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A natural selection of a graphic contraction transformation in fuzzy metric spaces

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Abstract

In this paper, we study sufficient conditions to find a vertex v of a graph such that Tv is a terminal vertex of a path which starts from v, where T is a self graphic contraction transformation defined on the set of vertices. Some examples are presented to support the results proved herein. Our results widen the scope of various results in the existing literature.

Keywords: Graphic contraction, fuzzy metric space, natural selection.

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1. Introduction and preliminaries

Zadeh [20] introduced the notion of fuzzy sets, a new way to represent vagueness and uncertainties in daily life. Kramosil and Michalek [14] introduced the notion of a fuzzy metric by using continuous t-norms, which generalizes the concept of a probabilistic metric space to fuzzy situation. Moreover, George and Veeramani ([5, 6]) modified the concept of fuzzy metric spaces and obtained a Hausdorff topology for this kind of fuzzy metric spaces.

Romaguera [17] introduced Hausdorff fuzzy metric on a set of nonempty compact subsets of a fuzzy metric space.

In the sequel, the letters $\mathbb{N}$, $\mathbb{R}^+$, and $\mathbb{R}$ will denote the set of natural numbers, the set of positive real numbers, and the set of real numbers, respectively.

Following definitions and known results will be needed in the sequel.

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Definition 1.1 ([19]). A binary operation $*: [0,1]^2 \longrightarrow [0,1]$ is called a continuous t-norm if

(1) $*$ is associative and commutative;
(2) $*: [0,1]^2 \longrightarrow [0,1]$ is continuous (it is continuous as a mapping under the usual topology on $[0,1]^2$);
(3) $a * 1 = a$ for all $a \in [0,1]$;
(4) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$.

Some basic examples of continuous t-norms are $\wedge$ (minimum t-norm), $\cdot$ (product t-norm), and $*_L$(Lukasiewicz t-norm), where, for all $a, b \in [0,1]$,

$$a \wedge b = \min\{a, b\},\ a \cdot b = ab,\ a *_L b = \max\{a + b - 1, 0\}.$$

It is easy to check that $*_L \leqslant \cdot \leqslant \wedge$. In fact $* \leqslant \wedge$ for all continuous t-norm $*$.

Definition 1.2 ([5, 6]). Let X be a nonempty set and $*$ a continuous t-norm. A fuzzy set M on $X \times X \times (0,\infty)$ is said to be a fuzzy metric on X if for any $x, y, z \in X$ and $s, t > 0$, the following conditions hold

(i) $M(x, y, t) > 0$;
(ii) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, z, t + s) \geqslant M(x, y, t) * M(y, z, s)$ for all $t, s > 0$;
(v) $M(x, y, \cdot) : (0,\infty) \to (0, 1]$ is continuous.

The triplet $(X, M, *)$ is called a fuzzy metric space. Each fuzzy metric M on X generates Hausdorff topology τ_M on X whose base is the family of open M-balls $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$, where

$$B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}.$$

Note that a sequence $\{x_n\}$ converges to $x \in X$ (with respect to τ_M) if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

Since for each $x, y \in X$, $M(x, y, \cdot)$ is a nondecreasing function on $(0,\infty)$ (see [7]). Moreover every fuzzy metric space X (in the sense of George and Veeramani [5]) is metrizable, that is, there exists a metric d on X which induces a topology that agrees with τ_M ([8]). Conversely, if (X, d) is a metric space and $M_d : X \times X \times (0,\infty) \to (0, 1]$ is defined as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$, then $(X, M_d, \wedge)$ is a fuzzy metric space, called the standard fuzzy metric space induced by the metric d (see [5]). The topologies induced by the standard fuzzy metric and the corresponding metric are the same ([9]).

A sequence $\{x_n\}$ in a fuzzy metric space X is said to be a Cauchy sequence if for each $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geqslant n_0$. A fuzzy metric space X is complete ([6]) if every Cauchy sequence converges in X. A subset A of X is closed if for each convergent sequence $\{x_n\}$ in A with $x_n \longrightarrow x$, we have $x \in A$. A subset A of X is compact if each sequence in A has a convergent subsequence.

Definition 1.3 ([17]). A fuzzy metric M is said to be continuous on $X^2 \times (0,\infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0,\infty)$ which converges to a point $(x, y, t) \in X^2 \times (0,\infty)$.

Proposition 1.4 ([17]). *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0,\infty)$.*

Lemma 1.5 ([17]). *Let $(X, M, *)$ be a fuzzy metric space. Then for each $a \in X$, $B \in \mathcal{K}(X)$ and $t > 0$, there is $b_0 \in B$ such that $M(a, B, t) = M(a, b_0, t)$.*

A sequence $\{t_n\}$ of positive real numbers is said to be s-increasing ([9]) if there exists $n_0 \in \mathbb{N}$ such that

$$t_{m+1} \geqslant t_m + 1$$

for all $m \geqslant n_0$. In a fuzzy metric space $(X, M, \wedge)$, an infinite product (compare [11]) is denoted by

$$M(x, y, t_1) \wedge M(x, y, t_2) \wedge \cdots \wedge M(x, y, t_n) \wedge \cdots = \prod_{i=1}^{\infty} M(x, y, t_i)$$

for all $x, y \in X$.

Definition 1.6 ([18]). Let $\Omega = \{\eta : [0, 1] \to [0, 1], \eta$ is continuous, nondecreasing and $\eta(t) > t$ for $t \in (0, 1)$, further $\eta(t) = 1$ if and only if $t = 1$ or $\eta(t) = 0$ if and only if $t = 0$, overall $\eta(t) \geqslant t$, for all $t \in [0, 1]\}$.

Let Ψ be a collection of all continuous and decreasing functions $\psi : [0, 1] \to [0, 1]$ with $\psi(t) = 0$ if and only if $t = 1$.

A function $\psi \in \Psi$ is said to have property (α) if for all $r, t > 0$,

$$r * t > 0, \text{ we have } \psi(r * t) \leqslant \psi(r) + \psi(t),$$

where $*$ is any continuous t-norm.

Now, an example is provided to explain the property (α).

Example 1.7. Define the mapping $\psi : [0, 1] \to [0, 1]$ by $\psi(t) = 1 - t$. Note that, it admits property (α) for different continuous t-norms. Take $* = \wedge$. Suppose that $(s * t) = \min\{s, t\} = (s \wedge t) = s \leqslant t$, where s, t $\in \mathbb{R} \cup \{0\}$. Then

$$\psi(s \wedge t) = \psi(s) \leqslant \psi(s) + \psi(t).$$

Similarly, if $\min\{s, t\} = t \leqslant s$, then we have $\psi(s \wedge t) = \psi(s) \leqslant \psi(s) + \psi(t)$. This shows that property (α) holds for minimum t-norm. If $* = \cdot$, then $(s * t) = (s \cdot t) = st$, where $s, t \in \mathbb{R} \cup \{0\}$. Note that

$$\psi(s \cdot t) = \psi(st) \leqslant \psi(s) + \psi(t).$$

Thus property (α) holds for a product t-norm. Suppose that $* = *_L$, that is, $s *_L t = \max\{s + t - 1, 0\}$. If $s *_L t = \max\{s + t - 1, 0\} = 0$, then $\psi(s *_L t) = \psi(0) \leqslant \psi(s) + \psi(t)$. If $s *_L t = \max\{s + t - 1, 0\} = s + t - 1$, then we have

$$\psi(s *_L t) = \psi(s + t - 1) \leqslant \psi(s) + \psi(t),$$

which shows property (α) holds for Lukasiewicz t-norm.

On the other hand, the interplay between the preference relation of abstract objects of underlying mathematical structure and fixed point theory is very strong and fruitful. This gives rise to an interesting branch of nonlinear functional analysis called order oriented fixed point theory. This theory is studied in the framework of a partially ordered sets along with appropriate mappings satisfying certain order conditions and has many applications in economics, computer science and other related disciplines.

Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [16], and then by Nieto and Lopez [15].

Recently, Azam [4] obtained coincidence points of mappings and relations satisfying certain contractive conditions in the setup of a metric space.

Jachymski [13] introduced a new approach in metric fixed point theory by replacing order structure with a graph structure on a metric space. In this way, the results obtained in ordered metric spaces are generalized (see also [12] and the reference therein); in fact, Gwodzdz-lukawska and Jachymski [10] developed the Hutchinson-Barnsley theory for finite families of mappings on a metric space endowed with a directed graph, further Abbas et al. obtained results using graphical contractions (for details see [1–3]).

The following definitions and notations will be needed in the sequel.

Let X be any set and Δ denotes the diagonal of $X \times X$. Let $G(V, E)$ be a undirected graph such that the set V of its vertices is a subset of X and E the set of edges of the graph which contains all loops, that is, $\Delta \subseteq E$. Also assume that the graph G has no parallel edges and, thus one can identify G with the pair (V, E).

Motivated by the work in [4], we introduce the following concept of a natural selection of a transformation $T : V \to V$.

If $x, y \in V$, then (x, y) denotes an edge between x and y.

If vertices x and y of a graph are connected by certain edges, we say there exists a path between x and y. In this case, we denote $[x, y]$ a path which starts from x and terminates at y (we call vertex y a terminal vertex and vertex x a reference vertex). Set

$$E^x = \text{ The collection of all terminal vertices of edges starting from } x \quad \text{and} \quad E^X = \cup_{x \in X} E^x.$$

A vertex $w \in V$ is called a natural selection of $T : V \to V$ if $Tw \in E^w$, that is, Tw is a terminal vertex of $[w, Tw]$.

Let x and y be two vertices of a graph G. It is a common practice to assign a certain weight to each edge of a graph. The positive real number obtained by calculating the distance between x and y can be used as a weight of an edge joining x and y.

In this paper, we assign a fuzzy weight $M(x, y, t)$ (a number between 0 and 1) to an edge (x, y) at t, where t is interpreted as a time. In this case we have a larger flexibility in choosing the weights specially when one is uncertain or confused at a certain point of time in assigning a weight to an edge (x, y) at a time t.

We establish an existence of a vertex ν of a graph such that its image under a graphic transformation satisfying certain contraction conditions becomes a terminal vertex of a path starting from ν.

We give examples to support our results and to show that our results are potential generalization of comparable results in the existing literature.

$M(x, y, t) = 1$ for all $t > 0$ if and only if a path $[x, y]$ defines a loop. Define

$$D\left(E^x, E^y, t\right) = \sup\{M(u, \nu, t), u \in E^x, \nu \in E^y\}.$$

2. Natural selection of graphic contractions

We start with the following result.

Theorem 2.1. *Let $T : V \to V$. If there exists a functions $\psi \in \Psi$ having property (α) such that*

$$\psi(M(Tx, Ty, t)) \leqslant k\psi(D(E^x, E^y, t)) \tag{2.1}$$

holds for all vertices $x, y \in V$ and $0 < k < 1$, then there exists $w \in V$ such that $Tw \in E^w$ provided that $T(V) \subseteq E^X$ and E^X is complete subspace of fuzzy metric space $(X, M, *)$.

Proof. Let x_0 be an arbitrarily fixed vertex of graph $G(V, E)$ (for simplicity G). We shall construct sequences of vertices $\{x_n\} \subset V, \{y_n\} \subset E^X$ as follows: Let $y_1 = Tx_0$. Since $T(V) \subseteq E^X$, we can choose a vertex x_1 in V such that $y_1 \in E^{x_1}$. Let $y_2 = Tx_1$. If $\psi(D(E^{x_0}, E^{x_1}, t) = 0$, then we have $Tx_0 = Tx_1$ which implies that $y_2 \in E^{x_1}$ and hence x_1 becomes the required vertex of G. If $\psi(D(E^{x_0}, E^{x_1}, t) \neq 0$, then by inequality (2.1), we have

$$\psi(M(Tx_0, Tx_1, t)) \leqslant k\psi(D(E^{x_0}, E^{x_1}, t)) \neq 0.$$

Choose another vertex x_2 in V such that $y_2 \in E^{x_2}$. If $\psi(D(E^{x_1}, E^{x_2}, t) = 0$, then x_2 is the required vertex in V. If $\psi(D(E^{x_1}, E^{x_2}, t) \neq 0$, then by inequality (2.1), we obtain that

$$\psi(M(Tx_1, Tx_2, t)) \leqslant k\psi(D(E^{x_1}, E^{x_2}, t) \neq 0.$$

Continuing this way, we can obtain two sequences of vertices $\{x_n\} \subset V$ and $\{y_n\} \subset E^X$ such that $y_n = Tx_{n-1}$, $y_n \in E^{x_n}$ and it satisfies:

$$\psi(M(y_n, y_{n+1}, t)) \leqslant k\psi(D(E^{x_{n-1}}, E^{x_n}, t) \neq 0, \ n = 1, 2, 3, \ldots.$$

Since $y_{n-1} \in E^{x_{n-1}}$, $y_n \in E^{x_n}$, we have

$$D(E^{x_{n-1}}, E^{x_n}, t) \geqslant M(y_{n-1}, y_n, t),$$

which further implies that

$$\psi(M(y_n, y_{n+1}, t)) \leqslant k\psi(M(y_{n-1}, y_n, t)).$$

Note that

$$\begin{aligned}\psi(M(y_n, y_{n+1}, t)) &\leqslant k\psi(M(y_{n-1}, y_n, t))\\ &\leqslant k^2\psi(M(y_{n-2}, y_{n-1}, t))\\ &\ \vdots\\ &\leqslant k^n\psi(M(y_0, y_1, t)), \ n = 1, 2, 3, \ldots.\end{aligned}$$

That is,

$$\psi(M(y_n, y_{n+1}, t)) \leqslant k^n\psi(M(y_0, y_1, t)).$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we have

$$\lim_{n\to\infty} \psi(M(y_n, y_{n+1}, t)) = 0.$$

Now, we show that $\{y_n\}$ is a Cauchy sequence. Suppose that there exist some $n_0 \in \mathbb{N}$ with $m > n > n_0$ such that

$$\psi(M(y_n, y_m, t)) = \psi(M(y_n, y_m, \sum_{i=n}^{m-1} a_i t),$$

where $\{a_i\}$ is a decreasing sequence of positive real numbers satisfying $\sum_{i=n}^{m-1} a_i = 1$. Thus

$$\begin{aligned}M(y_n, y_m, t) &= M(y_n, y_m, \sum_{i=n}^{m-1} a_i t)\\ &\geqslant M(y_n, y_{n+1}, a_n t) * M(y_{n+1}, y_{n+2}, a_{n+1} t) * \cdots * M(y_{m-1}, y_m, a_{m-1} t).\end{aligned}$$

Further, we obtain that

$$\begin{aligned}&\psi(M(y_n, y_m, t))\\ &\quad= \psi(M(y_n, y_m, \sum_{i=n}^{m-1} a_i t))\\ &\quad\leqslant \psi[M(y_n, y_{n+1}, a_n t) * M(y_{n+1}, y_{n+2}, a_{n+1} t) * \cdots * M(y_{m-1}, y_m, a_{m-1} t)]\\ &\quad\leqslant \psi(M(y_n, y_{n+1}, a_n t)) + \psi(M(y_{n+1}, y_{n+2}, a_{n+1} t)) + \cdots + \psi(M(y_{m-1}, y_m, a_{m-1} t))\\ &\quad\leqslant k^n\psi(M(y_0, y_1, a_n t)) + k^{n+1}\psi(M(y_0, y_1, a_{n+1} t)) + \cdots + k^{m-1}\psi(M(y_0, y_1, a_{m-1} t)).\end{aligned} \tag{2.2}$$

If

$$\max\{\psi(M(y_0,y_1,a_nt)),\psi(M(y_0,y_1,a_{n+1}t)),\cdots,\psi(M(y_0,y_1,a_{m-1}t))\}=\psi(M(y_0,y_1,bt))$$

for some $b\in\{a_i : n\leqslant i\leqslant m-1\}$, then the inequality (2.2) becomes

$$\begin{aligned}\psi(M(y_n,y_m,t))&\leqslant k^n\psi(M(y_0,y_1,a_nt))+k^{n+1}\psi(M(y_0,y_1,a_{n+1}t))+\cdots+k^{m-1}\psi(M(y_0,y_1,a_{m-1}t))\\&\leqslant k^n\psi(M(y_0,y_1,bt))+k^{n+1}\psi(M(y_0,y_1,bt))+\cdots+k^{m-1}\psi(M(y_0,y_1,bt))\\&\leqslant(k^n+k^{n+1}+\cdots+k^{m-1})\psi(M(y_0,y_1,bt))\\&\leqslant k^n(1+k+\cdots+k^{m-n-1})\psi(M(y_0,y_1,bt))\\&\leqslant\frac{k^n}{1-k}\psi(M(y_0,y_1,bt)),\end{aligned}$$

that is, for all $n\in\mathbb{N}$,

$$\psi(M(y_n,y_m,t))\leqslant\frac{k^n}{1-k}\psi(M(y_0,y_1,bt)).$$

On taking limit as $n\to\infty$ on both sides of the above inequality, we have

$$0\leqslant\lim_{n,m\to\infty}\psi(M(y_n,y_m,t))\leqslant 0.$$

By continuity of ψ, we obtain that

$$\lim_{n,m\to\infty}M(y_n,y_m,t)=1.$$

Hence $\{y_n\}$ is a Cauchy sequence in E^X. Next we assume that there exists a vertex z in E^X such that $\lim_{n\to\infty}M(y_n,z,t)=1$. Moreover, $z\in E^w$ for some $w\in X$. Also,

$$M(z,Tw,t)\geqslant M(z,y_{n+1},\frac{t}{2})*M(y_{n+1},Tw,\frac{t}{2}).$$

As $\psi\in\Psi$, so we have

$$\begin{aligned}\psi(M(z,Tw,t))&\leqslant\psi[M(z,y_{n+1},\frac{t}{2})*M(y_{n+1},Tw,\frac{t}{2})]\\&\leqslant\psi[M(z,y_{n+1},\frac{t}{2})]+\psi[M(y_{n+1},Tw,\frac{t}{2})]\\&=\psi[M(z,y_{n+1},\frac{t}{2})]+\psi[M(Tx_n,Tw,\frac{t}{2})]\\&\leqslant\psi[M(z,y_{n+1},\frac{t}{2})]+k\psi[D(E^{x_n},E^w,\frac{t}{2})]\\&\leqslant\psi[M(z,y_{n+1},\frac{t}{2})]+k\psi[M(y_n,z,\frac{t}{2})].\end{aligned}$$

On taking limit as $n\to\infty$, we have $z=Tw$, that is, $Tw\in E^w$. □

Example 2.2. Let $V=\mathbb{Q}\cup\mathbb{Q}'=\mathbb{R}$ and $M:X\times X\times(0,\infty)\to(0,1]$ be the fuzzy metric defined by $M(x,y,t)=\dfrac{t}{t+d(x,y)}$, where d is the usual metric on X. Suppose that $\psi(t)=1-t$ for all $t\in(0,1)$. Define the mapping $T:\mathbb{R}\to\mathbb{R}$ by

$$Tx=\begin{cases}2, & \text{if } x\in\mathbb{Q},\\ 0, & \text{if } x\in\mathbb{Q}'.\end{cases}$$

If $x,y\in\mathbb{Q}$, $Tx=Ty=2$ and $E^x=E^y=[0,4]$. Note that

$$\psi(M(Tx,Ty,t))=\psi(1)\leqslant k\psi(D(E^x,E^y,t))=k\psi(1)=0.$$

If $x,y\in\mathbb{Q}'$, then $Tx=Ty=0$ and $E^x=E^y=[7,9]$. In this case, we have

$$\psi(M(Tx,Ty,t))=\psi(1)\leqslant k\psi(D(E^x,E^y,t))=k\psi(1)=0.$$

If $x\in\mathbb{Q}$ and $y\in\mathbb{Q}'$ or $x\in\mathbb{Q}'$ and $y\in\mathbb{Q}$, then $E^x=[0,4]$ and $E^y=[7,9]$ or $E^x=[7,9]$ and $E^y=[0,4]$.

For, $k \geqslant \frac{3}{4}$, we have

$$\psi(M(Tx,Ty,t)) = \psi(\frac{t}{t+2}) = 1-\frac{t}{t+2} \leqslant k\psi(D(E^x,E^y,t)).$$

Also, $T(V) = \{0,2\} \subset E^x = [0,4] \cup [7,9]$. Thus all the conditions of Theorem (2.1) are satisfied. However, Theorem 3.1 in [4] does not hold in last case.

Theorem 2.3. *Let $T : V \to V$ with $T(V) \subseteq E^X$ and E^X a complete subspace of fuzzy metric space $(X, M, *)$. If there exists $\eta \in \Omega$ and $k \in (0,1)$ such that for all vertices $x, y \in V$, we have*

$$M(Tx,Ty,kt) \geqslant \eta(D(E^x,E^y,t)), \tag{2.3}$$

then there exists a vertex $w \in V$ such that $Tw \in E^w$ provided that for each $\varepsilon > 0$ and an s-increasing sequence $\{t_n\}$, there exists n_0 in $\mathbb{N}_0$ such that $\prod_{n\geqslant n_0}^{\infty} M(x,y,t_n) \geqslant 1-\varepsilon$ for all $n \geqslant n_0$.

Proof. Let x_0 be an arbitrarily fixed vertex of graph $G(V,E)$ (for simplicity G). We shall construct sequences of vertices $\{x_n\} \subset V, \{y_n\} \subset E^X$ as follows: Let $y_1 = Tx_0$. Since $T(V) \subseteq E^X$, we can choose a vertex x_1 in V such that $y_1 \in E^{x_1}$. Let $y_2 = Tx_1$. If $\eta(D(E^{x_0},E^{x_1},t) = 1$, then we have $Tx_0 = Tx_1$ which implies that $y_2 \in E^{x_1}$ and hence x_1 becomes the required vertex of G. If $\eta(D(E^{x_0},E^{x_1},t) \neq 1$, then by inequality (2.3), we have

$$M(Tx_0,Tx_1,t) \geqslant \eta(D(E^{x_0},E^{x_1},t)) \neq 1.$$

Choose another vertex x_2 in V such that $y_2 \in E^{x_2}$. If $\eta(D(E^{x_1},E^{x_2},t) = 1$, then x_2 is the required vertex in V. If $\eta(D(E^{x_1},E^{x_2},t) \neq 1$, then by inequality (2.3), we obtain that

$$M(Tx_1,Tx_2,kt) \geqslant \eta(D(E^{x_1},E^{x_2},t)) \neq 1.$$

Continuing this way, we can obtain two sequences of vertices $\{x_n\} \subset V$ and $\{y_n\} \subset E^X$ such that $y_n = Tx_{n-1}$, $y_n \in E^{x_n}$ and it satisfies:

$$M(y_n,y_{n+1},kt) \geqslant \eta(D(E^{x_{n-1}},E^{x_n},t)) \neq 1, \ n=1,2,3,\ldots.$$

As $y_n \in E^{x_n}$ and $y_{n+1} \in E^{x_{n+1}}$, we have

$$D(E^{x_n},E^{x_{n+1}},t) \geqslant M(y_n,y_{n+1},t).$$

Thus

$$M(y_n,y_{n+1},kt) \geqslant \eta(M(y_{n-1},y_n,t))$$

implies that

$$M(y_n,y_{n+1},t) \geqslant \eta(M(y_{n-1},y_n,\frac{t}{k})) \geqslant M(y_{n-1},y_n,\frac{t}{k}).$$

Continuing this way, we have

$$M(y_n,y_{n+1},t) \geqslant M(y_{n-1},y_n,\frac{t}{k}) \geqslant M(y_{n-2},y_{n-1},\frac{t}{k^2}) \geqslant \cdots \geqslant M(y_0,y_1,\frac{t}{k^n}).$$

Let $t > 0$, $\varepsilon > 0$, $m,n \in \mathbb{N}$ such that $m > n$ and $h_i = \dfrac{1}{i(i+1)}$ for $i \in \{n,n+1,\ldots,m-1\}$. As $h_n + h_{n+1} + \cdots + h_{m-1} < 1$, we have

$$\begin{aligned}
M(y_n,y_m,t) &\geqslant M(y_n,y_m,(h_n+h_{n+1}+\cdots+h_{m-1})t)\\
&\geqslant M(y_n,y_{n+1},h_nt) * M(y_{n+1},y_{n+2},h_{n+1}t) * \cdots * M(y_{m-1},y_m,h_{m-1}t)\\
&\geqslant M(y_0,y_1,\frac{h_n}{k^n}t) * M(y_0,y_1,\frac{h_{n+1}}{k^{n+1}}t) * \cdots * M(y_0,y_1,\frac{h_{m-1}}{k^{m-1}}t)
\end{aligned}$$

$$= M(y_0, y_1, \frac{1}{n(n+1)k^n}t) * M(y_0, y_1, \frac{1}{(n+1)(n+2)k^{n+1}}t) * \cdots * M(y_0, y_1, \frac{1}{m(m-1)k^{m-1}}t)$$
$$\geqslant \prod_{i=n}^{\infty} M(y_0, y_1, \frac{t}{i(i+1)k^i}) = \prod_{i=n}^{\infty} M(y_0, y_1, t_i),$$

where $t_i = \frac{t}{i(i+1)k^i}$. Since $\lim_{n\to\infty}(t_{n+1} - t_n) = \infty$, therefore $\{t_i\}$ is an s-increasing sequence. Consequently, there exists $n_0 \in \mathbb{N}$, such that for each $\varepsilon > 0$, we have $\prod_{n=1}^{\infty} M(y_0, y_1, t_n) \geqslant 1 - \varepsilon$ for all $n \geqslant n_0$. Hence $M(y_n, y_m, t) \geqslant 1 - \varepsilon$ for all $n, m \geqslant n_0$. Thus $\{y_n\}$ is a Cauchy sequence in E^X. Next we assume that there exists a vertex z in E^X such that $\lim_{n\to\infty} M(y_n, z, t) = 1$. Moreover, $z \in E^w$ for some $w \in V$. Now,

$$M(z, Tw, t) \geqslant M(z, y_{n+1}, (1-k)t) * M(y_{n+1}, Tw, kt)$$

implies

$$\begin{aligned} M(z, Tw, t) &\geqslant M(z, y_{n+1}, (1-k)t) * M(y_{n+1}, Tw, kt) \\ &\geqslant M(z, y_{n+1}, (1-k)t) * M(Tx_n, Tw, kt)] \\ &\geqslant M(z, y_{n+1}, (1-k)t) * \eta[D(E^{x_n}, E^w, kt)] \\ &\geqslant M(z, y_{n+1}, (1-k)t) * \eta[M(y_n, z, kt)]. \end{aligned}$$

On taking limit as $n \to \infty$, we have $z = Tw$ and hence $Tw \in E^w$. □

Example 2.4. Let $V = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$ and $M : X \times X \times (0, \infty) \to (0, 1]$ be the fuzzy metric defined by $M(x, y, t) = \frac{t}{t + d(x, y)}$, where d is the usual metric on X. Define the transformation $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{Q}'. \end{cases}$$

Suppose that $\eta(t) = \sqrt{t}$ for all $t \in (0, 1)$. If $x, y \in \mathbb{Q}$, then $Tx = Ty = 1$ and $E^x = E^y = [0, 4]$. Also,

$$M(Tx, Ty, kt) = 1 \geqslant \eta(D(E^x, E^y, t)),$$

when $x, y \in \mathbb{Q}'$. Then $Tx = Ty = 0$, and $E^x = E^y = [7, 9]$. In this case, we have

$$M(Tx, Ty, kt) = 1 \geqslant \eta(D(E^x, E^y, t)).$$

If $x \in \mathbb{Q}$ and $y \in \mathbb{Q}'$ or $x \in \mathbb{Q}'$ and $y \in \mathbb{Q}$, then $E^x = [0, 4]$ and $E^y = [7, 9]$ or $E^x = [7, 9]$ and $E^y = [0, 4]$. For $k \geqslant \frac{3}{4}$, we have

$$M(Tx, Ty, \frac{3}{4}t) = \frac{\frac{3}{4}t}{\frac{3}{4}t + 1} \geqslant \eta(D(E^x, E^y, t)).$$

Note that $T(V) = \{0, 1\} \subset E^x = [0, 4] \cup [7, 9]$. Thus all the conditions of Theorem (2.1) are satisfied.

In the next theorem, we prove the existence of a unique coincidence point of a pair of mappings under a contractive condition.

Theorem 2.5. *Let $T, S : V \to V$ be continuous mapping with $T(V) \subseteq S(V) \subseteq E^X$ and E^X a complete subspace of fuzzy metric space $(X, M, *)$. If there exists $\eta \in \Omega$ and $k \in (0, 1)$ such that for all vertices $x, y \in V$, we have*

$$M(Tx, Ty, kt) \geqslant \eta(M(Sx, Sy, t)),$$

then there exists a vertex $w \in V$ such that Tw, $Sw \in E^w$ provided that for each $\varepsilon > 0$ and an s-increasing sequence $\{t_n\}$, there exists n_0 in $\mathbb{N}_0$ such that $\prod_{n\geqslant n_0}^{\infty} M(x, y, t_n) \geqslant 1 - \varepsilon$ for all $n \geqslant n_0$. Moreover, if either T or S is injective, then the vertex w is unique.

Proof. By Theorem (2.3) and $y_n = Tx_{n-1} \in T(V) \subseteq S(V) \subseteq E^X$, then there exists y_n such that $Tx_{n-1} = Sy_n \in S(V)$, we obtain $\lim_{n\to\infty} y_n = w \in V$ such that $Tw = Sw$, where,

$$Sw = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_{n-1} = Tw, x_0 \in V.$$

For uniqueness, assume that $w_1, w_2 \in V$, $w_1 \neq w_2$, $Tw_1 = Sw_1$, and $Tw_2 = Sw_2$. Then $M(Tw_1, Tw_2, kt) \geqslant \eta(M(Sw_1, Sw_2, t))$. If S or T is injective, then

$$M(Sw_1, Sw_2, t) \neq 1,$$

and

$$M(Sw_1, Sw_2, t) > M(Sw_1, Sw_2, kt) = M(Tw_1, Tw_2, kt) \geqslant \eta(M(Sw_1, Sw_2, t)) \geqslant M(Sw_1, Sw_2, t),$$

which is a contradiction. □

Theorem 2.6. *Let $T, S : V \to V$ are continuous mappings with $T(V) \subseteq S(V) \subseteq E^X$ and E^X is a complete subspace of fuzzy metric space $(X, M, *)$. If there exists a $k \in (0, 1)$ such that for all $x, y \in V$, we have*

$$M(Tx, Ty, kt) \geqslant \eta(M(Sx, Sy, t)),$$

where $\eta \in \Omega$, then S and T have a coincidence point in X. Moreover, if either T or S is injective, then T and S have a unique coincidence point in X .

Proof. This uniqueness can be proved on the same lines as in proof of Theorem 2.5. □

Remark 2.7. If in the above theorem we choose $X = Y$ and $R = I$ (the identity mapping on X), we obtain the Banach contraction theorem.

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