Positive solutions for a class of fractional boundary value problems with fractional boundary conditions

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Abstract

In this paper, we study the solvability of a nonlinear fractional differential equation under fractional integral boundary conditions. Via a mixed monotone operator method, some new results on the existence and uniqueness of a positive solution for the considered model are obtained. Moreover, we provide iterative sequences for approximating the solution. Some examples are also presented in order to illustrate the obtained result.

Keywords: Fractional boundary value problem, fractional integral boundary condition, mixed monotone operator.

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1. Introduction

Fractional differential equations have received increasing attention during recent years due to their wide range of applications in a variety of disciplines in science and engineering, such as physics, control theory, chemistry, biology, ecology, aerodynamics, etc. It was shown that derivatives and integrals of fractional type provide an adequate mathematical modeling of real objects and processes. For details, we refer the reader to [6, 12, 13, 15, 19, 21–23] and references therein.

Recently, basing on nonlinear analysis tools, several existence results for different classes of fractional boundary value problems were established. In this direction, we refer to [1–5, 7, 8, 10, 17, 18, 20, 22, 26–30] and references therein. In particular, many works have been devoted to the study of the existence of solutions for fractional differential equations under different kinds of integral (or fractional integral) boundary conditions, see [2–5, 7, 8, 17, 26–29] and references therein. The motivation of a such study

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is due to the importance and numerous applications of integral boundary conditions in different fields, such as blood flow problems, chemical engineering, thermo-elasticity, etc. See, for example [2, 11, 24, 25].

In this paper, we are concerned with the solvability of a class of nonlinear fractional differential equations with fractional integral boundary conditions. More precisely, we are concerned with the fractional boundary value problem

$$\begin{cases} D_0^\alpha u(t) + f(t, u(t), (Hu)(t)) + g(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, \ u(1) = \lambda \left((I_0^\beta u)(1) + (I_0^\gamma u)(1) \right), \end{cases} \tag{1.1}$$

where $2 < \alpha < 3$, $\lambda > 0$, $\beta, \gamma \geqslant 0$, D_0^α is the Riemann-Liouville left-sided fractional derivative of order α, I_0^q, $q \in \{\beta, \gamma\}$ is the Riemann-Liouville left-sided fractional integral of order q, $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$, $g : [0,1] \times [0,\infty) \to [0,\infty)$, and $H : C([0,1]; \mathbb{R}) \to C([0,1]; \mathbb{R})$ is a certain operator (not necessarily linear). We provide sufficient conditions for the existence and uniqueness of positive solutions for Problem (1.1). Moreover, we present an iterative algorithm that converges to the solution. Some examples are also presented in order to illustrate the obtained result. Our approach is based on a mixed monotone operator method introduced in [29].

The paper is organized as follows. In Section 2, we recall briefly some basic concepts on fractional calculus. In Section 3, we define the adequate functional space on which problem (1.1) is posed, we compute its Green's function, and then we obtain an integral formulation of the problem. In Section 4, we recall the basic tools related to the mixed monotone operator method. In Section 5, we state and prove our main result (Theorem 5.1). In Section 6, several examples are presented in order to illustrate the obtained result. Finally, in Section 7, a comparison with existence results from the literature is presented.

2. Preliminaries on fractional calculus

Let $(a, b) \in \mathbb{R}^2$ with $a < b$.

Definition 2.1 ([19]). Let $f \in L^1([a, b]; \mathbb{R})$. The Riemann-Liouville left-sided fractional integral of order $\alpha > 0$ of f is defined by

$$(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} \, ds, \quad \text{a.e. } t \in [a, b],$$

where Γ is the Gamma function. For $\alpha = 0$, we take $I_a^0 f = f$.

Lemma 2.2 ([19]). *If $\alpha_1, \alpha_2 > 0$, then*

$$(I_a^{\alpha_1} I_a^{\alpha_2} f)(t) = (I_a^{\alpha_1 + \alpha_2} f)(t), \quad a.e. \ t \in [a, b],$$

for every $f \in L^1([a, b]; \mathbb{R})$.

Lemma 2.3 ([16]). *If $\alpha > 0$, $p \geqslant 1$, and $p > \frac{1}{\alpha}$, then $I_a^\alpha f \in C([a, b]; \mathbb{R})$ for every $f \in L^p([a, b]; \mathbb{R})$.*

Lemma 2.4 ([19]). *Let $\mu, \alpha > 0$. Then*

$$I_a^\mu (t-a)^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha + \mu)} (t-a)^{\alpha + \mu - 1}, \quad t > a.$$

Let $n \in \mathbb{N}$, $n \geqslant 1$. By $AC^n([a, b], \mathbb{R})$, we denote the space of all functions $f : [a, b] \to \mathbb{R}$ that are absolutely continuous together with the classical derivatives $f, Df, \ldots, D^{n-1}f$.

Definition 2.5 ([16]). Let $n - 1 < \alpha < n$ for some $n \in \mathbb{N}$, $n \geqslant 1$, and $f \in L^1([a, b]; \mathbb{R})$. We say that f possesses the left-sided Riemann-Liouville derivative D_a^α of order α on the interval $[a, b]$ if $I_a^{n-\alpha}f \in AC^n([a, b], \mathbb{R})$. By this derivative we mean the derivative $D^n(I_a^{n-\alpha}f)$, i.e.,

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} D^n \left(\int_a^t \frac{f(s)}{(t-s)^{1-n+\alpha}} \, ds \right), \quad \text{a.e. } t \in [a, b].$$

Lemma 2.6 ([16]). *Let $n - 1 < \alpha < n$ for some $n \in \mathbb{N}$, $n \geqslant 2$, and $f \in L^1([a, b]; \mathbb{R})$. Then f has the left-sided Riemann-Liouville derivative $D_a^\alpha f$ of order α on the interval $[a, b]$ if and only if there exist constants $c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$ and a function $\varphi \in L^1([a, b]; \mathbb{R})$ such that*

$$f(t) = \frac{c_0}{\Gamma(\alpha - n + 1)}(t - a)^{\alpha - n} + \frac{c_1}{\Gamma(\alpha - n + 2)}(t - a)^{\alpha - n + 1}$$
$$+ \cdots + \frac{c_{n-1}}{\Gamma(\alpha)}(t - a)^{\alpha - 1} + I_a^\alpha \varphi(t), \quad a.e. \ t \in [a, b]. \tag{2.1}$$

In such a case, we have

$$c_i = D^i(I_a^{n-\alpha}f)(a), \quad i = 0, 1, 2, \ldots, n - 1$$

and

$$D_a^\alpha f(t) = \varphi(t), \quad a.e. \ t \in [a, b].$$

3. Integral representation of solutions

In order to obtain an integral formulation of problem (1.1), we need the computation of its Green's function. Here, we compute rigorously the Green's function associated to problem (1.1) with the adequate functional spaces.

By $AC_a^\alpha([a, b]; \mathbb{R})$, we denote the set of functions $f : [a, b] \to \mathbb{R}$ that belong to $L^1([a, b]; \mathbb{R})$, and having the representation (2.1). We introduce the functional space $X_a^\alpha([a, b]; \mathbb{R})$ defined by

$$X_a^\alpha([a, b]; \mathbb{R}) = AC_a^\alpha([a, b]; \mathbb{R}) \cap C^1([a, b]; \mathbb{R}).$$

We have the following characterization of the above functional space in the case $2 < \alpha < 3$.

Lemma 3.1. *Let $2 < \alpha < 3$. For every $u \in X_0^\alpha([0, 1]; \mathbb{R})$, we have*

$$u(0) = u'(0) = 0.$$

Proof. Let $u \in X_0^\alpha([0, 1]; \mathbb{R})$. Then the function u has the representation (2.1) with $n = 3$, that is,

$$u(t) = \frac{c_0}{\Gamma(\alpha - 2)}t^{\alpha - 3} + \frac{c_1}{\Gamma(\alpha - 1)}t^{\alpha - 2} + \frac{c_2}{\Gamma(\alpha)}t^{\alpha - 1} + \frac{1}{\Gamma(\alpha)}\int_0^t \varphi(s)(t - s)^{\alpha - 1} \, ds, \quad a.e. \ t \in [0, 1],$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2$, and $\varphi = D_0^\alpha u \in L^1([0, 1]; \mathbb{R})$. Since u is continuous at 0, we have

$$\lim_{t \to 0^+} u(t) = u(0).$$

On the other hand, since $2 < \alpha < 3$, we have

$$\lim_{t \to 0^+} u(t) = \lim_{t \to 0^+} \left(\frac{c_0}{\Gamma(\alpha - 2)}t^{\alpha - 3} + \frac{1}{\Gamma(\alpha)}\int_0^t \varphi(s)(t - s)^{\alpha - 1} \, ds \right).$$

Observe that

$$\left| \int_0^t \varphi(s)(t - s)^{\alpha - 1} \, ds \right| \leqslant \int_0^t |\varphi(s)| \, ds.$$

Therefore,

$$\lim_{t \to 0^+} \int_0^t \varphi(s)(t - s)^{\alpha - 1} \, ds = 0.$$

Hence, we have

$$u(0) = \lim_{t \to 0^+} u(t) = \lim_{t \to 0^+} \frac{c_0}{\Gamma(\alpha - 2)}t^{\alpha - 3},$$

which yields $c_0 = 0$ and $u(0) = 0$. Next, it follows that

$$u(t) = \frac{c_1}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{c_2}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t \varphi(s)(t-s)^{\alpha-1}\,ds, \quad \text{a.e. } t \in [0,1].$$

Differentiating with respect to t, we get

$$u'(t) = \frac{c_1(\alpha-2)}{\Gamma(\alpha-1)}t^{\alpha-3} + \frac{c_2(\alpha-1)}{\Gamma(\alpha)}t^{\alpha-2} + \frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^t \varphi(s)(t-s)^{\alpha-2}\,ds, \quad \text{a.e. } t \in [0,1].$$

Since u' is continuous at 0, we have

$$\lim_{t \to 0^+} u'(t) = u'(0).$$

On the other hand, we have

$$\lim_{t \to 0^+} u'(t) = \lim_{t \to 0^+}\left(\frac{c_1(\alpha-2)}{\Gamma(\alpha-1)}t^{\alpha-3} + \frac{(\alpha-1)}{\Gamma(\alpha)}\int_0^t \varphi(s)(t-s)^{\alpha-2}\,ds\right).$$

Similarly, we have

$$\lim_{t \to 0^+}\int_0^t \varphi(s)(t-s)^{\alpha-2}\,ds = 0,$$

which yields

$$u'(0) = \lim_{t \to 0^+}\frac{c_1(\alpha-2)}{\Gamma(\alpha-1)}t^{\alpha-3}.$$

Hence, we obtain $c_1 = 0$ and $u'(0) = 0$. $\qquad\square$

From Lemma 3.1, we deduce immediately the following characterization of the functional space $X_0^\alpha([0,1];\mathbb{R})$ for $2 < \alpha < 3$.

Lemma 3.2. *Let $2 < \alpha < 3$. For every $u \in X_0^\alpha([0,1];\mathbb{R})$, there exists $c(u) \in \mathbb{R}$ such that*

$$u(t) = \frac{c(u)}{\Gamma(\alpha)}t^{\alpha-1} + (I_0^\alpha\varphi)(t), \quad \text{a.e. } t \in [0,1],$$

where $\varphi = D_0^\alpha u \in L^1([a,b];\mathbb{R})$.

Lemma 3.3. *Let $2 < \alpha < 3$ and $\beta, \gamma \geqslant 0$. For every $u \in X_0^\alpha([0,1];\mathbb{R})$, we have*

$$(I_0^\beta u)(1) + (I_0^\gamma u)(1) = \left(\frac{1}{\Gamma(\alpha+\beta)} + \frac{1}{\Gamma(\alpha+\gamma)}\right)c(u) + \int_0^1 (1-s)^{\alpha-1}\left[\frac{(1-s)^\beta}{\Gamma(\alpha+\beta)} + \frac{(1-s)^\gamma}{\Gamma(\alpha+\gamma)}\right]\varphi(s)\,ds,$$

where $\varphi = D_0^\alpha u \in L^1([a,b];\mathbb{R})$.

Proof. Let $u \in X_0^\alpha([0,1];\mathbb{R})$. Using Lemma 3.2, we obtain

$$(I_0^\beta u)(t) = \frac{c(u)}{\Gamma(\alpha)}I_0^\beta t^{\alpha-1} + (I_0^\beta I_0^\alpha\varphi)(t), \quad \text{a.e. } t \in [0,1].$$

On the other hand, from Lemmas 2.4 and 2.2, we have

$$I_0^\beta t^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}t^{\alpha+\beta-1}$$

and

$$(I_0^\beta I_0^\alpha\varphi)(t) = (I_0^{\alpha+\beta}\varphi)(t), \quad \text{a.e. } t \in [0,1].$$

Therefore,

$$(I_0^\beta u)(t) = \frac{c(u)}{\Gamma(\alpha + \beta)} t^{\alpha+\beta-1} + (I_0^{\alpha+\beta}\varphi)(t), \quad \text{a.e. } t \in [0,1].$$

From Lemma 2.3, we have $I_0^\beta u \in C([0,1];\mathbb{R})$. Hence,

$$(I_0^\beta u)(1) = \lim_{t\to 1^-} (I_0^\beta u)(t) = \frac{c(u)}{\Gamma(\alpha+\beta)} + \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} \varphi(s)\, ds. \tag{3.1}$$

Similarly, we obtain

$$(I_0^\gamma u)(1) = \frac{c(u)}{\Gamma(\alpha+\gamma)} + \frac{1}{\Gamma(\alpha+\gamma)} \int_0^1 (1-s)^{\alpha+\gamma-1} \varphi(s)\, ds. \tag{3.2}$$

Adding (3.1) to (3.2), the desired result follows. □

Lemma 3.4. *Let* $2 < \alpha < 3$, $\lambda > 0$, *and* $\beta, \gamma \geqslant 0$. *Suppose that*

$$[C(\lambda, \alpha, \beta, \gamma)]^{-1} := \frac{1}{\Gamma(\alpha)} - \frac{\lambda}{\Gamma(\alpha+\beta)} - \frac{\lambda}{\Gamma(\alpha+\gamma)} \neq 0. \tag{3.3}$$

Let $u \in X_0^\alpha([0,1];\mathbb{R})$ *be such that*

$$u(1) = \lambda \left((I_0^\beta u)(1) + (I_0^\gamma u)(1) \right). \tag{3.4}$$

Then

$$u(t) = \int_0^1 G(t,s)[-\varphi(s)]\, ds, \quad t \in [0,1],$$

where $\varphi = D_0^\alpha u \in L^1([a,b];\mathbb{R})$ *and*

$$G(t,s) = \frac{C(\lambda, \alpha, \beta, \gamma)}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1}\psi(s) - \frac{1}{C(\lambda,\alpha,\beta,\gamma)}(t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}\psi(s), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

where

$$\psi(s) = \left[\frac{1}{\Gamma(\alpha)} - \frac{\lambda(1-s)^\beta}{\Gamma(\alpha+\beta)} - \frac{\lambda(1-s)^\gamma}{\Gamma(\alpha+\gamma)} \right], \quad s \in [0,1].$$

Proof. From Lemma 3.2, we have

$$u(t) = \frac{c(u)}{\Gamma(\alpha)} t^{\alpha-1} + (I_0^\alpha \varphi)(t), \quad \text{a.e. } t \in [0,1],$$

where $\varphi = D_0^\alpha u \in L^1([a,b];\mathbb{R})$. On the other hand, since u is continuous in $[0,1]$, the above equality can be extended to all the inteval $[0,1]$, i.e.,

$$u(t) = \frac{c(u)}{\Gamma(\alpha)} t^{\alpha-1} + (I_0^\alpha \varphi)(t), \quad t \in [0,1].$$

Taking $t = 1$, we obtain

$$u(1) = \frac{c(u)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi(s)\, ds.$$

Hence, by(3.4) and Lemma 3.3, we obtain

$$\frac{c(u)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi(s)\, ds$$

$$= \left(\frac{\lambda}{\Gamma(\alpha + \beta)} + \frac{\lambda}{\Gamma(\alpha + \gamma)} \right) c(u) + \int_0^1 (1 - s)^{\alpha - 1} \left[\frac{\lambda(1 - s)^\beta}{\Gamma(\alpha + \beta)} + \frac{\lambda(1 - s)^\gamma}{\Gamma(\alpha + \gamma)} \right] \varphi(s) \, ds,$$

which from (3.3) yields

$$c(u) = C(\lambda, \alpha, \beta, \gamma) \int_0^1 (1 - s)^{\alpha - 1} \psi(s)[-\varphi(s)] \, ds.$$

Therefore, we obtain

$$u(t) = \frac{C(\lambda, \alpha, \beta, \gamma)}{\Gamma(\alpha)} \int_0^1 t^{\alpha - 1} (1 - s)^{\alpha - 1} \psi(s)[-\varphi(s)] \, ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} [-\varphi(s)] \, ds$$

for every $t \in [0, 1]$. Next, we have

$$u(t) = \int_0^t \frac{C(\lambda, \alpha, \beta, \gamma)}{\Gamma(\alpha)} \left[t^{\alpha - 1} (1 - s)^{\alpha - 1} \psi(s) - \frac{1}{C(\lambda, \alpha, \beta, \gamma)} (t - s)^{\alpha - 1} \right] [-\varphi(s)] \, ds$$
$$+ \int_t^1 \frac{C(\lambda, \alpha, \beta, \gamma)}{\Gamma(\alpha)} t^{\alpha - 1} (1 - s)^{\alpha - 1} \psi(s)[-\varphi(s)] \, ds, \quad t \in [0, 1],$$

which yields the desired result.   □

Lemma 3.5. *Let $2 < \alpha < 3$ and $\beta, \gamma \geqslant 0$. Suppose that*

$$0 < \lambda \left(\frac{1}{\Gamma(\alpha + \beta)} + \frac{1}{\Gamma(\alpha + \gamma)} \right) < \frac{1}{\Gamma(\alpha)}. \tag{3.5}$$

Then

  (i) *G is continuous in $[0, 1] \times [0, 1]$;*
 (ii) *for every $(t, s) \in [0, 1] \times [0, 1]$, we have*

$$t^{\alpha - 1} \mu(s) \leqslant G(t, s) \leqslant t^{\alpha - 1} \nu(s),$$

   where

$$\mu(s) = \frac{C(\lambda, \alpha, \beta, \gamma)}{\Gamma(\alpha)} (1 - s)^{\alpha - 1} \left[\frac{\lambda}{\Gamma(\alpha + \beta)} (1 - (1 - s)^\beta) + \frac{\lambda}{\Gamma(\alpha + \gamma)} (1 - (1 - s)^\gamma) \right]$$

   and

$$\nu(s) = \frac{C(\lambda, \alpha, \beta, \gamma)}{\Gamma(\alpha)} (1 - s)^{\alpha - 1} \psi(s);$$

(iii) *$G(t, s) \geqslant 0$ for every $(t, s) \in [0, 1] \times [0, 1]$.*

Proof. Properties (i) and (ii) follow immediately from the definition of the Green's function G. On the other hand, from (3.5), we have

$$C(\lambda, \alpha, \beta, \gamma) > 0.$$

Therefore, by (ii) we deduce (iii).   □

The following result is an immediate consequence of Lemma 3.4.

Lemma 3.6. *Let $2 < \alpha < 3 \ \lambda > 0$, and $\beta, \gamma \geqslant 0$ be such that (3.3) is satisfied. Let $y \in L^1([a, b]; \mathbb{R})$. Then the fractional boundary value problem*

$$\begin{cases} D_0^\alpha u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, \ u(1) = \lambda \left((I_0^\beta u)(1) + (I_0^\gamma u)(1) \right) \end{cases}$$

has a unique solution in $X_0^\alpha([0, 1]; \mathbb{R})$, which is given by

$$u(t) = \int_0^1 G(t, s) y(s) \, ds, \quad t \in [0, 1].$$

Now, we are able to give an equivalent integral formulation of problem (1.1).
problem (1.1) is investigated under the following assumptions:

(A1) $2 < \alpha < 3$ and $\beta, \gamma \geqslant 0$;
(A2) (3.5) is satisfied;
(A3) $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ and $g : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous functions;
(A4) $H : C([0,1]; \mathbb{R}) \to C([0,1]; \mathbb{R})$ satisfies

$$u \in C([0,1]; [0,\infty)) \implies Hu \in C([0,1]; [0,\infty)).$$

For $u \in C([0,1]; [0,\infty))$, we denote by Tu the function defined by

$$(Tu)(t) = \int_0^1 G(t,s) \left[f(s, u(s), (Hu)(s)) + g(s, u(s)) \right] ds, \quad t \in [0,1]. \tag{3.6}$$

From the above assumptions and the properties (i) and (iii) of Lemma 3.5, it can be easily seen that

$$T : C([0,1]; [0,\infty)) \to C([0,1]; [0,\infty))$$

is a well-defined operator. Moreover, from Lemma 3.6, $u \in X_0^\alpha([0,1]; \mathbb{R})$ is a solution of (1.1) if and only if $u \in C([0,1]; [0,\infty))$ is a fixed point of T.

4. The mixed monotone operator method

From the above study, we know that $u \in X_0^\alpha([0,1]; \mathbb{R})$ is a solution of (1.1) if and only if $u \in C([0,1]; [0,\infty))$ is a fixed point of the operator T defined by (3.6). In order to study the existence of fixed points of the operator T, we shall use the mixed monotone operator method. In this section, we recall the basic tools related to this method.

Let $(E, \| \cdot \|)$ be a real Banach space. We denote by θ_E the zero vector of E.

Definition 4.1. A nonempty closed convex subset P of E is said to be a cone in E if it satisfies the following conditions:

(i) $(r, x) \in [0,\infty) \times P \implies rx \in P$;
(ii) $(x, -x) \in P \times P \implies x = \theta_E$.

Let P be a cone in E. Then P induces a partial order $\leqslant_P$ in E defined by

$$(x, y) \in E \times E, \quad x \leqslant_P y \iff y - x \in P.$$

By $x <_P y$, $(x, y) \in E \times E$, we mean $x \leqslant_P y$ and $x \neq y$.

Definition 4.2. Let P be a cone in E. If there exists a constant $C > 0$ such that

$$(x, y) \in E \times E, \ \theta_E \leqslant_P x \leqslant_P y \implies \|x\| \leqslant C\|y\|,$$

then the cone P is said to be normal. In this case, the smallest constant C satisfying the above property is called the normal constant of P.

For $(x, y) \in E \times E$, the notation $x \sim y$ means that there exist constants $\rho, \tau > 0$ such that

$$\tau y \leqslant_P x \leqslant_P \rho y.$$

It can be easily seen that $\sim$ is an equivalence relation in E.
For $h \in E$, $\theta_E <_P h$, we denote by P_h the subset of P defined by

$$P_h = \{x \in P : x \sim h\}.$$

For more details on cones in Banach spaces, we refer the reader to [14].

Definition 4.3. An operator $S : E \to E$ is said to be increasing (Resp. decreasing) if for every $(x,y) \in E \times E$, $x \leqslant_P y$ implies that $Sx \leqslant_P Sy$ (Resp. $Sy \leqslant_P Sx$).

Definition 4.4. An operator $A : P \times P \to P$ is said to be mixed monotone if $A(x,y)$ is increasing in x, and decreasing in y, i.e.,

$$(x,y),(u,v) \in P \times P, \ x \leqslant_P u, \ y \geqslant_P v \implies A(x,y) \leqslant_P A(u,v).$$

Definition 4.5. An operator $B : P \to P$ is said to be subhomogeneous if

$$(t,x) \in (0,1) \times P \implies B(tx) \geqslant_P tBx.$$

The following result appears in [29], and it is the main tool used in our study.

Lemma 4.6. *Let* $(E, \| \cdot \|)$ *be a real Banach space, and let* P *be a normal cone in* E. *Let* $h \in E$, $\theta_E <_P h$, *and* $\tau \in (0,1)$. *Let us consider two operators* $A : P \times P \to P$ *and* $B : P \to P$. *We suppose that* A *is a mixed monotone operator satisfying*

$$A(tx, t^{-1}y) \geqslant_P t^\tau A(x,y), \quad t \in (0,1), (x,y) \in P \times P. \tag{4.1}$$

We suppose also that B *is an increasing subhomogeneous operator satisfying the following conditions:*

  (i) *there exists* $h_0 \in P$ *such that* $A(h_0, h_0) \in P_h$ *and* $Bh_0 \in P_h$;
  (ii) *there exists a constant* $\delta_0 > 0$ *such that*

$$A(x,y) \geqslant_P \delta_0 Bx, \quad (x,y) \in P \times P.$$

Then

  (a) $A : P_h \times P_h \to P_h$ *and* $B : P_h \to P_h$;
  (b) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that*

$$rv_0 \leqslant_P u_0 <_P v_0, \ u_0 \leqslant_P A(u_0, v_0) + Bu_0 \leqslant_P A(v_0, u_0) + Bv_0 \leqslant_P v_0;$$

  (c) *there exists a unique solution* $x^* \in P_h$ *such that*

$$x^* = A(x^*, x^*) + Bx^*;$$

  (d) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \ y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

  we have $\|x_n - x^*\| \to 0$ *and* $\|y_n - x^*\| \to 0$ *as* $n \to \infty$.

Now, we are ready to prove our main result.

5. Main result

In this section, sufficient conditions are provided for the existence and uniqueness of positive solutions for problem (1.1). An iterative algorithm is also provided in order to approximate the solution.

We will work in the Banach space $C([0,1]; \mathbb{R})$ equipped with the standard norm $\| \cdot \|_\infty$ defined by

$$\|u\|_\infty = \max\{|u(t)| : 0 \leqslant t \leqslant 1\}, \quad u \in C([0,1]; \mathbb{R}).$$

In $C([0,1]; \mathbb{R})$, we consider the cone $P = C([0,1]; [0,\infty))$. It can be easily seen that P is a normal cone with normal constant equal to 1. The partial order $\leqslant_P$ induced by the cone P is defined by

$$(u,v) \in C([0,1]; \mathbb{R}) \times C([0,1]; \mathbb{R}), \quad u \leqslant_P v \iff u(t) \leqslant v(t), 0 \leqslant t \leqslant 1.$$

The following additional assumptions are needed.

(A5) There exists $t_0 \in [0,1]$ such that $g(t_0, 0) > 0$.

(A6) For fixed $t \in [0,1]$, $f(t,x,y)$ is increasing in x, and decreasing in y; and $g(t,x)$ is increasing in x.

(A7) For every $(k,t,x) \in (0,1) \times [0,1] \times [0,\infty)$, we have

$$g(t, kx) \geqslant kg(t,x).$$

(A8) There exists a constant $\tau \in (0,1)$ such that

$$f(t, kx, k^{-1}y) \geqslant k^\tau f(t,x,y), \quad (k,t,x,y) \in (0,1) \times [0,1] \times [0,\infty) \times [0,\infty).$$

(A9) There exists a constant $\delta_0 > 0$ such that

$$f(t,x,y) \geqslant \delta_0 g(t,x), \quad (t,x,y) \in [0,1] \times [0,\infty) \times [0,\infty).$$

(A10) $H : C([0,1]; \mathbb{R}) \to C([0,1]; \mathbb{R})$ satisfies
   (H1) $(u,v) \in P \times P$, $u \leqslant_P v \implies Hu \leqslant_P Hv$;
   (H2) $H(ku) \geqslant_P kHu$ for every $(k,u) \in (0,1) \times P$.

Our main result in this paper is the following.

Theorem 5.1. *Suppose that assumptions* (A1)-(A10) *are satisfied. Let* $h \in C([0,1]; \mathbb{R})$ *be the function defined by* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *Then*

(i) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that*

$$rv_0 \leqslant_P u_0 <_P v_0,$$

and, moreover, for every $t \in [0,1]$, *we have*

$$u_0(t) \leqslant \int_0^1 G(t,s) \left[f(s, u_0(s), (Hv_0)(s)) + g(s, u_0(s)) \right] ds,$$

$$v_0(t) \geqslant \int_0^1 G(t,s) \left[f(s, v_0(s), (Hu_0)(s)) + g(s, v_0(s)) \right] ds;$$

(ii) *problem* (1.1) *has a unique positive solution* $x^* \in P_h$;

(iii) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n(t) = \int_0^1 G(t,s) \left[f(s, x_{n-1}(s), (Hy_{n-1})(s)) + g(s, x_{n-1}(s)) \right] ds, \quad t \in [0,1], n = 1, 2, \ldots$$

and

$$y_n(t) = \int_0^1 G(t,s) \left[f(s, y_{n-1}(s), (Hx_{n-1})(s)) + g(s, y_{n-1}(s)) \right] ds, \quad t \in [0,1], n = 1, 2, \ldots,$$

we have $\|x_n - x^*\|_\infty \to 0$ *and* $\|y_n - x^*\|_\infty \to 0$ *as* $n \to \infty$.

Proof. It was shown in Section 3 that $u \in X_0^\alpha([0,1]; \mathbb{R})$ is a solution of (1.1) if and only if $u \in P$ is a fixed point of T, where $T : P \to P$ is defined by (3.6). Let us introduce the operators $A : P \times P \to P$ and $B : P \to P$ defined by

$$A(u,v)(t) = \int_0^1 G(t,s) f(s, u(s), (Hv)(s)) ds, \quad (u,v) \in P \times P, \ t \in [0,1]$$

and

$$(Bu)(t) = \int_0^1 G(t,s) g(s, u(s)) ds, \quad u \in P, \ t \in [0,1].$$

Observe that $u \in P$ is a fixed point of T if and only if $u \in P$ satisfies the operator equation $A(u, u) + Bu = u$.

In the sequel, we check that assumptions of Lemma 4.6 are satisfied. From (A6), (H1), and the positivity of the Green's function G (see property (iii) in Lemma 3.5), it is easily seen that A is a mixed monotone operator and B is an increasing operator. Let $k \in (0, 1)$ and $(x, y) \in P \times P$. We have

$$A(kx, k^{-1}y)(t) = \int_0^1 G(t, s)f(s, kx(s), (H(k^{-1}y))(s))\,ds, \quad t \in [0, 1].$$

On the other hand, from (H2) we have

$$Hy = H(k(k^{-1}y)) \geqslant_P kH(k^{-1}y),$$

that is,

$$H(k^{-1}y)(s) \leqslant k^{-1}(Hy)(s), \quad s \in [0, 1].$$

Using the above inequality, (A6), and the positivity of the Green's function G, we obtain

$$A(kx, k^{-1}y)(t) \geqslant \int_0^1 G(t, s)f(s, kx(s), k^{-1}(Hy)(s))\,ds, \quad t \in [0, 1],$$

which implies by (A8) that

$$A(kx, k^{-1}y)(t) \geqslant k^\tau \int_0^1 G(t, s)f(s, x(s), (Hy)(s))\,ds = k^\tau A(x, y)(t), \quad t \in [0, 1].$$

Therefore, condition (4.1) of Lemma 4.6 is satisfied. Next, by assumption (A7) and the positivity of the Green's function G, we see easily that B is a subhomogeneous operator .

Next, we take the function

$$h(t) = t^{\alpha-1}, \quad t \in [0, 1].$$

Clearly, we have $h \in P$ and $h \not\equiv 0$. Taking into account property (ii) in Lemma 3.5, assumptions (A4) and (A6), and the positivity of the Green's function G, we get

$$A(h, h)(t) = \int_0^1 G(t, s)f(s, h(s), (Hh)(s))\,ds \leqslant \int_0^1 G(t, s)f(s, 1, 0)\,ds \leqslant t^{\alpha-1}\int_0^1 v(s)f(s, 1, 0)\,ds, \quad t \in [0, 1],$$

that is,

$$A(h, h) \leqslant_P \left(\int_0^1 v(s)f(s, 1, 0)\,ds\right) h. \tag{5.1}$$

Similarly, using the above assumptions and (H1), we obtain

$$A(h, h)(t) = \int_0^1 G(t, s)f(s, h(s), (Hh)(s))\,ds$$

$$\geqslant \int_0^1 G(t, s)f(s, 0, (H1)(s))\,ds \geqslant t^{\alpha-1}\int_0^1 \mu(s)f(s, 0, (H1)(s))\,ds, \quad t \in [0, 1],$$

that is,

$$A(h, h) \geqslant_P \left(\int_0^1 \mu(s)f(s, 0, (H1)(s))\,ds\right) h. \tag{5.2}$$

Combining (5.1) with (5.2), we obtain

$$\alpha_1 h \leqslant_P A(h, h) \leqslant_P \alpha_2 h, \tag{5.3}$$

where

$$\alpha_1 = \int_0^1 \mu(s)f(s,0,(H1)(s))\,ds \quad \text{and} \quad \alpha_2 = \int_0^1 \nu(s)f(s,1,0)\,ds.$$

In what follows, we shall prove that $\alpha_i > 0$, $i = 1, 2$. To do this, it is sufficient to prove that $\alpha_1 > 0$ (since $\alpha_1 \leqslant \alpha_2$). In fact, since $g(t_0, 0) > 0$ for some $t_0 \in [0,1]$ (from assumption (A5)), the continuity of g implies the existence of a certain subest E of $[0,1]$ with a non-zero Lebesgue measure such that

$$g(t,0) > 0, \quad t \in E. \tag{5.4}$$

Now, taking in consideration assumption (A9), we have

$$f(s,0,(H1)(s)) \geqslant \delta_0 g(s,0), \quad s \in [0,1]. \tag{5.5}$$

Therefore, by (5.4) and (5.5), we have

$$\begin{aligned}
\alpha_1 &= \int_0^1 \mu(s)f(s,0,(H1)(s))\,ds \\
&= \int_0^1 \frac{C(\lambda,\alpha,\beta,\gamma)}{\Gamma(\alpha)}(1-s)^{\alpha-1}\left[\frac{\lambda}{\Gamma(\alpha+\beta)}(1-(1-s)^\beta) + \frac{\lambda}{\Gamma(\alpha+\gamma)}(1-(1-s)^\gamma)\right] \times f(s,0,(H1)(s))\,ds \\
&\geqslant \frac{\delta_0 C(\lambda,\alpha,\beta,\gamma)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left[\frac{\lambda}{\Gamma(\alpha+\beta)}(1-(1-s)^\beta) + \frac{\lambda}{\Gamma(\alpha+\gamma)}(1-(1-s)^\gamma)\right] \times g(s,0)\,ds \\
&\geqslant \frac{\delta_0 C(\lambda,\alpha,\beta,\gamma)}{\Gamma(\alpha)}\int_E (1-s)^{\alpha-1}\left[\frac{\lambda}{\Gamma(\alpha+\beta)}(1-(1-s)^\beta) + \frac{\lambda}{\Gamma(\alpha+\gamma)}(1-(1-s)^\gamma)\right] \times g(s,0)\,ds > 0.
\end{aligned}$$

Hence, we deduce by (5.3) that $A(h,h) \in P_h$.

Using similar arguments as above, for any $t \in [0,1]$, we have

$$\beta_1 h \leqslant_P Bh \leqslant_P \beta_2 h,$$

where

$$\beta_1 = \int_0^1 \mu(s)g(s,0)\,ds > 0 \quad \text{and} \quad \beta_2 = \int_0^1 \nu(s)g(s,1)\,ds.$$

Therefore, we have $Bh \in P_h$. Hence, assumption (i) of Lemma 4.6 is satisfied with $h_0 = h$.

Now, let $(x,y) \in P \times P$. Using assumption (A9), we obtain

$$A(x,y)(t) = \int_0^1 G(t,s)f(s,x(s),(Hy)(s))\,ds \geqslant \delta_0 \int_0^1 G(t,s)g(s,x(s))\,ds = \delta_0(Bx)(t), \quad t \in [0,1].$$

Hence, we have

$$A(x,y) \geqslant_P \delta_0 Bx, \quad (x,y) \in P \times P,$$

and assumption (ii) of Lemma 4.6 is satisfied.

Finally, applying Lemma 4.6, the desired result follows. $\qquad\square$

6. Examples

In order to illustrate the obtained result given by Theorem 5.1, some examples are presented in this section.

Example 6.1. Consider the fractional boundary value problem

$$\begin{cases} D_0^{\frac{5}{2}}u(t) + t + t^3 + 2\sqrt{u(t)} + \frac{1}{[u(t)]^p+1} = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, \ u(1) = \displaystyle\int_0^1 u(s)\,ds, \end{cases} \tag{6.1}$$

where $0 < p < 1$.

Problem (6.1) is a particular case of problem (1.1) with $\alpha = \frac{5}{2}$, $\lambda = \frac{1}{2}$, $\beta = \gamma = 1$, $f(t, x, y) = t + \sqrt{x} + \frac{1}{y^p+1}$, $g(t, x) = t^3 + \sqrt{x}$, and $H : C([0, 1]; \mathbb{R}) \to C([0, 1]; \mathbb{R})$ is the operator defined by $Hu = u$ for every $u \in C([0, 1]; \mathbb{R})$.

In this considered case, we have

$$\lambda \left(\frac{1}{\Gamma(\alpha + \beta)} + \frac{1}{\Gamma(\alpha + \gamma)} \right) = \frac{1}{\Gamma\left(\frac{7}{2}\right)} \approx 0.3095 \quad \text{and} \quad \frac{1}{\Gamma(\alpha)} = \frac{1}{\Gamma\left(\frac{5}{2}\right)} \approx 0.7524.$$

Observe that

$$0 < \lambda \left(\frac{1}{\Gamma(\alpha + \beta)} + \frac{1}{\Gamma(\alpha + \gamma)} \right) < \frac{1}{\Gamma(\alpha)}.$$

Therefore, assumptions (A1) and (A2) of Theorem 5.1 are satisfied. Moreover, it is clear that $f : [0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ and $g : [0, 1] \times [0, \infty) \to [0, \infty)$ are continuous functions, and then assumption (A3) is satisfied. Assumptions concerning the operator H ((A4) and (A10)) are trivial. On the other hand, we have $g(1, 0) = 1 > 0$. Therefore, assumption (A5) is satisfied. It is easily seen that assumption (A6) is satisfied. In order to check assumption (A7), let us take $(k, t, x) \in (0, 1) \times [0, 1] \times [0, \infty)$. We have

$$g(t, kx) = t^3 + \sqrt{kx} \geqslant kt^3 + \sqrt{k}\sqrt{x} \geqslant kt^3 + k\sqrt{x} = k\left(t^3 + \sqrt{x}\right) = kg(t, x).$$

Then assumption (A7) is satisfied. Next, let $(k, t, x, y) \in (0, 1) \times [0, 1] \times [0, \infty) \times [0, \infty)$. We have

$$\begin{aligned}
f(t, kx, k^{-1}y) &= t + \sqrt{kx} + \frac{1}{k^{-p}y^p + 1} = t + \sqrt{k}\sqrt{x} + \frac{k^p}{y^p + k^p} \\
&\geqslant t + \sqrt{k}\sqrt{x} + \frac{k^p}{y^p + 1} \\
&\geqslant k^{\max\{\frac{1}{2}, p\}} \left(t + \sqrt{x} + \frac{1}{y^p + 1} \right) = k^{\max\{\frac{1}{2}, p\}} f(t, x, y).
\end{aligned}$$

Hence assumption (A8) is satisfied with $\tau = \max\{\frac{1}{2}, p\}$. Finally, it can be easily seen that assumption (A9) is satisfied with $\delta_0 = 1$. Therefore, all assumptions of Theorem 5.1 are satisfied. As consequence, problem 6.1 admits a unique solution $x^* \in P_h$, where $h(t) = t^{\frac{3}{2}}$, $t \in [0, 1]$.

Next, we present some examples of operators $H : C([0, 1]; \mathbb{R}) \to C([0, 1]; \mathbb{R})$ satisfying assumptions (A4) and (A10) of Theorem 5.1.

Example 6.2. Let $\varphi : [0, 1] \to [0, 1]$ be a continuous function. Define the operator $H : C([0, 1]; \mathbb{R}) \to C([0, 1]; \mathbb{R})$ by

$$(Hu)(t) = u(\varphi(t)), \quad t \in [0, 1]$$

for every $u \in C([0, 1]; \mathbb{R})$. Clearly, such operator satisfies assumptions (A4) and (A10).

Example 6.3. Let $\varphi : [0, 1] \to [0, \infty)$ be a continuous function. Define the operator $H : C([0, 1]; \mathbb{R}) \to C([0, 1]; \mathbb{R})$ by

$$(Hu)(t) = \varphi(t)u(t), \quad t \in [0, 1]$$

for every $u \in C([0, 1]; \mathbb{R})$. Then the operator H satisfies assumptions (A4) and (A10).

Example 6.4. Let $\varphi : [0, 1] \to [0, \infty)$ be a continuous function. Define the operator $H : C([0, 1]; \mathbb{R}) \to C([0, 1]; \mathbb{R})$ by

$$(Hu)(t) = \varphi(t)u(t), \quad t \in [0, 1]$$

for every $u \in C([0, 1]; \mathbb{R})$. Then the operator H satisfies assumptions (A4) and (A10).

Notice that the composition and multiplication operators on $C([0, 1]; \mathbb{R})$ are linear. The next examples present nonlinear operators $H : C([0, 1]; \mathbb{R}) \to C([0, 1]; \mathbb{R})$ satisfying the required assumptions.

Example 6.5. For $u \in C([0,1]; \mathbb{R})$, we define the mapping $Hu : [0,1] \to \mathbb{R}$ by

$$(Hu)(t) = \max\{|u(s)| : 0 \leqslant s \leqslant t\}, \quad t \in [0,1].$$

It was proved in [9] that H maps $C([0,1]; \mathbb{R})$ into itself. Moreover, it can be easily seen that assumptions (A4) and (A10) are satisfied by the operator H.

Example 6.6. Let $r \in (0,1)$. For $u \in C([0,1]; \mathbb{R})$, we define the mapping $Hu : [0,1] \to \mathbb{R}$ by

$$(Hu)(t) = u^r(t), \quad t \in [0,1].$$

It can be easily seen that assumptions (A4) and (A10) are satisfied by the operator H.

Example 6.7. Using the same arguments as in Example 6.1, if $H : C([0,1]; \mathbb{R}) \to C([0,1]; \mathbb{R})$ is any operator between the ones appearing in Examples 6.2-6.6, then the fractional boundary value problem

$$\begin{cases} D_0^{\frac{5}{2}} u(t) + t + t^3 + 2\sqrt{u(t)} + \frac{1}{[(Hu)(t)]^p + 1} = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 u(s) \, ds, \end{cases}$$

where $0 < p < 1$, admits a unique solution $x^* \in P_h$, where $h(t) = t^{\frac{3}{2}}$, $t \in [0,1]$.

7. Comparison with existing results

In [7], the authors considered the fractional boundary value problem

$$\begin{cases} D_0^\alpha u(t) + F(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, \ u(1) = \mu \int_0^1 u(s) \, ds, \end{cases} \tag{7.1}$$

where $2 < \alpha < 3$, $0 < \mu < \alpha$ and $F : [0,1] \times [0,\infty) \to [0,\infty)$. Observe that problem (6.1) is a particular case of problem (7.1) with $\alpha = \frac{5}{2}$, $\mu = 1$ and $F(t,x) = t + t^3 + 2\sqrt{x} + \frac{1}{x^p + 1}$.

Sufficient conditions for the existence of solutions for problem (7.1) were provided in [7]. Before recalling the main result in [7], we need to introduce some notations.

We put

$$F_0 := \lim_{x \to 0^+} \left(\min_{t \in [\frac{1}{2}, 1]} \frac{F(t,x)}{x} \right), \quad F_\infty := \lim_{x \to \infty} \left(\min_{t \in [\frac{1}{2}, 1]} \frac{F(t,x)}{x} \right)$$

and

$$F^0 := \lim_{x \to 0^+} \left(\max_{t \in [0,1]} \frac{F(t,x)}{x} \right), \quad F^\infty := \lim_{x \to \infty} \left(\max_{t \in [0,1]} \frac{F(t,x)}{x} \right).$$

Theorem 7.1 ([7]). *Suppose that*

(a) $F : [0,1] \times [0,\infty) \to [0,\infty)$ *is continuous;*
(b) $(F_0, F^\infty) = (\infty, 0)$ *(sublinear case) or* $(F^0, F_\infty) = (0, \infty)$ *(superlinear case).*

Then problem (7.1) admits at least one positive solution in $C([0,1]; \mathbb{R})$.

Notice that in Example 6.1, we have

$$\min_{t \in [\frac{1}{2}, 1]} \frac{F(t,x)}{x} = \frac{\frac{1}{2} + \frac{1}{8} + 2\sqrt{x} + \frac{1}{x^p + 1}}{x} \quad \text{and} \quad \max_{t \in [0,1]} \frac{F(t,x)}{x} = \frac{2 + 2\sqrt{x} + \frac{1}{x^p + 1}}{x}.$$

In such a case, it can be easily seen that

$$(F_0, F^\infty) = (\infty, 0) \quad \text{and} \quad (F^0, F_\infty) = (\infty, 0).$$

Therefore, we are in the sublinear case. Hence, by Theorem 7.1, problem (6.1) admits at least one positive solution in $C([0, 1]; \mathbb{R})$.

Observe that Theorem 7.1 provides only the existence of positive solutions for problem (6.1), while by our result (Theroem 5.1), we have an additional information, that is, a unique solution of problem 6.1 exists in the set P_h, where $h(t) = t^{\frac{3}{2}}$, $t \in [0, 1]$.

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