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Integral transforms and partial sums of certain meromorphically p-valent starlike functions

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Abstract

In this paper, we introduce two new subclasses of meromorphically p-valent starlike functions. Inclusion relation, integral transforms, and partial sums for each of these classes are discussed.

Keywords: Analytic function, meromorphic function, p-valent function, starlike function, subordination, inclusion relation, integral transforms, partial sum.

2010 MSC: 30C45, 30C80.

1. Introduction

In this paper, we assume that

$$-1 \leqslant B < 0, \quad B < A \leqslant -B, \quad \lambda \geqslant 1 \quad \text{and} \quad k \in \mathbb{N} \setminus \{1\}. \tag{1.1}$$

For functions f and g analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, the function f is said to be subordinate to g, written $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function w in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$.

A function f which is analytic in a domain $D \subset \mathbb{C}$ is called p-valent in D if for every complex number w, the equation $f(z) = w$ has at most p roots in D and there will be a complex number w_0 such that the equation $f(z) = w_0$ has exactly p roots in D. Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad (p \in \mathbb{N}), \tag{1.2}$$

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which are analytic in the punctured open unit disk $\mathbb{U}_0 = \mathbb{U} \setminus \{0\}$. We denote by S_p^* the well-known class of meromorphically p-valent starlike functions. It is defined as follows

$$S_p^* = \left\{ f \in \Sigma_p : \mathrm{Re}\frac{zf'(z)}{f(z)} < 0, z \in \mathbb{U} \right\}.$$

Let

$$f_j(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,j} z^n \in \Sigma_p \quad (j = 1, 2).$$

Then the Hadamard product (or convolution) of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z).$$

Lemma 1.1. *Let $f \in \Sigma_p$ defined by (1.2) satisfies*

$$\sum_{n=p}^{\infty} \{p(1-A) + (1-B)[\lambda n + p(\lambda - 1)\delta_{n,p,k}]\} \leqslant p(A-B). \tag{1.3}$$

Then

$$\frac{p(1-\lambda)f_{p,k}(z) - \lambda z f'(z)}{p f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{1.4}$$

where

$$f_{p,k}(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{jp} f(\varepsilon_k^j z), \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right) \tag{1.5}$$

and

$$\delta_{n,p,k} = \begin{cases} 0, & \left(\frac{n+p}{k} \notin \mathbb{N}\right), \\ 1, & \left(\frac{n+p}{k} \in \mathbb{N}\right). \end{cases} \tag{1.6}$$

Proof. The function $f_{p,k}$ in (1.5) can be expressed as

$$f_{p,k}(z) = z^{-p} + \sum_{n=p}^{\infty} \delta_{n,p,k} a_n z^n \tag{1.7}$$

with

$$\delta_{n,p,k} = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{j(n+p)} = \begin{cases} 0 & \left(\frac{n+p}{k} \notin \mathbb{N}\right), \\ 1 & \left(\frac{n+p}{k} \in \mathbb{N}\right). \end{cases}$$

According to (1.1) and (1.6), we see that

$$pA - B[p(1-\lambda)\delta_{n,p,k} - \lambda n] \leqslant -B[p - p(\lambda-1)\delta_{n,p,k} - \lambda n] \leqslant 0 \quad (n \geqslant p). \tag{1.8}$$

Let the inequality (1.3) hold. Then from (1.7) and (1.8), we deduce that

$$\begin{aligned} \left| \frac{\frac{p(1-\lambda)f_{p,k}(z)-\lambda z f'(z)}{pf(z)} - 1}{A - B\frac{p(1-\lambda)f_{p,k}(z)-\lambda z f'(z)}{pf(z)}} \right| &= \left| \frac{-\sum_{n=p}^{\infty}[p(\lambda-1)\delta_{n,p,k} + \lambda n + p]a_n z^{n+p}}{p(A-B) + \sum_{n=p}^{\infty}\{pA - B[p(1-\lambda)\delta_{n,p,k} - \lambda n]\}a_n z^{n+p}} \right| \\ &\leqslant \frac{\sum_{n=p}^{\infty}[p(\lambda-1)\delta_{n,p,k} + \lambda n + p]|a_n|}{p(A-B) + \sum_{n=p}^{\infty}\{pA - B[p(1-\lambda)\delta_{n,p,k} - \lambda n]\}|a_n|} \leqslant 1. \end{aligned}$$

Hence, by the maximum modulus theorem, we have (1.4). The proof is completed. □

We now introduce the following two subclasses of Σ_p.

Definition 1.2. A function $f \in \Sigma_p$ defined by (1.2) is said to be in the class $M_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality (1.3).

Definition 1.3. A function $f \in \Sigma_p$ defined by (1.2) is said to be in the class $R_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality

$$\sum_{n=p}^{\infty} n\{p(1-A)+(1-B)[\lambda n + p(\lambda-1)\delta_{n,p,k}]\} \leqslant p^2(A-B).$$

For $f \in \Sigma_p$, we have

$$2z^{-p} + \frac{zf'(z)}{p} = z^{-p} + \sum_{n=p}^{\infty} \frac{n}{p} a_n z^n,$$

which implies that

$$f \in R_{p,k}(\lambda, A, B) \quad \text{if and only if} \quad 2z^{-p} + \frac{zf'(z)}{p} \in M_{p,k}(\lambda, A, B). \tag{1.9}$$

If we write

$$\alpha_n = \frac{p(1-A)+(1-B)[\lambda n + p(\lambda-1)\delta_{n,p,k}]}{p(A-B)} \quad \text{and} \quad \beta_n = \frac{n}{p}\alpha_n \quad (n \geqslant p), \tag{1.10}$$

then it is easy to verify that

$$\frac{\partial \beta_n}{\partial \lambda} = \frac{n}{p}\frac{\partial \alpha_n}{\partial \lambda} > 0, \quad \frac{\partial \beta_n}{\partial A} = \frac{n}{p}\frac{\partial \alpha_n}{\partial A} < 0, \quad \text{and} \quad \frac{\partial \beta_n}{\partial B} = \frac{n}{p}\frac{\partial \alpha_n}{\partial B} \geqslant 0.$$

Thus, we obtain the following inclusion relations. If

$$1 \leqslant \lambda_0 \leqslant \lambda, \quad -1 \leqslant B_0 \leqslant B < 0 \quad B < A \leqslant -B, \quad \text{and} \quad A \leqslant A_0 \leqslant -B_0,$$

then

$$R_{p,k}(\lambda, A, B) \subset M_{p,k}(\lambda, A, B) \subset M_{p,k}(\lambda_0, A_0, B_0) \subset M_{p,k}(1, 1, -1) \subseteq S_p^* = \left\{ f \in \Sigma_p : \operatorname{Re}\frac{zf'(z)}{f(z)} < 0, z \in U \right\}.$$

Therefore, by Lemma 1.1, we see that each function in the classes $M_{p,k}(\lambda, A, B)$ and $R_{p,k}(\lambda, A, B)$ is meromorphically p-valent starlike function. Meromorphic (and analytic) functions which are starlike have been extensively investigated by several authors (see, e.g., [1–22] and the references therein). In this paper we study some properties such as inclusion relation, integral transforms, and partial sums for the above-defined classes $M_{p,k}(\lambda, A, B)$ and $R_{p,k}(\lambda, A, B)$.

2. Inclusion relation

In this section we shall generalize the above mentioned inclusion relation

$$R_{p,k}(\lambda, A, B) \subset M_{p,k}(\lambda, A, B). \tag{2.1}$$

Theorem 2.1. *If* $-1 \leqslant D \leqslant B$, *then*

$$R_{p,k}(\lambda, A, B) \subset M_{p,k}(\lambda, C(D), D),$$

where

$$C(D) = D + \frac{(1-D)(A-B)}{1-B}.$$

The number $C(D)$ *cannot be decreased for each* D.

Proof. Since $-1 \leqslant D \leqslant B < 0$ and $B < A \leqslant -B$, we see that

$$D < C(D) \leqslant D - \frac{2B(1-D)}{1-B} \leqslant -D.$$

Let $f \in \mathcal{R}_{p,k}(\lambda, A, B)$. In order to prove that $f \in \mathcal{M}_{p,k}(\lambda, C(D), D)$, we need only to find the smallest C $(D < C \leqslant -D)$ such that

$$\frac{p(1-C)+(1-D)[\lambda n + p(\lambda-1)\delta_{n,p,k}]}{p(C-D)} \leqslant \frac{n\{p(1-A)+(1-B)[\lambda n + p(\lambda-1)\delta_{n,p,k}]\}}{p^2(A-B)} \tag{2.2}$$

for all $n \geqslant p$, that is, that

$$\frac{(1-D)[\lambda n + p + p(\lambda-1)\delta_{n,p,k}]}{p(C-D)} - 1 \leqslant \frac{n}{p}\left\{\frac{(1-B)[\lambda n + p + p(\lambda-1)\delta_{n,p,k}]}{p(A-B)} - 1\right\}. \tag{2.3}$$

For $n \geqslant p$ and $\frac{n+p}{k} \notin \mathbb{N}$, (2.3) becomes

$$C \geqslant D + \frac{1-D}{\frac{n(1-B)}{p(A-B)} - \frac{n-p}{\lambda n + p}} := \varphi(n).$$

Noting that (1.1), a simple calculation shows that $\varphi(n)$ $(n \geqslant p)$ is decreasing in n. Therefore,

$$\varphi(n) \leqslant \begin{cases} \varphi(p+1), & \left(\frac{2p}{k} \in \mathbb{N}\right), \\ \varphi(p), & \left(\frac{2p}{k} \notin \mathbb{N}\right). \end{cases}$$

For $n \geqslant p$ and $\frac{n+p}{k} \in \mathbb{N}$, (2.3) is equivalent to

$$C \geqslant D + \frac{1-D}{\frac{n(1-B)}{p(A-B)} - \frac{n-p}{\lambda(n+p)}} := \psi(n).$$

Also, $\psi(n)(n \geqslant p)$ is decreasing in n. Thus

$$\psi(n) \leqslant \begin{cases} \psi(p), & \left(\frac{2p}{k} \in \mathbb{N}\right), \\ \psi\left(k\left(\left[\frac{2p}{k}\right]+1\right)-p\right), & \left(\frac{2p}{k} \notin \mathbb{N}\right), \end{cases} \tag{2.4}$$

where $[x]$ in (2.4) denotes the integer part of a given real number x. Consequently, by taking

$$C = \varphi(p) = \psi(p) = D + \frac{(1-D)(A-B)}{1-B} = C(D), \tag{2.5}$$

it follows from (2.2) to (2.5) that $f \in \mathcal{M}_{p,k}(\lambda, C(D), D)$. Furthermore, for $\frac{2p}{k} \in \mathbb{N}$ and $D < C_0 < C(D)$, we see that

$$\begin{aligned} &\frac{1-C_0+(2\lambda-1)(1-D)}{C_0-D} \cdot \frac{A-B}{1-A+(2\lambda-1)(1-B)} \\ &> \frac{1-C(D)+(2\lambda-1)(1-D)}{C(D)-D} \cdot \frac{A-B}{1-A+(2\lambda-1)(1-B)} = 1, \end{aligned}$$

which implies that the function

$$f(z) = z^{-p} + \frac{A-B}{1-A+(2\lambda-1)(1-B)} z^p \in \mathcal{R}_{p,k}(\lambda, A, B)$$

is not in the class $M_{p,k}(\lambda, C_0, D)$. Also, for $\frac{2p}{k} \notin \mathbb{N}$ and $D < C_0 < C(D)$, we have

$$\frac{1-C_0+\lambda(1-D)}{C_0-D} \cdot \frac{A-B}{1-A+\lambda(1-B)} > \frac{1-C_0+\lambda(1-D)}{C_0-D} \cdot \frac{A-B}{1-A+\lambda(1-B)} = 1,$$

which implies that the function

$$f(z) = z^{-p} + \frac{A-B}{1-A+\lambda(1-B)} z^p \in R_{p,k}(\lambda, A, B) \tag{2.6}$$

is not in the class $M_{p,k}(\lambda, C_0, D)$. The proof of the theorem is completed. □

Remark 2.2. Putting $D = B$ in Theorem 2.1, we have the inclusion relation (2.1).

3. Integral transforms

Theorem 3.1. *Let* $p < \mu < p(2\lambda+1)$. *Suppose that* $f \in M_{p,k}(\lambda, A, B)$ *and*

$$I_\mu(z) = \frac{\mu-p}{z^\mu} \int_0^z t^{\mu-1} f(t)dt. \tag{3.1}$$

Then $I_\mu \in M_{p,k}(\lambda, C_1(D), D)$, *where* $-1 \leqslant D \leqslant B$ *and*

$$C_1(D) = D + \frac{(\lambda+1)(\mu-p)(A-B)(1-D)}{(\lambda+1)(\mu+p)(1-B) - 2p(A-B)}.$$

The number $C_1(D)$ *cannot be decreased for each* D.

Proof. Since $-1 \leqslant D \leqslant B < 0$, $B < A \leqslant -B$ and $p < \mu < p(2\lambda+1)$, we can see that

$$D < C_1(D) \leqslant D + \frac{(\lambda+1)(\mu-p)(A-B)(1-D)}{\lambda(\mu+p)(1-B)} \leqslant D - \frac{2B(1-D)}{1-B} \leqslant -D.$$

For

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in M_{p,k}(\lambda, A, B),$$

it follows from (3.1) that

$$I_\mu(z) = z^{-p} + \sum_{n=p}^{\infty} \frac{\mu-p}{\mu+n} a_n z^n. \tag{3.2}$$

In order to prove that $I_\mu \in M_{p,k}(\lambda, C_1(D), D)$, we need only to find the smallest C $(D < C \leqslant -D)$ such that

$$\frac{p(1-C)+(1-D)[\lambda n + p(\lambda-1)\delta_{n,p,k}]}{p(C-D)} \cdot \frac{\mu-p}{\mu+n} \leqslant \frac{p(1-A)+(1-B)[\lambda n + p(\lambda-1)\delta_{n,p,k}]}{p(A-B)} \tag{3.3}$$

for all $n \geqslant p$.

For $n \geqslant p$ and $\frac{n+p}{k} \notin \mathbb{N}$, (3.3) becomes

$$C \geqslant D + \frac{1-D}{\frac{(\mu+n)(1-B)}{(\mu-p)(A-B)} - \frac{p(n+p)}{(\mu-p)(\lambda n+p)}} := \varphi_1(n).$$

It is easy to show that $\varphi_1(n)$ $(n \geqslant p)$ is a decreasing function of n and so

$$\varphi_1(n) \leqslant \begin{cases} \varphi_1(p+1), & \left(\frac{2p}{k} \in \mathbb{N}\right), \\ \varphi_1(p), & \left(\frac{2p}{k} \notin \mathbb{N}\right). \end{cases}$$

For $n \geqslant p$ and $\frac{n+p}{k} \in \mathbb{N}$, (3.3) reduces to

$$C \geqslant D + \frac{1-D}{\frac{(\mu+n)(1-B)}{(\mu-p)(A-B)} - \frac{p}{\lambda(\mu-p)}} := \psi_1(n)$$

and we have

$$\psi_1(n) \leqslant \begin{cases} \psi_1(p), & \left(\frac{2p}{k} \in \mathbb{N}\right), \\ \psi_1\left(k\left(\left[\frac{2p}{k}\right]+1\right)-p\right), & \left(\frac{2p}{k} \notin \mathbb{N}\right). \end{cases} \tag{3.4}$$

A simple calculation shows that $\psi_1(p) \leqslant \varphi_1(p)$. Therefore, by taking

$$C = \varphi_1(p) = C_1(D),$$

it follows from (3.3) to (3.4) that $I_\mu \in M_{p,k}(\lambda, C_1(D), D)$.

Furthermore, the number $C_1(D)$ is best possible for the function defined by (2.6). The proof of the theorem is completed. □

Theorem 3.2. *Let* $p < \mu < p(2\lambda+1)$. *Also let* I_μ *and* $C_1(D)$ *be the same as in Theorem 3.1. If* $f \in R_{p,k}(\lambda, A, B)$, *then* $I_\mu \in R_{p,k}(\lambda, C_1(D), D)$ *and the number* $C_1(D)$ *cannot be decreased for each* D.

Proof. By (3.2) we have

$$I_\mu(z) = \left(z^{-p} + \sum_{n=p}^{\infty} \frac{\mu-p}{\mu+n} z^n\right) * f(z)$$

and so

$$2z^{-p} + \frac{z(I_\mu(z))'}{p} = \left(z^{-p} + \sum_{n=p}^{\infty} \frac{\mu-p}{\mu+n} z^n\right) * \left(2z^{-p} + \frac{zf'(z)}{p}\right). \tag{3.5}$$

In view of (3.5) and (1.9), an application of Theorem 3.1 yields Theorem 3.2. The proof of the theorem is completed. □

4. Partial sums

In this section, we let $f \in \Sigma_p$ be given by (1.2) and define the partial sums $s_1(z)$ and $s_m(z)$ by

$$s_1(z) = z^{-p} \quad \text{and} \quad s_m(z) = z^{-p} + \sum_{n=p}^{p+m-2} a_n z^n \quad (m \in \mathbb{N} \setminus \{1\}).$$

For simplicity we use the notation α_n $(n \geqslant p)$ defined by (1.10).

Theorem 4.1. *Let* $p \geqslant 2$ *and* $1 \leqslant \lambda \leqslant \frac{p}{p-1}$. *Suppose that* $f \in M_{p,k}(\lambda, A, B)$. *Then for* $m \in \mathbb{N}$, *we have*

$$\mathrm{Re}\frac{f(z)}{s_m(z)} > 1 - \frac{1}{\alpha_{p+m-1}} \quad (z \in U) \tag{4.1}$$

and

$$\mathrm{Re}\frac{s_m(z)}{f(z)} > \frac{\alpha_{p+m-1}}{1+\alpha_{p+m-1}} \quad (z \in U). \tag{4.2}$$

The bounds in (4.1) and (4.2) are sharp for each m.

Proof. In view of the assumptions of the theorem, we see that

$$\alpha_n = \frac{p(1-A) + (1-B)[\lambda n + p(\lambda-1)\delta_{n,p,k}]}{p(A-B)} \geqslant \frac{2-A-B}{A-B} \geqslant 1 \tag{4.3}$$

and

$$\alpha_{n+1} = \alpha_n + \frac{(1-B)[\lambda + p(\lambda-1)(\delta_{n+1,p,k} - \delta_{n,p,k})]}{p(A-B)} \geqslant \alpha_n + \frac{(1-B)[\lambda - p(\lambda-1)]}{p(A-B)} \geqslant \alpha_n. \tag{4.4}$$

Let $f \in \mathcal{M}_{p,k}(\lambda, A, B)$. Then it follows from (4.3) and (4.4) that

$$\sum_{n=p}^{p+m-2} |a_n| + \alpha_{p+m-1} \sum_{n=p+m-1}^{\infty} |a_n| \leqslant \sum_{n=p}^{\infty} \alpha_n |a_n| \leqslant 1 \quad (m \in \mathbb{N} \setminus \{1\}). \tag{4.5}$$

If we put

$$p_1(z) = 1 + \alpha_{p+m-1} \left(\frac{f(z)}{s_m(z)} - 1 \right)$$

for $z \in \mathbb{U}$ and $m \in \mathbb{N} \setminus \{1\}$, then $p_1(0) = 1$ and we deduce from (4.5) that

$$\begin{aligned}
\left| \frac{p_1(z) - 1}{p_1(z) + 1} \right| &= \left| \frac{\alpha_{p+m-1} \sum_{n=p+m-1}^{\infty} a_n z^{n+p}}{2 \left(1 + \sum_{n=p}^{p+m-2} a_n z^{n+p} \right) + \alpha_{p+m-1} \sum_{n=p+m-1}^{\infty} a_n z^{n+p}} \right| \\
&\leqslant \frac{\alpha_{p+m-1} \sum_{n=p+m-1}^{\infty} |a_n|}{2 - 2 \sum_{n=p}^{p+m-2} |a_n| - \alpha_{p+m-1} \sum_{n=p+m-1}^{\infty} |a_n|} \leqslant 1.
\end{aligned}$$

This implies that $\operatorname{Re} p_1(z) > 0$ $(z \in \mathbb{U})$, and so (4.1) holds for $m \in \mathbb{N} \setminus \{1\}$.

Similarly, by setting

$$p_2(z) = (1 + \alpha_{p+m-1}) \frac{s_m(z)}{f(z)} - \alpha_{p+m-1},$$

it follows from (4.5) that

$$\begin{aligned}
\left| \frac{p_2(z) - 1}{p_2(z) + 1} \right| &= \left| \frac{-(1 + \alpha_{p+m-1}) \sum_{n=p+m-1}^{\infty} a_n z^{n+p}}{2 \left(1 + \sum_{n=p}^{p+m-2} a_n z^{n+p} \right) + (1 - \alpha_{p+m-1}) \sum_{n=p+m-1}^{\infty} a_n z^{n+p}} \right| \\
&\leqslant \frac{(1 + \alpha_{p+m-1}) \sum_{n=p+m-1}^{\infty} |a_n|}{2 - 2 \sum_{n=p}^{p+m-2} |a_n| - (\alpha_{p+m-1} - 1) \sum_{n=p+m-1}^{\infty} |a_n|} \leqslant 1.
\end{aligned}$$

Hence, we have (4.2) for $m \in \mathbb{N} \setminus \{1\}$.

For $m = 1$, replacing (4.5) by

$$\alpha_p \sum_{n=p}^{\infty} |a_n| \leqslant \sum_{n=p}^{\infty} \alpha_n |a_n| \leqslant 1$$

and proceeding as the above, we see that (4.1) and (4.2) are also true.

Furthermore, taking the function

$$f(z) = z^{-p} + \frac{z^{p+m-1}}{\alpha_{p+m-1}} \in \mathcal{M}_{p,k}(\lambda, A, B),$$

we have $s_m(z) = z^{-p}$,

$$\operatorname{Re} \frac{f(z)}{s_m(z)} \to 1 - \frac{1}{\alpha_{p+m-1}} \quad \text{as} \quad z \to \exp\left(\frac{\pi i}{2p + m - 1} \right)$$

and

$$\operatorname{Re} \frac{s_m(z)}{f(z)} \to \frac{\alpha_{p+m-1}}{1 + \alpha_{p+m-1}} \quad \text{as} \quad z \to 1.$$

The proof of the theorem is completed. □

Theorem 4.2. *Let* $p \geqslant 2$ *and* $1 \leqslant \lambda \leqslant \frac{p}{p-1}$. *Suppose that* $f \in R_{p,k}(\lambda, A, B)$. *Then for* $m \in \mathbb{N}$, *we have*

$$\mathrm{Re}\frac{f(z)}{s_m(z)} > 1 - \frac{p}{(p+m-1)\alpha_{p+m-1}} \quad (z \in U) \tag{4.6}$$

and

$$\mathrm{Re}\frac{s_m(z)}{f(z)} > \frac{(p+m-1)\alpha_{p+m-1}}{p+(p+m-1)\alpha_{p+m-1}} \quad (z \in U). \tag{4.7}$$

The bounds in (4.6) *and* (4.7) *are sharp for the function*

$$f(z) = z^{-p} + \frac{pz^{p+m-1}}{(p+m-1)\alpha_{p+m-1}} \in R_{p,k}(\lambda, A, B). \tag{4.8}$$

Proof. According to the assumptions of the theorem, it follows from (4.3) and (4.4) that

$$\sum_{n=p}^{p+m-2} |a_n| + \frac{(p+m-1)\alpha_{p+m-1}}{p} \sum_{n=p+m-1}^{\infty} |a_n| \leqslant \sum_{n=p}^{\infty} \frac{n}{p}\alpha_n |a_n| \leqslant 1 \quad (m \in \mathbb{N}\setminus\{1\}) \tag{4.9}$$

and

$$\alpha_p \sum_{n=p}^{\infty} |a_n| \leqslant \sum_{n=p}^{\infty} \frac{n}{p}\alpha_n |a_n| \leqslant 1. \tag{4.10}$$

If we put

$$p_1(z) = 1 + \frac{(p+m-1)\alpha_{p+m-1}}{p}\left[\frac{f(z)}{s_m(z)} - 1\right]$$

and

$$p_2(z) = \left[1 + \frac{(p+m-1)\alpha_{p+m-1}}{p}\right]\frac{s_m(z)}{f(z)} - \frac{(p+m-1)\alpha_{p+m-1}}{p},$$

then (4.9) and (4.10) lead to $\mathrm{Re}\, p_j(z) > 0$ $(z \in U; m \in \mathbb{N}; j = 1, 2)$. The proof of the theorem is completed. □

Theorem 4.3. *Let* $p \geqslant 2$ *and* $1 \leqslant \lambda \leqslant \frac{p}{p-1}$. *Suppose that* $f \in R_{p,k}(\lambda, A, B)$. *Then for* $m \in \mathbb{N}$, *we have*

$$\mathrm{Re}\frac{f'(z)}{s'_m(z)} > 1 - \frac{1}{\alpha_{p+m-1}} \quad (z \in U) \tag{4.11}$$

and

$$\mathrm{Re}\frac{s'_m(z)}{f'(z)} > \frac{\alpha_{p+m-1}}{1+\alpha_{p+m-1}} \quad (z \in U). \tag{4.12}$$

The bounds in (4.11) *and* (4.12) *are sharp.*

Proof. By virtue of the assumptions of the theorem, it follows from (4.3) and (4.4) that

$$\frac{1}{p}\sum_{n=p}^{p+m-2} n|a_n| + \frac{\alpha_{p+m-1}}{p} \sum_{n=p+m-1}^{\infty} n|a_n| \leqslant \sum_{n=p}^{\infty} \frac{n}{p}\alpha_n |a_n| \leqslant 1 \quad (m \in \mathbb{N}\setminus\{1\}) \tag{4.13}$$

and

$$\frac{\alpha_p}{p} \sum_{n=p}^{\infty} n|a_n| \leqslant \sum_{n=p}^{\infty} \frac{n}{p}\alpha_n |a_n| \leqslant 1. \tag{4.14}$$

By considering the functions

$$p_1(z) = 1 + \alpha_{p+m-1}\left(\frac{f'(z)}{s'_m(z)} - 1\right) \quad \text{and} \quad p_2(z) = (1 + \alpha_{p+m-1})\frac{s'_m(z)}{f'(z)} - \alpha_{p+m-1},$$

we deduce from (4.13) and (4.14) that Re $p_j(z) > 0$ $(z \in \mathbb{U}; m \in \mathbb{N}; j = 1, 2)$. Thus (4.11) and (4.12) hold true.

Furthermore, the bounds in (4.11) and (4.12) are best possible for the function defined by (4.8). The proof of the theorem is completed. □

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