ and its critical points are solutions of (2.1).

We call that $u \in E$ is a weak solution of (2.1), if

$$\begin{aligned}&\left[a + b[u]_{s,p}^{(\theta-1)p}\right] \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(v(x)-v(y))dxdy \\ &= -\lambda \int_{\mathbb{R}^N} V(x)|u|^{p-2}uvdx + \lambda \int_{\mathbb{R}^N} |u|^{p_s^*-2}uvdx + \lambda \int_{\mathbb{R}^N} h(x,u)vdx,\end{aligned}$$

where $v \in E$.

Now we will prove the following result.

Theorem 2.3. *Let* (V) *and* (H) *be satisfied. Thus:*

(1) *For any $\kappa > 0$ there is $\Lambda_\kappa > 0$ such that if $\lambda \leqslant \Lambda_\kappa$, problem* (2.1) *has at least one solution u_λ satisfying*

$$\frac{\mu - p}{p}\int_{\mathbb{R}^N} H(x,u_\lambda)dx + \frac{s}{N}\int_{\mathbb{R}^N} |u_\lambda|^{p_s^*} dx \leqslant \kappa \lambda^{-\frac{N}{ps}}, \tag{2.3}$$

$$\left(\frac{1}{p} - \frac{1}{\mu}\right)\left[\iint_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^p}{|x-y|^{N+sp}} dxdy + \lambda \int_{\mathbb{R}^N} V(x)|u_\lambda|^p dx\right] \leqslant \kappa \lambda^{1-\frac{N}{ps}}. \tag{2.4}$$

Moreover, $u_\lambda \to 0$ in E as $\lambda \to \infty$.

(2) *Assume additionally that $h(x,t)$ is odd in t, for any $m \in \mathbb{N}$ and $\kappa > 0$ there is $\Lambda_{m\kappa} > 0$ such that if $\lambda \geqslant \Lambda_{m\kappa}$, problem* (2.1) *has at least m pairs of solutions $u_{\lambda,i}$, $u_{\lambda,-i}$, $i = 1, 2, \cdots, m$ which satisfy the estimates* (2.3) *and* (2.4). *Moreover, $u_{\lambda,i} \to 0$ in E as $\lambda \to \infty$, $i = 1, 2, \cdots, m$.*

3. Behaviors of (PS) sequences

In this section, in order to overcome the lack of some compactness, we use the fractional version of the principle of concentration compactness of Lions [21] in fractional Sobolev spaces. Let

$$C_c(\mathbb{R}^N) = \{u \in C(\mathbb{R}^N) : \text{supp}(u) \text{ is a compact subset of } \mathbb{R}^N\}$$

and denote by $C_0(\mathbb{R}^N)$ the closure of $C_c(\mathbb{R}^N)$ with respect to the norm $|\eta|_\infty = \sup_{x\in\mathbb{R}^N} |\eta(x)|$. As is well known, a finite measure on $\mathbb{R}^N$ is a continuous linear functional on $C_0(\mathbb{R}^N)$. For a measure μ we give the norm

$$\|\mu\| = \sup_{C_0(\mathbb{R}^N),\ |\eta|_\infty = 1} |(\mu, \eta)|,$$

where $(\mu, \eta) = \int_{\mathbb{R}^N} \eta d\mu$.

Definition 3.1. Let $\mathcal{M}(\mathbb{R}^N)$ denote the finite nonnegative Borel measure space on $\mathbb{R}^N$. For any $\mu \in \mathcal{M}(\mathbb{R}^N)$, the equation $\mu(\mathbb{R}^N) = \|\mu\|$ holds. We say that $\mu_n \rightharpoonup \mu$ weakly * in $\mathcal{M}(\mathbb{R}^N)$, if $(\mu_n, \eta) \to (\mu, \eta)$ holds for all $\eta \in C_0(\mathbb{R}^N)$ as $n \to \infty$.

Lemma 3.2 ([41]). *Let $\{u_n\}_n \subset W^{s,p}(\mathbb{R}^N)$ with upper bound $C > 0$ for all $n \geqslant 1$ and*

$$\begin{aligned} u_n &\rightharpoonup u \ \text{ weakly in } W^{s,p}(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} dy &\rightharpoonup \mu \ \text{ weakly} * \text{ in } \mathcal{M}(\mathbb{R}^N), \\ |u_n(x)|^{p_s^*} &\rightharpoonup \nu \ \text{ weakly} * \text{ in } \mathcal{M}(\mathbb{R}^N). \end{aligned}$$

Then

$$\begin{aligned} \mu &= \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dy + \sum_{j \in J} \mu_j \delta_{x_j} + \widetilde{\mu}, \quad \mu(\mathbb{R}^N) \leqslant C^p, \\ \nu &= |u|^{p_s^*} + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \nu(\mathbb{R}^N) \leqslant S^{p_s^*} C^p, \end{aligned}$$

where J is at most countable, sequences $\{\mu_j\}_j, \{\nu_j\}_j \subset \mathbb{R}_0^+$, $\{x_j\}_j \subset \mathbb{R}^N$, δ_{x_j} is the Dirac mass centered at $\{x_j\}_j$, $\widetilde{\mu}$ is a non-atomic measure,

$$\nu(\mathbb{R}^N) \leqslant S^{-p_s^*/p} \mu(\mathbb{R}^N)^{p_s^*/p}, \quad \nu_j \leqslant S^{-p_s^*/p} \mu_j^{p_s^*/p} \text{ for all } j \in J,$$

and $S > 0$ is the best constant of $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$.

Actually, Lemma 3.2 does not provide any information about the possible loss of mass at infinity for a weakly convergent sequence. The following theorem expresses this fact in quantitative terms.

Lemma 3.3 ([41]). *Let $\{u_n\}_n \subset W^{s,p}(\mathbb{R}^N)$ be a bounded sequence such that*

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} dy &\rightharpoonup \mu \ \text{ weakly} * \text{ in } \mathcal{M}(\mathbb{R}^N), \\ |u_n|^{p_s^*} &\rightharpoonup \nu \ \text{ weakly} * \text{ in } \mathcal{M}(\mathbb{R}^N), \end{aligned}$$

and define

$$\mu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\{x \in \mathbb{R}^N : |x| > R\}} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} dy dx$$

and

$$\nu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\{x \in \mathbb{R}^N : |x| > R\}} |u_n|^{p_s^*} dx.$$

Then the quantities μ_∞ and ν_∞ are well defined and satisfy

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} dy dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty$$

and

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{p_s^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty.$$

Moreover, the following inequality holds

$$S \nu_\infty^{p/p_s^*} \leqslant \mu_\infty.$$

In the following, we use the fractional version of the principle of concentration compactness to show that J_λ satisfies the $(PS)_c$ at energy levels c below some constant.

Lemma 3.4. *Assume that* (V) *and* (H) *be satisfied. Let* $\{u_n\} \subset E$ *be a* $(PS)_c$ *sequence for* J_λ*. Then there exists a constant* $M(c)$ *which is independent of* $\lambda \geqslant 0$ *such that* $c \geqslant 0$ *and*

$$\limsup_{n\to\infty} \|u_n\|_\lambda^p \leqslant M(c).$$

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence for J_λ, thus

$$J_\lambda(u_n) - \frac{1}{\mu} J'_\lambda(u_n)u_n = c + o(1) + \epsilon_n \|u_n\|_\lambda, \tag{3.1}$$

where $\epsilon_n \to 0$ as $n \to \infty$. On the other hand,

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{\mu} J'_\lambda(u_n)u_n = a\left(\frac{1}{p} - \frac{1}{\mu}\right)[u]_{s,p}^p + b\left(\frac{1}{\theta p} - \frac{1}{\mu}\right)[u]_{s,p}^{\theta p} \\ + \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} \lambda V(x)|u_n|^p\,dx + \left(\frac{1}{\mu} - \frac{1}{p_s^*}\right)\lambda \int_{\mathbb{R}^N} |u_n|^{p_s^*}\,dx \\ + \lambda \int_{\mathbb{R}^N} \left[\frac{1}{\mu} h(x,u_n)u_n - H(x,u_n)\right] dx. \end{aligned} \tag{3.2}$$

Condition (h_3) implies that

$$\frac{1}{\mu} h(x,u_n)u_n - H(x,u_n) \geqslant 0 \quad \text{and} \quad b\left(\frac{1}{\theta p} - \frac{1}{\mu}\right)[u]_{s,p}^{\theta p} \geqslant 0.$$

Thus, it follows from (3.1) and (3.2) that

$$\min\left\{\left(\frac{1}{p} - \frac{1}{\mu}\right)a, \left(\frac{1}{p} - \frac{1}{\mu}\right)\right\} \|u_n\|_\lambda^p \leqslant c + o(1) + \epsilon_n \|u_n\|_\lambda,$$

hence for n large enough, there exists constant $M(c) := \left(\min\left\{\left(\frac{1}{p} - \frac{1}{\mu}\right)a, \left(\frac{1}{p} - \frac{1}{\mu}\right)\right\}\right)^{-1} c$ such that

$$\|u_n\|_\lambda^p \leqslant M(c).$$

Thus $\|u_n\|_\lambda$ is bounded as $n \to \infty$. Taking the limit in (3.2) shows that $c \geqslant 0$. □

Lemma 3.5. *Suppose that* (V) *and* (H) *hold. For any* $\lambda \geqslant 1$, J_λ *satisfies* $(PS)_c$ *condition for all* $c \in \left(0, \sigma_0 \lambda^{1 - \frac{p_s^*}{p_s^* - \theta p}}\right)$, *where* $\sigma_0 := \left(\frac{1}{\mu} - \frac{1}{p_s^*}\right)(bS^\theta)^{\frac{p_s^*}{p_s^* - \theta p}}$, *that is any* $(PS)_c$*-sequence* $(u_n) \subset E$ *has a strongly convergent subsequence in* E.

Proof. Since $\{u_n\}_n \subset E$ is bounded and nonnegative, up to a subsequence, there exists a nonnegative function $u \in E$ such that $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^\sigma_{loc}(\mathbb{R}^N)$ for $\sigma \in [1, p_s^*)$, and $u_n \to u$ a.e. in $\mathbb{R}^N$. By Lemma 3.2, up to a subsequence, there exists a (at most) countable set J, non-atomic measure $\widetilde{\mu}$, points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$, and $\{\mu_j\}_{j\in J}, \{\nu_j\}_{j\in J} \subset \mathbb{R}^+$ such that as $n \to \infty$

$$\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\,dy \rightharpoonup \mu = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\,dy + \sum_{j\in J} \mu_j \delta_{x_j} + \widetilde{\mu}$$

and

$$|u_n|^{p_s^*} \rightharpoonup \nu = |u|^{p_s^*} + \sum_{j\in J} \nu_j \delta_{x_j}$$

in the measure sense, where δ_{x_j} is the Dirac measure concentrated x_j. Moreover,

$$\nu_j \leqslant S^{-p_s^*/p} \mu_j^{p_s^*/p}, \quad \forall j \in J, \tag{3.3}$$

and $S > 0$ is the best constant of $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$. Next we claim that $J = \emptyset$. Suppose by contradiction that $J \neq \emptyset$. Fix $j \in J$. For $\varepsilon > 0$, choose $\varphi_{\varepsilon,j} \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi_{\varepsilon,j} = 1 \text{ for } |x - x_j| \leqslant \varepsilon; \; \varphi_{\varepsilon,j} = 0 \text{ for } |x - x_j| \geqslant 2\varepsilon,$$

and $|\nabla \varphi_{\varepsilon,j}| \leqslant 2/\varepsilon$. Obviously, $\varphi_{\varepsilon,j} u_n \in E$. It follows from $\langle J'_\lambda(u_n), \varphi_{\varepsilon,j} u_n\rangle \to 0$ that

$$\begin{aligned} &\left(a + b[u_n]_{s,p}^{(\theta-1)p}\right) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi_{\varepsilon,j}(x)u_n(x) - \varphi_{\varepsilon,j}(y)u_n(y))}{|x-y|^{N+ps}} dxdy \\ &\quad = -\lambda \int_{\mathbb{R}^N} V(x)|u_n|^p \varphi_{\varepsilon,j} dx + \lambda \int_{\mathbb{R}^N} u_n^{p_s^*} \varphi_{\varepsilon,j} dx + \lambda \int_{\mathbb{R}^N} h(x,u_n)\varphi_{\varepsilon,j} u_n dx + o(1). \end{aligned} \tag{3.4}$$

Using the Hölder inequality, we deduce

$$\begin{aligned} &\lim_{\varepsilon\to 0}\lim_{n\to\infty} \left| \left(a + b[u_n]_{s,p}^{(\theta-1)p}\right) \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y))u_n(x)}{|x-y|^{N+ps}} dxdy \right| \\ &\leqslant C \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{(p-1)/p} \left(\iint_{\mathbb{R}^{2N}} \frac{|(\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y))u_n(x)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p} \\ &\leqslant C \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left(\iint_{\mathbb{R}^{2N}} \frac{|(\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y))u_n(x)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}. \end{aligned} \tag{3.5}$$

Similar to the proof of Lemma 2.1 in [41], we claim that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|(\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y))u_n(x)|^p}{|x-y|^{N+ps}} dxdy = 0. \tag{3.6}$$

In the following, we just give a sketch of the proof for the reader's convenience.

On the one hand, we have

$$\begin{aligned} \mathbb{R}^N \times \mathbb{R}^N &= ((\mathbb{R}^N \setminus B(x_j,2\varepsilon)) \cup B(x_j,2\varepsilon)) \times ((\mathbb{R}^N \setminus B(x_j,2\varepsilon)) \cup B(x_j,2\varepsilon)) \\ &= ((\mathbb{R}^N \setminus B(x_j,2\varepsilon)) \times (\mathbb{R}^N \setminus B(x_j,2\varepsilon))) \cup (B(x_j,2\varepsilon) \times \mathbb{R}^N) \cup ((\mathbb{R}^N \setminus B(x_j,2\varepsilon)) \times B(x_j,2\varepsilon)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p |\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)|^p}{|x-y|^{N+ps}} dxdy \\ &\quad = \iint_{B(x_j,2\varepsilon)\times\mathbb{R}^N} \frac{|u_n(x)|^p |\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)|^p}{|x-y|^{N+ps}} dxdy \\ &\qquad + \iint_{(\mathbb{R}^N\setminus B(x_j,2\varepsilon))\times B(x_j,2\varepsilon)} \frac{|u_n(x)|^p |\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)|^p}{|x-y|^{N+ps}} dxdy \\ &\quad \leqslant C\varepsilon^{-ps} \int_{B(x_j,2\varepsilon)} |u_n(x)|^p dx + C\varepsilon^{-ps} \int_{B(x_j,3\varepsilon)} |u_n(x)|^p dx + C\varepsilon^{-ps} \int_{B(x_j,K\varepsilon)} |u_n(x)|^p dx \\ &\qquad + CK^{-N} \left(\int_{\mathbb{R}^N\setminus B(x_j,K\varepsilon)} |u_n(x)|^{p_s^*} dx \right)^{p/p_s^*} \\ &\quad \leqslant C\varepsilon^{-ps} \int_{B(x_j,3\varepsilon)} |u_n(x)|^p dx + CK^{-N} \left(\int_{\mathbb{R}^N\setminus B(x_j,K\varepsilon)} |u_n(x)|^{p_s^*} dx \right)^{p/p_s^*} \end{aligned}$$

$$\leqslant C\varepsilon^{-2s}\int_{B(x_j,K\varepsilon)}|u_n(x)|^p\,dx+CK^{-N}.$$

Note that $u_n \rightharpoonup u$ weakly in E, then $u_n \to u$ in $L^{\sigma}_{loc}(\mathbb{R}^N), 1\leqslant \sigma < p_s^*$, which implies that

$$C\varepsilon^{-ps}\int_{B(x_j,K\varepsilon)}|u_n(x)|^p\,dx+CK^{-N}\to C\varepsilon^{-ps}\int_{B(x_j,K\varepsilon)}|u(x)|^p\,dx+CK^{-N},$$

as $n\to\infty$. By the Hölder inequality, we obtain

$$\begin{aligned}C\varepsilon^{-ps}\int_{B(x_j,K\varepsilon)}|u(x)|^p\,dx+CK^{-N}&\leqslant C\varepsilon^{-ps}\left(\int_{B(x_j,K\varepsilon)}|u(x)|^{p_s^*}dx\right)^{p/p_s^*}\left(\int_{B(x_j,K\varepsilon)}dx\right)^{1-p/p_s^*}+CK^{-N}\\&=CK^{ps}\left(\int_{B(x_j,K\varepsilon)}|u(x)|^{p_s^*}dx\right)^{p/p_s^*}+CK^{-N}\to CK^{-N}\end{aligned}$$

as $\varepsilon\to 0$. Furthermore, we have

$$\begin{aligned}&\limsup_{\varepsilon\to0}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^p|\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y)|^p}{|x-y|^{N+ps}}dxdy\\&=\lim_{K\to\infty}\limsup_{\varepsilon\to0}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^p|\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y)|^p}{|x-y|^{N+ps}}dxdy=0.\end{aligned}$$

Hence the claim holds.

We deduce from (3.4)-(3.6) that

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to\infty}\left(a+b[u_n]_{s,p}^{(\theta-1)p}\right)\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\varphi_{\varepsilon,j}(y)\,dydx\\&\geqslant\lim_{\varepsilon\to0}\lim_{n\to\infty}\left[a\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\varphi_{\varepsilon,j}(y)\,dxdy+b\left(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\varphi_{\varepsilon,j}(y)\,dxdy\right)^{\theta}\right]\\&\geqslant b\mu_j^{\theta},\end{aligned}\tag{3.7}$$

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N}u_n^{p_s^*}\varphi_{\varepsilon,j}\,dx=\lim_{\varepsilon\to0}\int_{\mathbb{R}^N}u^{p_s^*}\varphi_{\varepsilon,j}\,dx+\nu_j=\nu_j,\tag{3.8}$$

and

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N}h(x,u_n)u_n\phi_{\varepsilon,j}(x)\,dx=\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N}h(x,u)u\phi_{\varepsilon,j}(x)\,dx=0.\tag{3.9}$$

It follows from (3.7)-(3.9) that

$$\nu_j\geqslant b\mu_j^{\theta}.$$

Combining this inequality with (3.3), we obtain $\nu_j\geqslant b\lambda^{-1}S^{\theta}\nu_j^{\frac{\theta p}{p_s^*}}$. This result implies that

(I) $\nu_j=0$ or

(II) $\nu_j\geqslant(b\lambda^{-1}S^{\theta})^{\frac{p_s^*}{p_s^*-\theta p}}$.

To obtain the possible concentration of mass at infinity, similarly, we define a cut off function $\chi_R\in C^{\infty}(\mathbb{R}^{\mathbb{N}})$ which satisfies $\chi_R\in[0,1]$ and $\chi_R(x)=0$ for $|x|<R$, $\chi_R(x)=1$ for $|x|>2R$, and $|\nabla\chi_R|\leqslant 2/R$.

Note that $\langle J'_\lambda(u_n), u_n\chi_R\rangle \to 0$, this fact implies that

$$\begin{aligned}\left(a+b[u_n]_{s,p}^{(\theta-1)p}\right)&\left[\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p\chi_R(x)}{|x-y|^{N+ps}}dxdy+\lambda\int_{\mathbb{R}^N}V(x)|u_n|^p\chi_R dx\right.\\&\left.+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))u_n(y)(\chi_R(x)-\chi_R(y))}{|x-y|^{N+ps}}dxdy\right]\\&=\lambda\int_{\mathbb{R}^N}u_n^{p_s^*}\chi_R dx+\lambda\int_{\mathbb{R}^N}f(x,u_n)u_n\chi_R dx+o(1).\end{aligned}\tag{3.10}$$

Similarly, by the Hölder inequality, it is easy to prove that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))u_n(y)(\chi_R(x)-\chi_R(y))}{|x-y|^{N+ps}}dxdy=0,\tag{3.11}$$

and

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}h(x,u_n)u_n\chi_R dx=0.\tag{3.12}$$

Hence we deduce from (3.10)-(3.12) that

$$\begin{aligned}\lim_{R\to\infty}\limsup_{n\to\infty}&\left(a+b[u_n]_{s,p}^{(\theta-1)p}\right)\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p\chi_R(x)}{|x-y|^{N+ps}}dxdy\\&=\left[a+b\left(\int_{\mathbb{R}^N}d\mu+\mu_\infty\right)^{\theta-1}\right]\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{\{x\in\mathbb{R}^N:|x|>R\}}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dydx\right)\\&\geqslant\left(a+b\mu_\infty^\theta\right)\mu_\infty\geqslant b\mu_\infty^\theta\end{aligned}$$

and

$$\nu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n(x)\chi_R(x)|^{p_s^*}dx.$$

Letting $R\to\infty$ in (3.10), we obtain $\nu_\infty\geqslant b\lambda^{-1}S^\theta\nu_\infty^{\frac{\theta p}{p_s^*}}$. By Lemma 3.3, we obtain $\nu_\infty\geqslant\lambda^{-1}S\nu_\infty^{\frac{p}{p_s^*}}$. This result implies that

(III) $\nu_\infty=0$ or

(IV) $\nu_\infty\geqslant(b\lambda^{-1}S^\theta)^{\frac{p_s^*}{p_s^*-\theta p}}$.

Next, we claim that (II) and (IV) cannot occur. If the case (IV) holds, for some $j\in J$, then by using Lemma 3.3 and condition (h_3), we have that

$$\begin{aligned}c&=\lim_{n\to\infty}\left(J_\lambda(u_n)-\frac{1}{\mu}\langle J'_\lambda(u_n),u_n\rangle\right)\\&=\lim_{n\to\infty}\left(a\left(\frac{1}{p}-\frac{1}{\mu}\right)[u]_{s,p}^p+b\left(\frac{1}{\theta p}-\frac{1}{\mu}\right)[u]_{s,p}^{\theta p}\right)+\left(\frac{1}{p}-\frac{1}{\mu}\right)\lambda\int_{\mathbb{R}^N}V(x)|u_n|^p dx\\&\quad+\left(\frac{1}{\mu}-\frac{1}{p_s^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{p_s^*}dx+\lambda\int_{\mathbb{R}^N}\left[\frac{1}{\mu}h(x,u_n)u_n-H(x,u_n)\right]dx\\&\geqslant\left(\frac{1}{\mu}-\frac{1}{p_s^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{p_s^*}dx\geqslant\left(\frac{1}{\mu}-\frac{1}{p_s^*}\right)\lambda\int_{\mathbb{R}^N}|u_n|^{p_s^*}\chi_R dx\\&=\left(\frac{1}{\mu}-\frac{1}{p_s^*}\right)\lambda\nu_\infty\geqslant\sigma_0\lambda^{1-\frac{p_s^*}{p_s^*-\theta p}},\end{aligned}$$

where $\sigma_0 = \left(\frac{1}{\mu} - \frac{1}{p_s^*}\right)(bS^\theta)^{\frac{p_s^*}{p_s^*-\theta p}}$. This is impossible. Consequently, $\nu_j = 0$ for all $j \in J$. Similarly, we can prove that (II) cannot occur for each j. Thus

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} u_n^{p_s^*} dx = \int_{\mathbb{R}^N} u^{p_s^*} dx.$$

From the weak lower semicontinuity of the norm and Brezis-Lieb Lemma [6], we have

$$\begin{aligned}
o(1)\|u_n\| &= \langle J'(u_n), u_n\rangle \\
&= a[u_n]_{s,p}^p + b[u_n]_{s,p}^{\theta p} + \lambda \int_{\mathbb{R}^N} V(x)|u_n|^p dx - \lambda \int_{\mathbb{R}^N} |u_n|^{p_s^*} dx - \lambda \int_{\mathbb{R}^N} h(x,u_n)u_n dx \\
&\geqslant a[u_n - u]_{s,p}^p + \lambda \int_{\mathbb{R}^N} V(x)|u_n - u|^p dx + a[u]_{s,p}^p + b[u]_{s,p}^{\theta p} + \lambda \int_{\mathbb{R}^N} V(x)|u|^p dx \\
&\quad - \lambda \int_{\mathbb{R}^N} |u|^{p_s^*} dx - \lambda \int_{\mathbb{R}^N} h(x,u)u dx \\
&= \min\{a,1\}\|u_n - u\|_\lambda^p + o(1)\|u\|,
\end{aligned}$$

here we use $J'(u) = 0$. Thus we prove that $\{u_n\}$ strongly converges to u in $Z(\Omega)$. This completes the proof of Lemma 3.5. □

4. Proofs of Theorem 2.3

In the following, we always consider $\lambda \geqslant 1$. By the assumptions (V) and (H), one can see that $J_\lambda(u)$ has mountain pass geometry.

Lemma 4.1. *Assume* (V) *and* (H) *hold. There exist* $\alpha_\lambda, \rho_\lambda > 0$ *such that* $J_\lambda(u) > 0$ *if* $u \in B_{\rho_\lambda} \setminus \{0\}$ *and* $J_\lambda(u) \geqslant \alpha_\lambda$ *if* $u \in \partial B_{\rho_\lambda}$, *where* $B_{\rho_\lambda} = \{u \in E : \|u\|_\lambda \leqslant \rho_\lambda\}$.

Proof. By (h_1)-(h_3), for $\delta \leqslant \left(2p\lambda\mu_p^p\right)^{-1}$ there is $C_\delta > 0$ such that

$$\frac{1}{p_s^*} \int_{\mathbb{R}^N} |u|^{p_s^*} dx + \int_{\mathbb{R}^N} H(x,u) dx \leqslant \delta\|u\|_p^p + C_\delta \|u\|_{p_s^*}^{p_s^*},$$

where μ_s is the embedding constant of (2.2). So, from condition (H) it follows that

$$J_\lambda(u) \geqslant \min\left\{\frac{a}{p}, 1\right\}\|u\|_\lambda^p - \lambda\delta\|u\|_p^p - \lambda C_\delta \|u\|_{p_s^*}^{p_s^*} \geqslant \frac{1}{2}\min\left\{\frac{a}{p}, 1\right\}\|u\|_\lambda^p - \lambda C_\delta \mu_{p_s^*}^{p_s^*}\|u\|_{p_s^*}^{p_s^*}.$$

Since $p_s^* > p$, we know that the conclusion of Lemma 4.1 holds. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Under the assumption of Lemma 4.1, for any finite dimensional subspace* $F \subset X^s$,

$$J_\lambda(u) \to -\infty \quad as \quad u \in F,\ \|u\|_\lambda \to \infty.$$

Proof. Using conditions (V) and (h_1)-(h_3), we can get

$$J_\lambda(u) \leqslant \max\left\{\frac{a}{p}, 1\right\}\|u\|_\lambda^p + \frac{b}{\theta p}\|u\|_\lambda^{\theta p} - \frac{\lambda}{p_s^*}\|u\|_{p_s^*}^{p_s^*} - \lambda l_0 \|u\|_\nu^\nu$$

for all $u \in F$. Since all norms in a finite-dimensional space are equivalent and $p < p_s^*$. This completes the proof of Lemma 4.2. □

Since $J_\lambda(u)$ does not satisfy $(PS)_c$ condition for all $c > 0$. Thus, in the following we will find a special finite-dimensional subspaces by which we construct sufficiently small minimax levels.

Recall that the assumption (V) implies there is $x_0 \in \mathbb{R}^N$ such that $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$. Without loss of generality we assume from now on that $x_0 = 0$.

From condition (h_3) it follows that

$$\frac{\lambda}{p_s^*}\int_{\mathbb{R}^N} |u|^{p_s^*}dx + \lambda\int_{\mathbb{R}^N} H(x,u)dx \geqslant l_0\lambda\int_{\mathbb{R}^N}|u|^\nu dx.$$

Definite the function $I_\lambda \in C^1(X^s,\mathbb{R})$ by

$$I_\lambda(u) := \frac{a}{p}[u]_{s,p}^p + \frac{b}{\theta p}[u]_{s,p}^{\theta p} + \frac{\lambda}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - l_0\lambda\int_{\mathbb{R}^N}|u|^\nu dx.$$

Then $J_\lambda(u) \leqslant I_\lambda(u)$ for all $u \in E$ and it suffices to construct small minimax levels for I_λ.

Note that

$$\inf\left\{\iint_{\mathbb{R}^{2N}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}dxdy : \phi \in C_0^\infty(\mathbb{R}^N), |\phi|_\nu = 1\right\} = 0.$$

For any $1 > \delta > 0$ one can choose $\phi_\delta \in C_0^\infty(\mathbb{R}^N)$ with $|\phi_\delta|_p = 1$ and $\operatorname{supp}\phi_\delta \subset B_{r_\delta}(0)$ so that

$$\iint_{\mathbb{R}^{2N}} \frac{|\phi_\delta(x)-\phi_\delta(y)|^p}{|x-y|^{N+sp}}dxdy < \delta.$$

Set

$$\frac{1}{l} := \frac{p}{N-sp}\cdot\frac{1}{p_s^*-\theta p}$$

and

$$e_\lambda = \phi_\delta(\lambda^{\frac{1}{l}}x), \tag{4.1}$$

then

$$\operatorname{supp} e_\lambda \subset B_{\lambda^{-\frac{1}{l}}r_\delta}(0).$$

Thus, for $t \geqslant 0$,

$$\begin{aligned}
I_\lambda(te_\lambda) &= \frac{at^p}{p}[e_\lambda]_{s,p}^p + \frac{bt^{\theta p}}{\theta p}[e_\lambda]_{s,p}^{\theta p} + \frac{t^p}{p}\lambda\int_{\mathbb{R}^N} V(x)|e_\lambda|^p dx - l_0 t^\nu\lambda\int_{\mathbb{R}^N}|e_\lambda|^\nu dx\\
&= \lambda^{1-\frac{N}{l}}\left[\frac{at^p}{p}[\phi_\delta]_{s,p}^p + \frac{bt^{\theta p}}{\theta p}\lambda^{(\theta-1)(1-\frac{N}{l})}[\phi_\delta]_{s,p}^{\theta p} + \frac{t^p}{p}\int_{\mathbb{R}^N} V\left(\lambda^{-\frac{1}{l}}x\right)|\phi_\delta|^p dx - t^\nu l_0\int_{\mathbb{R}^N}|\phi_\delta|^\nu dx\right]\\
&\leqslant \lambda^{1-\frac{N}{l}}\left[\frac{at^p}{p}[\phi_\delta]_{s,p}^p + \frac{bt^{\theta p}}{\theta p}[\phi_\delta]_{s,p}^{\theta p} + \frac{t^p}{p}\int_{\mathbb{R}^N} V\left(\lambda^{-\frac{1}{l}}x\right)|\phi_\delta|^p dx - t^\nu l_0\int_{\mathbb{R}^N}|\phi_\delta|^\nu dx\right]\\
&= \lambda^{1-\frac{N}{l}}\Psi_\lambda(t\phi_\delta) = \lambda^{1-\frac{p_s^*}{p_s^*-\theta p}}\Psi_\lambda(t\phi_\delta),
\end{aligned}$$

where $\Psi_\lambda \in C^1(E,\mathbb{R})$ defined by

$$\Psi_\lambda(u) := \frac{a}{p}[u]_{s,p}^p + \frac{b}{\theta p}[u]_{s,p}^{\theta p} + \frac{1}{p}\int_{\mathbb{R}^N} V\left(\lambda^{-\frac{1}{l}}x\right)|u|^p dx - l_0\int_{\mathbb{R}^N}|u|^s dx.$$

We obtain by $\nu > p$ that

$$\max_{t\geqslant 0}\Psi_\lambda(t\phi_\delta) = \frac{p-2}{2p(\nu l_0)^{\frac{p}{\nu-p}}}\left(\iint_{\mathbb{R}^{2N}} \frac{|\phi_\delta(x)-\phi_\delta(y)|^p}{|x-y|^{N+sp}}dxdy + \int_{\mathbb{R}^N} V\left(\lambda^{-\frac{1}{ps}}x\right)|\phi_\delta|^p dx\right)^{\frac{\nu}{\nu-p}}.$$

On the one hand, since $V(0) = 0$ and note that suup $\phi_\delta \subset B_{r_\delta}(0)$, there is $\Lambda_\delta > 0$ such that

$$V\left(\lambda^{-\frac{1}{ps}}x\right) \leqslant \frac{\delta}{|\phi_\delta|_p^p} \quad \text{for all } \ |x| \leqslant r_\delta \text{ and } \lambda \geqslant \Lambda_\delta.$$

This implies that

$$\max_{t\geqslant 0} \Psi_\lambda(t\phi_\delta) \leqslant \frac{p-2}{2p(\nu l_0)^{\frac{p}{\nu-p}}}(2\delta)^{\frac{\nu}{\nu-p}}.$$

Therefore, for all $\lambda \geqslant \Lambda_\delta$,

$$\max_{t\geqslant 0} J_\lambda(t\phi_\delta) \leqslant \frac{p-2}{2p(\nu l_0)^{\frac{p}{\nu-p}}}(2\delta)^{\frac{\nu}{\nu-p}}\lambda^{1-\frac{N}{ps}}. \tag{4.2}$$

Thus we have the following lemma.

Lemma 4.3. *Under the assumption of Lemma 4.1, for any $\kappa > 0$ there exists $\Lambda_\kappa > 0$ such that for each $\lambda \geqslant \Lambda_\kappa$, there is $\widehat{e}_\lambda \in X^s$ with $\|\widehat{e}_\lambda\|_\lambda > \rho_\lambda$, $J_\lambda(\widehat{e}_\lambda) \leqslant 0$ and*

$$\max_{t\in[0,1]} J_\lambda(t\widehat{e}_\lambda) \leqslant \kappa\lambda^{1-\frac{N}{ps}}.$$

Proof. Choose $\delta > 0$ so small that $\frac{p-2}{2p(\nu l_0)^{\frac{p}{\nu-p}}}(2\delta)^{\frac{\nu}{\nu-p}} \leqslant \kappa$. Let $e_\lambda \in X^s$ be the function defined by (4.1). Take $\Lambda_\kappa = \Lambda_\delta$. Let $\widehat{t}_\lambda > 0$ be such that $\widehat{t}_\lambda\|e_\lambda\|_\lambda > \rho_\lambda$ and $J_\lambda(te_\lambda) \leqslant 0$ for all $t \geqslant \widehat{t}_\lambda$. By (4.2), let $\widehat{e}_\lambda = \widehat{t}_\lambda e_\lambda$, we know that the conclusion of Lemma 4.3 holds. □

For any $m^* \in \mathbb{N}$, one can choose m^* functions $\phi_\delta^i \in C_0^\infty(\mathbb{R}^N)$ such that supp $\phi_\delta^i \cap$ supp $\phi_\delta^k = \emptyset$, $i \neq k$, $|\phi_\delta^i|_s = 1$, and

$$\int\int_{\mathbb{R}^{2N}} \frac{|\phi_\delta^i(x) - \phi_\delta^i(y)|^p}{|x-y|^{N+sp}} dxdy < \delta.$$

Let $r_\delta^{m^*} > 0$ be such that supp $\phi_\delta^i \subset B_{r_\delta}^i(0)$ for $i = 1, 2, \cdots, m^*$. Set

$$e_\lambda^i(x) = \phi_\delta^i(\lambda^{\frac{1}{ps}}x) \quad \text{for } i = 1, 2, \cdots, m^* \tag{4.3}$$

and

$$H_{\lambda\delta}^{m^*} = \text{span}\{e_\lambda^1, e_\lambda^2, \cdots, e_\lambda^{m^*}\}.$$

Observe that for each $u = \sum_{i=1}^{m^*} c_i e_\lambda^i \in H_{\lambda\delta}^{m^*}$,

$$[u]_{s,p}^p = \sum_{i=1}^{m^*} |c_i|^p [e_\lambda^i]_{s,p}^p,$$

$$\int_{\mathbb{R}^N} V(x)|u|^p dx = \sum_{i=1}^{m^*} |c_i|^p \int_{\mathbb{R}^N} V(x)|e_\lambda^i|^p dx,$$

$$\frac{1}{p_s^*}\int_{\mathbb{R}^N} |u|^{p_s^*} dx = \frac{1}{p_s^*}\sum_{i=1}^{m^*} |c_i|^{p_s^*} \int_{\mathbb{R}^N} |e_\lambda^i|^{p_s^*} dx,$$

and

$$\int_{\mathbb{R}^N} H(x,u)dx = \sum_{i=1}^{m^*} \int_{\mathbb{R}^N} H(x, c_i e_\lambda^i)dx.$$

Therefore,

$$J_\lambda(u) = \sum_{i=1}^{m^*} J_\lambda(c_i e_\lambda^i)$$

and as before

$$J_\lambda(c_i e_\lambda^i) \leqslant \lambda^{1-\frac{N}{p}} \Psi(|c_i| e_\lambda^i).$$

Set

$$\beta_\delta := \max\{|\phi_\delta^i|_p^p : j = 1, 2, \cdots, m^*\}$$

and choose $\Lambda_{m^*\delta} > 0$ so that

$$V(\lambda^{-\frac{1}{ps}} x) \leqslant \frac{\delta}{\beta_\delta} \text{ for all } |x| \leqslant r_\delta^{m^*} \text{ and } \lambda \geqslant \Lambda_{m^*\delta}.$$

As before, we can obtain the following

$$\max_{u \in H_{\lambda\delta}^{m^*}} J_\lambda(u) \leqslant m^* \frac{p-2}{2p(\nu l_0)^{\frac{p}{\nu-p}}} (2\delta)^{\frac{\nu}{\nu-p}} \lambda^{1-\frac{N}{ps}} \tag{4.4}$$

for all $\lambda \geqslant \Lambda_{m^*\delta}$.

Using this estimate, we have the following.

Lemma 4.4. *Under the assumptions of Lemma 4.1, for any $m^* \in \mathbb{N}$ and $\kappa > 0$ there exists $\Lambda_{m^*\kappa} > 0$ such that for each $\lambda \geqslant \Lambda_{m^*\kappa}$, there exists an m^*-dimensional subspace $F_{\lambda m^*}$ satisfying*

$$\max_{u \in F_{\lambda m^*}} J_\lambda(u) \leqslant \kappa \lambda^{1-\frac{N}{ps}}.$$

Proof. Choose $\delta > 0$ so small that $m^* \frac{p-2}{2p(\nu l_0)^{\frac{p}{\nu-p}}} (2\delta)^{\frac{\nu}{\nu-p}} \leqslant \kappa$. Taking $F_{\lambda m^*} = H_{\lambda\delta}^{m^*} = \operatorname{span}\{e_\lambda^1, e_\lambda^2, \cdots, e_\lambda^{m^*}\}$, where $e_\lambda^i(x) = \phi_\delta^i(\lambda^{\frac{1}{ps}} x)$, for $i = 1, 2, \cdots, m^*$ are given by (4.3). From (4.4), we know that the conclusion of Lemma 4.4 holds. □

We now establish the existence and multiplicity results.

Proof of Theorem 2.3. For any $o < \kappa < \sigma_0$, by Lemma 3.4, we choose $\Lambda_\sigma > 0$ and define for $\lambda \geqslant \Lambda_\sigma$, the minimax value

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(t\widehat{e}_\lambda),$$

where

$$\Gamma_\lambda := \{\gamma \in C([0,1], X^s) : \gamma(0) = 0 \text{ and } \gamma(1) = \widehat{e}_\lambda\}.$$

By Lemma 4.1, we have $\alpha_\lambda \leqslant c_\lambda \leqslant \kappa \lambda^{1-\frac{N}{ps}}$. In virtue of Lemma 3.4, we know that J_λ satisfies the $(PS)_{c_\lambda}$ condition, there is $u_\lambda \in X^s$ such that $J'_\lambda(u_\lambda) = 0$ and $J_\lambda(u_\lambda) = c_\lambda$. Moreover, it is well known that such a Mountain-Pass solution is a least energy solution of problem (2.1).

Because u_λ is a critical point of J_λ, for $\rho \in [p, p_s^*]$,

$$\begin{aligned}
\kappa \lambda^{1-\frac{N}{ps}} \geqslant J_\lambda(u_\lambda) &= J_\lambda(u_\lambda) - \frac{1}{\rho} J'_\lambda(u_\lambda) u_\lambda \\
&= \left(\frac{1}{p} - \frac{1}{\rho}\right) \int\int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x) - u_\lambda(y)|^p}{|x-y|^{N+sp}} dxdy + \left(\frac{1}{p} - \frac{1}{\rho}\right) \lambda \int_{\mathbb{R}^N} V(x) |u_\lambda|^p dx
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{\rho}-\frac{1}{p_s^*}\right)\lambda\int_{\mathbb{R}^N}|u_\lambda|^{p_s^*}dx+\lambda\int_{\mathbb{R}^N}\left[\frac{1}{\mu}h(x,u_\lambda)u_\lambda-H(x,u_\lambda)\right]dx\\
\geqslant&\left(\frac{1}{p}-\frac{1}{\rho}\right)\iint_{\mathbb{R}^{2N}}\frac{|u_\lambda(x)-u_\lambda(y)|^p}{|x-y|^{N+sp}}dxdy+\left(\frac{1}{p}-\frac{1}{\rho}\right)\lambda\int_{\mathbb{R}^N}V(x)|u_\lambda|^p dx\\
&+\left(\frac{1}{\rho}-\frac{1}{p_s^*}\right)\lambda\int_{\mathbb{R}^N}|u_\lambda|^{p_s^*}dx+\left(\frac{\mu}{\rho}-1\right)\lambda\int_{\mathbb{R}^N}H(x,u_\lambda)dx.
\end{aligned}
$$

Taking $\rho=p$, we obtain the estimates (i) and taking $\rho=\mu$ we obtain the estimate (ii). This completes the proof of Theorem 2.3 (1).

Denote the set of all symmetric (in the sense that $-Z=Z$) and closed subsets of X^s by Σ for each $Z\in\Sigma$. Let $\mathrm{gen}(Z)$ be the Krasnoselkski genus and

$$j(Z):=\min_{\iota\in\Gamma_{m^*}}\mathrm{gen}(\iota(Z)\cap\partial B_{\rho_\lambda}),$$

where Γ_{m^*} is the set of all odd homeomorphisms $\iota\in C(X^s,X^s)$ and ρ_λ is the number from Lemma 4.1. Then j is a version of Benci's pseudoindex [5]. Let

$$c_{\lambda i}:=\inf_{j(Z)\geqslant i}\sup_{u\in Z}J_\lambda(u),\quad 1\leqslant i\leqslant m^*.$$

Since $J_\lambda(u)\geqslant\alpha_\lambda$ for all $u\in\partial B^+_{\rho\lambda}$ and since $j(F_{\lambda m^*})=\dim F_{\lambda m^*}=m^*$,

$$\alpha_\lambda\leqslant c_{\lambda 1}\leqslant\cdots\leqslant c_{\lambda m^*}\leqslant\sup_{u\in H_{\lambda m^*}}J_\lambda(u)\leqslant\kappa\lambda^{1-\frac{N}{ps}}.$$

It follows from Lemma 3.4 that J_λ satisfies the $(PS)_{c_\lambda}$ condition at all levels $c<\sigma_0\lambda^{1-\frac{N}{ps}}$. By the usual critical point theory, all $c_{\lambda i}$ are critical levels and J_λ has at least m^* pairs of nontrivial critical points. □

Acknowledgment

Y. Q. Song was supported by NSFC (No. 11301038), The Natural Science Foundation of Jilin Province (No. 20160101244JC), Research Foundation during the 13th Five-Year Plan Period of Department of Education of Jilin Province, China (JJKH20170648KJ).

S. Y. Shi was supported by NSFC grant (No. 11771177), China Automobile Industry Innovation and Development Joint Fund (No. U1664257), Program for Changbaishan Scholars of Jilin Province and Program for JLU Science, Technology Innovative Research Team (No. 2017TD-20).

References

[1] D. Applebaum, *Lévy processes–from probability to finance and quantum groups*, Notices Amer. Math. Soc., **51** (2004), 1336–1347. 1

[2] G. Autuori, A. Fiscella, P. Pucci, *Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity*, Nonlinear Anal., **125** (2015), 699–714. 1

[3] G. Autuori, P. Pucci, *Elliptic problems involving the fractional Laplacian in* $\mathbb{R}^N$, J. Differential Equations, **255** (2013), 2340–2362. 1

[4] B. Barrios, E. Colorado, A. de Pablo, U. Sánchez, *On some critical problems for the fractional Laplacian operator*, J. Differential Equations, **252** (2012), 6133–6162. 1

[5] V. Benci, *On critical point theory for indefinite functionals in the presence of symmetries*, Trans. Amer. Math. Soc., **274** (1982), 533–572. 4

[6] Brézis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **34** (1983), 437–477. 1, 2, 3

[7] L. Caffarelli, *Non-local diffusions, drifts and games*, Nonlinear partial differential equations, Abel Symp., Springer, Heidelberg, **7** (2012), 37–52. 1

[8] X.-J. Chang, *Ground state solutions of asymptotically linear fractional Schrödinger equations*, J. Math. Phys., **54** (2013), 10 pages. 2

- [9] M. de Souza, *On a class of nonhomogeneous fractional quasilinear equations in* $\mathbb{R}^n$ *with exponential growth*, NoDEA Nonlinear Differential Equations Appl., **22** (2015), 499–511. 1
- [10] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., **136** (2012), 521–573. 1, 1, 2.1
- [11] Y.-H. Ding, F.-H. Lin, *Solutions of perturbed Schrödinger equations with critical nonlinearity*, Calc. Var. Partial Differential Equations, **30** (2007), 231–249. 1
- [12] W. Dong, J.-F. Xu, Z.-L. Wei, *Infinitely many weak solutions for a fractional Schrödinger equation*, Bound. Value Probl., **2014** (2014), 14 pages. 1
- [13] T.-X. Gou, H.-R. Sun, *Solutions of nonlinear Schrödinger equation with fractional Laplacian without the Ambrosetti-Rabinowitz condition*, Appl. Math. Comput., **257** (2015), 409–416. 1
- [14] A. Iannizzotto, S.-B. Liu, K. Perera, M. Squassina, *Existence results for fractional* p*-Laplacian problems via Morse theory*, Adv. Calc. Var., **9** (2016), 101–125. 1
- [15] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A, **268** (2000), 298–305. 1
- [16] N. Laskin, *Fractional Schrödinger equation*, Phys. Rev. E, **66** (2002), 7 pages. 1
- [17] S.-H. Liang, S.-Y. Shi, *Soliton solutions to Kirchhoff type problems involving the critical growth in* $\mathbb{R}^N$, Nonlinear Anal., **81** (2013), 31–41. 1
- [18] S.-H. Liang, J.-H. Zhang, *Existence of solutions for Kirchhoff type problems with critical nonlinearity in* $\mathbb{R}^3$, Nonlinear Anal. Real World Appl., **17** (2014), 126–136.
- [19] S.-H. Liang, J.-H. Zhang, *Existence and multiplicity of solutions for fourth-order elliptic equations of Kirchhoff type with critical growth in* $\mathbb{R}^N$, J. Math. Phys., **57** (2016), 13 pages. 1
- [20] S.-H. Liang, J.-H. Zhang, *Multiplicity of solutions for the noncooperative Schrödinger-Kirchhoff system involving the fractional* p*-Laplacian in* $\mathbb{R}^N$, Z. Angew. Math. Phys., **68** (2017), 18 pages. 1
- [21] P.-L. Lions, *The concentration-compactness principle in the calculus of variations*, The locally compact case, II, Ann. Inst. H. Poincaré Anal. Non Linéaire, **1** (1984), 223–283. 3
- [22] J. Liu, J.-F. Liao, C.-L. Tang, *Positive solutions for Kirchhoff-type equations with critical exponent in* $\mathbb{R}^N$, J. Math. Anal. Appl., **429** (2015), 1153–1172. 1
- [23] G. Molica Bisci, V. D. Rădulescu, *Ground state solutions of scalar field fractional Schrödinger equations*, Calc. Var. Partial Differential Equations, **54** (2015), 2985–3008. 1
- [24] G. Molica Bisci, V. D. Rădulescu, *Multiplicity results for elliptic fractional equations with subcritical term*, NoDEA Nonlinear Differential Equations Appl., **22** (2015), 721–739.
- [25] G. Molica Bisci, D. Repovš, *Higher nonlocal problems with bounded potential*, J. Math. Anal. Appl., **420** (2014), 167–176. 1
- [26] A. Ourraoui, *On a* p*-Kirchhoff problem involving a critical nonlinearity*, C. R. Math. Acad. Sci. Paris, **352** (2014), 295–298. 1
- [27] P. Pucci, S. Saldi, *Critical stationary Kirchhoff equations in* $\mathbb{R}^N$ *involving nonlocal operators*, Rev. Mat. Iberoam., **32** (2016), 1–22. 1
- [28] P. Pucci, M.-Q. Xiang, B.-L. Zhang, *Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional* p*-Laplacian in* $\mathbb{R}^N$, Calc. Var. Partial Differential Equations, **54** (2015), 2785–2806. 1
- [29] P. Pucci, M.-Q. Xiang, B.-L. Zhang, *Existence and multiplicity of entire solutions for fractional* p*-Kirchhoff equations*, Adv. Nonlinear Anal., **5** (2016), 27–55. 1, 1
- [30] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, (1986). 2
- [31] X. Ros-Oton, J. Serra, *Nonexistence results for nonlocal equations with critical and supercritical nonlinearities*, Comm. Partial Differential Equations, **40** (2015), 115–133. 1
- [32] R. Servadei, E. Valdinoci, *Fractional Laplacian equations with critical Sobolev exponent*, Rev. Mat. Complut., **28** (2015), 655–676. 1
- [33] R. Servadei, E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc., **367** (2015), 67–102. 1
- [34] X.-D. Shang, J.-H. Zhang, *Concentrating solutions of nonlinear fractional Schrdinger equation with potentials*, J. Differential Equations, **258** (2015), 1106–1128. 1
- [35] K.-M. Teng, *Multiple solutions for a class of fractional Schrdinger equations in* $\mathbb{R}^N$, Nonlinear Anal. Real World Appl., **21** (2015), 76–86. 1
- [36] C. Torres, *On superlinear fractional* p*-Laplacian in* $\mathbb{R}^n$, ArXiv, **2014** (2014), 12 pages. 1
- [37] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston, Inc., Boston, MA, (1996). 2
- [38] M.-Q. Xiang, B.-L. Zhang, M. Ferrara, *Existence of solutions for Kirchhoff type problem involving the non-local fractional* p*-Laplacian*, J. Math. Anal. Appl., **424** (2015), 1021–1041. 1
- [39] M.-Q. Xiang, B.-L. Zhang, M. Ferrara, *Multiplicity results for the non-homogeneous fractional* p*-Kirchhoff equations with concave-convex nonlinearities*, Proc. A, **471** (2015), 14 pages. 1
- [40] M.-Q. Xiang, B.-L. Zhang, V. D. Rădulescu, *Existence of solutions for perturbed fractional* p*-Laplacian equations*, J. Differential Equations, **260** (2016), 1392–1413. 1

[41] M.-Q. Xiang, B.-L. Zhang, X. Zhang, *A nonhomogeneous fractional* p*-Kirchhoff type problem involving critical exponent in* $\mathbb{R}^N$, Adv. Nonlinear Stud., **17** (2017), 611–640. 3.2, 3.3, 3

[42] X. Zhang, B.-L. Zhang, M.-Q. Xiang, *Ground states for fractional Schrödinger equations involving a critical nonlinearity*, Adv. Nonlinear Anal., **5** (2016), 293–314. 1