Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 263–273
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

On the periodic solution of a class of stochastic nonlinear system with delays

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Communicated by K. Q. Lan

Abstract

This paper is devoted to investigating a class of stochastic nonlinear system with periodic coefficients. Some criteria on existence and uniqueness of periodic solution are established for the stochastic nonlinear system. Finally, a numerical example is given to show the effectiveness and merits of the present results.

Keywords: Periodic solution, stochastic, Itô's formula, existence.

2010 MSC: MSC 34K50.

1. Introduction

During the past years, the theory and applications of stochastic differential equations have been developed very quickly, see e.g. [3, 11–13]. However, few results have been obtained in the direction of the periodically stochastic differential equations. Till now, we only find that very few results for periodic solution of stochastic differential equations have been published in [5, 6, 19]. Recently, some results have been established for the periodically stochastic differential equations. In [4] and [1], the authors studied the existence of periodic solution of differential equations with random right sides. Xu et al. [17] considered the following periodic stochastic functional differential equation (SFDE)

$$\begin{cases} dx(t) = B(t, x_t)dt + \sigma(t, x_t)dW(t), \ t \geqslant t_0 \geqslant 0, \\ x_{t_0}(t_0 + s) = \phi(s) \in C, \ s \in [0, \tau] \end{cases} \tag{1.1}$$

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doi: 10.22436/jnsa.011.02.08

Received: 2017-11-17 *Revised:* 2017-12-03 *Accepted:* 2017-12-22

on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$. By using the properties of periodic Markov processes, they obtained some sufficient conditions for the existence of periodic solution of system (1.1). After that, when the delay is unbounded in system (1.1), Li and Xu [9] also obtained the existence of periodic solution of system (1.1). In [19], Zhang and Gopalsamy studied two n species stochastic population models with periodic coefficients. Based on Lemma 2.3 (see e.g. [2]), some existence results of periodic solution are obtained for the above system, but the conditions of Lemma 2.3 are very sharp. In [24], a class of impulsive stochastic Nicholson's blowflies model is investigated by applying Cauchy matrix. Under proper conditions, the existence and exponential stability of square-mean almost periodic solutions for the model with multiple nonlinear harvesting terms and delays.

For practical applications, we note that when designing the neural networks or in the implementation of neural systems, stochastic perturbations are almost inevitable [10]. Hence, stochastic modeling is a vital issue. Therefore, it is necessary to investigate effects of stochastic perturbations on the stability of neural networks. Stability analysis of various stochastic neural networks with time delays has become an attractive topic of research, see e.g. [14, 16, 22, 23]. In [8], the stability and stabilization were studied for a class of stochastic systems with impulsive effects. In [15], the pth moment ($p \geqslant 2$) and the almost-sure stability of stochastic Cohen-Grossberg neural networks with mixed time delays and nonlinear impulsive effects were investigated through the Razumikhin type technique. Zhang et al. [21] studied a class of stochastic neural networks with local impulsive effects. For deterministic neural network, see e.g. [18, 20].

One of our main objectives is to study a kind of stochastic nonlinear system and obtain the properties of periodic solution for the above system. Firstly, an effective existence theorem for stochastic periodic process is established. Then, some sufficient conditions for the uniqueness of periodic solution of nonlinear system are given. To overcome the difficulties created by the special features possessed by the periodic stochastic equations with delays, as one will see, several novel analysis methods are introduced. These existence theorems are rather general and therefore have great power in applications.

The distinctive contributions of this paper are outlined as follows: (1) few results are obtained for the existence and uniqueness of the periodic solutions of stochastic nonlinear system with activation functions, novel analysis technique is developed since the conventional analysis tool no longer applies; (2) we develop the methods in [17] and [9], our method for the proof of existence and uniqueness of periodic solutions can more easily be understood; (3) the existence theorems of stochastic periodic solutions in this paper are rather general and therefore have great power in applications.

The following sections are organized as follows. In Section 2, some useful Lemmas and Definitions are given. In Section 3, sufficient conditions are established for existence and uniqueness of periodic solutions of systems (3.1) and (3.9), respectively. In Section 4, an example is given to show the feasibility of our results.

2. Preliminaries

In [7], Kolmanovskii and Myshkis have considered the existence of periodic solutions of the SFDEs of Itô. The method of the shift and averaging of the initial distribution generating a solution with certain properties was used. Particularly, they considered SRDE (stochastic retard differential equation) of the form

$$\begin{cases} dx(t) = a(t, x_t)dt + b(t, x_t)d\xi(t), \quad t \geqslant 0, \\ x_t(\theta) := x(t+\theta), \ -h \leqslant \theta \leqslant 0, \ x(t) \in \mathbb{R}^n, \ \xi(t) \in \mathbb{R}^l, \end{cases} \tag{2.1}$$

with the initial condition

$$x_t(\theta) := \phi(\theta), \ -h \leqslant \theta \leqslant 0. \tag{2.2}$$

Here $\xi(t)$ is the standard Wiener process, the continuous functionals $a(t, \phi)$ and $b(t, \phi)$ are defined on $[0, \infty) \times C[-h, 0]$ and are T-periodic with respect to the first argument. The solution of system (2.1) for $t \geqslant 0$ is determined by initial condition (2.2). Now we give some basic results for the periodic solutions of the SFDEs of Itô.

Definition 2.1 ([7]). T-periodic solution of (2.1) is a T-periodic stochastic process periodically connected with $\xi(t)$ and satisfying (2.1) with probability 1. The form means that all finite-dimensional distributions of the process $(x(t), -\xi(\tau)+\xi(s))$, $\tau < s$, are invariant with respect to all shifts of the arguments by the quantity kT, where k is an arbitrary integer, i.e., that for any positive integers N, m and any $t_1, \dots, t_N, \tau_1, \dots, \tau_m, s_1, \dots, s_m$ such that $\tau_i < s_i, i = 1, \dots, m$, the distribution of the probabilities of the random variable $(x(t_1), \dots, x(t_N), \xi(s_1)-\xi(\tau_1), \dots, \xi(s_m)-\xi(\tau_m))$ coincides with the distribution of the probabilities of variable $(x(t_1+kT), \dots, x(t_N+kT), \xi(s_1+kT)-\xi(\tau_1+kT), \dots, \xi(s_m+kT)-\xi(\tau_m+kT))$.

Theorem 2.2 ([7]). *Assume that the the coefficients of system (2.1) satisfy the assumptions*

$$|a(t,\phi)|^2 \leqslant q_1 + \int_{-h}^{0} |\phi(s)|^2 dk_1(s), \quad |b(t,\phi)|^2 \leqslant q_2 + \int_{-h}^{0} |\phi(s)|^2 dk_2(s),$$

where q_1 and q_2 are some positive constants, and $k_1(s)$ and $k_2(s)$ are scalar non-decreasing functions on $[-h, 0]$. Let, further on, a solution $x(t,\phi)$ of (2.1) and (2.2) exists such that for $s_1, s_2 \in [-h, 0]$,

$$\mathbb{E}|x(s_1) - x(s_2)|^{\varepsilon_1} \leqslant C|s_1 - s_2|^{1+\varepsilon_2},$$

where $\mathbb{E}$ is mathematical expectation, $\varepsilon_1, \varepsilon_2$ and C are positive constants, and for $t \geqslant 0$ the second moment is bounded:

$$\mathbb{E}|x(t,\phi)|^2 \leqslant C < \infty.$$

Then there exists a T-periodic solution of system (2.1).

Lemma 2.3 ([7]). *Let the continuous scalar functions $z(t)$ and $f(t)$, $t \geqslant 0$ satisfy for some constant C and for any $t_1, t_2 :\ t_2 \geqslant t_1 \geqslant 0$ the inequality*

$$z(t_2) - z(t_1) \leqslant -C\int_{t_1}^{t_2} z(s)ds + \int_{t_1}^{t_2} f(s)ds.$$

Then

$$z(t) \leqslant z(0)e^{-Ct} + \int_0^t e^{-C(t-s)} f(s)ds.$$

Lemma 2.4 ([7]). *Let the continuous non-negative function $x(t)$ satisfies for some constants C_i the inequality*

$$x(t) \leqslant C_1 + C_2\int_0^t e^{-C_3(t-s)} x(s)ds.$$

Then, if $C_3 > C_2$, then $x(t) \leqslant C_3C_1(C_3 - C_2)^{-1}$.

Remark 2.5. Theorem 2.2 shows that the boundedness of the moments of some solution implies the existence of a periodic solution. In relation to Theorem 2.2 we shall note that for determinate equation of the form (2.1) for $b = 0$ the existence of a bounded solution does not imply the existence of a periodic solution.

3. Existence and stability of periodic solutions

In this section, we consider a stochastic nonlinear system

$$\begin{cases} dx_i(t) = [-a_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_i(x_i(t)) + \sum_{j=1}^n d_{ij}(t)g_i(x_i(t-\tau))]dt \\ \qquad\qquad +\sigma_i(t, x_i(t), x_i(t-\tau))d\xi_i(t), \\ x_{it}(\theta) = \phi_i(\theta), \quad \theta \in [-h, 0], \ i = 1, 2, \dots, n, \end{cases} \tag{3.1}$$

where $t \geqslant 0$, $\phi_i(\theta) \in C([-h, 0], \mathbb{R})$, $0 < \tau \leqslant h$, $x_i(t)$ is the state of the system, $f_i(x_i(t))$ and $g_i(x_i(t-\tau))$ are the activation functions, $a_i(t) \geqslant 0$ shows the rate of change for the state $x(t)$, τ represents the transmission delay, and $\xi(t) = (\xi_1(t), \dots, \xi_n(t))^\top$ is the standard Wiener process. We assume that $a_i(t), a_{ij}(t)$ and $d_{ij}(t)$ are defined on $[0, \infty)$, are T-periodic and continuous functions. Assume that the following conditions hold.

(H_1) For $i = 1, 2, \ldots, n$, there exist positive constants q_{1i} and q_{2i} such that

$$|g_i(x_i(t-\tau))|^2 \leqslant q_{1i} + \int_{-h}^{0} |\phi_i(s)|^2 dk_{1i}(s), \quad |\sigma_i(t, x_i(t), x_i(t-\tau))|^2 \leqslant q_{2i} + \int_{-h}^{0} |\phi_i(s)|^2 dk_{2i}(s),$$

where $k_{1i}(s)$ and $k_{2i}(s)$ are scalar non-decreasing bounded functions on $[-h, 0]$.

(H_2) For $i = 1, 2, \ldots, n,\ t \geqslant 0$, we have

$$\sum_{j=1}^{n} a_{ij}(t) \geqslant 0,\ x_i(t) f_i(x_i(t)) \leqslant 0.$$

(H_2)* For $i = 1, 2, \ldots, n,\ t \geqslant 0$, we have

$$\sum_{j=1}^{n} a_{ij}(t) \leqslant 0,\ x_i(t) f_i(x_i(t)) \geqslant 0.$$

Theorem 3.1. *Let the assumptions* (H_1) *and* (H_2) *hold. Furthermore, the following inequality holds:*

$$\gamma_{1i} = \inf_{t \geqslant 0} \left[a_i(t) - \alpha_{01}^{1/2} \sum_{j=1}^{n} |d_{ij}(t)|/2 - \alpha_{01}^{1/2}/2 - \alpha_{02}/2 \right] > 0,$$

where $j = 1, \ldots, n,\ \alpha_{0j} = \int_{-h}^{0} dk_{ji}(s),\ j = 1, 2, 3,$ *and* k_{1i} *and* k_{2i} *are scalar non-decreasing functions bounded on* $[-h, 0]$*. Then there exists a* T*-periodic solution of* (3.1)*.*

Proof. Let

$$k_{3i} = \alpha_{01}^{-1/2} \sum_{j=1}^{n} d_{ij}(s) k_{1i}(s) + k_{2i}(s),\ i = 1, 2, \ldots, n.$$

For $i = 1, 2, \ldots, n$, consider the following functional on the trajectories of (3.1):

$$V(t, x_{it}) = x_i^2(t) + \int_{-h}^{0} dk_{3i}(s) \int_{t+s}^{t} x_i^2(\tau) d\tau, \tag{3.2}$$

where $x_{it}(\theta) = x_i(t+\theta),\ \theta \in [-h, 0]$. From Itô's formula, ($H_2$), (3.1), and (3.2), we have

$$\begin{aligned} &dV(t, x_{it}) \\ &= \Bigg[\sigma_i^2(t, x_i(t), x_i(t-\tau)) + \Bigg(- a_i(t) x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_i(x_i(t)) + \sum_{j=1}^{n} d_{ij}(t) g_i(x_i(t-\tau)) \Bigg) 2x_i(t) \\ &\quad + x_i^2(t)(\sqrt{\alpha_{01}} + \alpha_{02}) - \int_{-h}^{0} x_i^2(t+s) dk_{3i}(s) \Bigg] dt + 2x_i(t) \sigma_i(t, x_i(t), x_i(t-\tau)) d\xi_i(t) \\ &\leqslant \Bigg[\sigma_i^2(t, x_i(t), x_i(t-\tau)) + 2 \sum_{j=1}^{n} d_{ij}(t) x_i(t) g_i(x_i(t-\tau)) \\ &\quad + x_i^2(t)(\sqrt{\alpha_{01}} + \alpha_{02} - 2a_i(t)) - \int_{-h}^{0} x_i^2(t+s) dk_{3i}(s) \Bigg] dt + 2x_i(t) \sigma_i(t, x_i(t), x_i(t-\tau)) d\xi_i(t). \end{aligned} \tag{3.3}$$

We shall transform the separate addends in (3.3). In view of (H_1), considering the term

$$2x_i(t) \sum_{j=1}^{n} d_{ij}(t) g_i(x_i(t-\tau)),$$

we have

$$\begin{aligned}
&2|x_i(t)\sum_{j=1}^{n} d_{ij}(t)g_i(x_i(t-\tau))| \\
&\quad\leqslant \alpha_{01}^{1/2}\sum_{j=1}^{n}|d_{ij}(t)|x_i^2(t)+\alpha_{01}^{-1/2}\sum_{j=1}^{n}|d_{ij}(t)|g_i^2(x_i(t-\tau)) \\
&\quad\leqslant \alpha_{01}^{1/2}\sum_{j=1}^{n}|d_{ij}(t)|x_i^2(t)+\alpha_{01}^{-1/2}\sum_{j=1}^{n}|d_{ij}(t)|\Big(q_{1i}+\int_{-h}^{0}|\phi_i(s)|^2dk_{1i}(s)\Big) \\
&\quad= \alpha_{01}^{1/2}\sum_{j=1}^{n}|d_{ij}(t)|x_i^2(t)+\alpha_{01}^{-1/2}\sum_{j=1}^{n}|d_{ij}(t)|\Big(q_{1i}+\int_{-h}^{0}|x_{it}(s)|^2dk_{1i}(s)\Big) \\
&\quad= \alpha_{01}^{1/2}\sum_{j=1}^{n}|d_{ij}(t)|x_i^2(t)+\alpha_{01}^{-1/2}\sum_{j=1}^{n}|d_{ij}(t)|\Big(q_{1i}+\int_{-h}^{0}|x_i(t+s)|^2dk_{1i}(s)\Big) \\
&\quad\leqslant \alpha_{01}^{1/2}\sum_{j=1}^{n}|d_{ij}(t)|x_i^2(t)+\alpha_{01}^{-1/2}\sum_{j=1}^{n}|d_{ij}(t)|q_{1i}+\alpha_{01}^{-1/2}\sum_{j=1}^{n}|d_{ij}(t)|\int_{-h}^{0}|x_i(t+s)|^2dk_{1i}(s)
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\sigma_i^2(t,x_i(t),x_i(t-\tau)) &\leqslant q_{2i}+\int_{-h}^{0}|\phi_i(s)|^2dk_{2i}(s) \\
&= q_{2i}+\int_{-h}^{0}|x_{it}(s)|^2dk_{2i}(s)\leqslant q_{2i}+\int_{-h}^{0}|x_i(t+s)|^2dk_{2i}(s),\ i=1,\dots,n,
\end{aligned} \tag{3.5}$$

where $x_{it}(s):=\phi_i(s),\ -h\leqslant s\leqslant 0$. In view of (3.3)-(3.5), we have

$$dV(t,x_{it})\leqslant -2\gamma_{1i}x_i^2(t)dt+C_{1i}dt+2x_i(t)\sigma_i(t,x_i(t),x_i(t-\tau))d\xi_i(t), \tag{3.6}$$

where

$$\begin{aligned}
\gamma_{1i} &= \inf_{t\geqslant 0}[a_i(t)-\alpha_{01}^{1/2}\sum_{j=1}^{n}|d_{ij}(t)|/2-\alpha_{01}^{1/2}/2-\alpha_{02}/2]>0,\ i=1,\dots,n, \\
C_{1i} &= q_{2i}+q_{1i}\alpha_{01}^{-1/2}\sup_{t\geqslant 0}\{\sum_{j=1}^{n}|d_{ij}(t)|\},\ i=1,\dots,n.
\end{aligned}$$

For $t_2\geqslant t_1\geqslant 0$, integrating the both sides of (3.6) from t_1 to t_2, and taking the mathematical expectation, we have

$$\mathbb{E}x_i^2(t_2)-\mathbb{E}x_i^2(t_1)\leqslant -\gamma_{2i}\int_{t_1}^{t_2}\mathbb{E}x_i^2(t)dt+C_{1i}(t_2-t_1)+\int_{t_1}^{t_2}dt\int_{-h}^{0}\mathbb{E}x_i^2(t+s)dk_{3i}(s), \tag{3.7}$$

where $\gamma_{2i}=2\gamma_{1i}+\alpha_{03}$. From (3.7) and Lemma 2.3, we have

$$\mathbb{E}x_i^2(t)\leqslant C_{2i}+\int_0^t d\tau\int_{-h}^{0}\mathbb{E}x_i^2(\tau+s)z_i(t-\tau)dk_{3i}(s),$$

where $C_{2i}=C_{1i}/\gamma_{2i}+\mathbb{E}x_i^2(0),\ \ z_i(t-\tau)=e^{-\gamma_{2i}(t-\tau)}$. Denote by ν_i a function such that $\nu_i(t)=0$ for $t<0$ and $\nu_i(t)=1$ for $t\geqslant 0$ and set

$$\Gamma_i(\tau)=\sup_{0\leqslant t\leqslant\tau}\mathbb{E}x_i^2(t),\ i=1,\dots,n.$$

Note that

$$\begin{aligned}\int_0^t d\tau \int_{-h}^0 \mathbb{E}x_i^2(\tau+s)z_i(t-\tau)dk_{3i}(s) &= \int_0^t d\tau \int_{-h}^0 \mathbb{E}\phi_i^2(\tau+s)z_i(t-\tau)[1-\nu_i(s+\tau)]dk_{3i}(s)\\ &\quad + \int_0^t d\tau \int_{-h}^0 \mathbb{E}x_i^2(\tau+s)z_i(t-\tau)\nu_i(s+\tau)dk_{3i}(s)\\ &\leqslant \int_0^h d\tau \int_{-h}^0 \mathbb{E}\phi_i^2(\tau+s)z_i(t-\tau)[1-\nu_i(s+\tau)]dk_{3i}(s)\\ &\quad + \alpha_{03}\int_0^t \Gamma_i(\tau)z_i(t-\tau)d\tau.\end{aligned}$$

Hence

$$\mathbb{E}x_i^2(t) \leqslant C_{3i} + \alpha_{03}\int_0^t \Gamma_i(\tau)z_i(t-\tau)d\tau$$

and

$$C_{3i} \geqslant C_{2i} + \int_0^h d\tau \int_{-h}^0 \mathbb{E}\phi_i^2(\tau+s)z_i(t-\tau)[1-\nu_i(s+\tau)]dk_{3i}(s).$$

Note that

$$\int_0^t \Gamma_i(\tau)z_i(t-\tau)d\tau = \frac{1}{\gamma_{2i}}(1-z_i(t))\gamma_{2i}(0) + \frac{1}{\gamma_{2i}}\int_0^t (1-z_i(t-\tau))d\Gamma_i(\tau).$$

Thus,

$$\Gamma_i(t) \leqslant C_{3i} + \alpha_{03}\int_0^t \Gamma_i(\tau)z_i(t-\tau)d\tau. \tag{3.8}$$

In view of (3.8), $\gamma_{2i} > \alpha_{03}$, and Lemma 2.4, we have

$$\Gamma_i(t) \leqslant \gamma_{2i} + \frac{\gamma_{2i}C_{3i}}{\gamma_{2i}-\alpha_{03}} := C_i,\ i=1,\dots,n,$$

and

$$\mathbb{E}x_i^2(t) \leqslant C_i,\ i=1,\dots,n.$$

Thus,

$$\mathbb{E}x^2(t) \leqslant \sum_{i=1}^n C_i.$$

By Theorem 2.2, there exists a T-periodic solution of system (3.1). □

From the proof of Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let the assumptions* (H_1) *and* $(H_2)^*$ *hold. Furthermore, the following inequality holds:*

$$\gamma_{1i} = \inf_{t\geqslant 0}\left[a_i(t) - \alpha_{01}^{1/2}\sum_{j=1}^n |d_{ij}(t)|/2 - \alpha_{01}^{1/2}/2 - \alpha_{02}/2\right] > 0,$$

where $\alpha_{0j} = \int_{-h}^0 dk_{ji}(s)$, $j=1,2,3$, *and* k_{1i} *and* k_{2i} *are scalar non-decreasing functions bounded on* $[-h,0]$. *Then there exists a* T*-periodic solution of* (3.1).

Remark 3.3. Compared with the methods in [17] and [9], our methods are easier than ones in the above two papers. In fact, in order to obtain existence of periodic solution for system (1.1), they obtained the following lemma.

Lemma 3.4. A sufficient condition for the existence of an ω-periodic Markov process with a given ω-periodic transition function $p(\nu, x_\nu, t, A)$ is that for some t_0, ϕ, x_t are uniformly stochastically continuous and

$$\lim_{r\to\infty}\lim_{T\to\infty}\int_{t_0}^{t_0+T} p(t_0, \phi, \bar{U}_r)dt = 0,$$

provided the transition function $p(\nu, x_\nu, t, A)$ satisfies the following not very restrictive assumption that

$$\alpha(r) = \sup_{\phi\in U_{\beta(r)}, 0<t_0, t-t_0<\omega} p(t_0, \phi, \bar{U}_r) \to 0 \text{ as } r \to \infty$$

for some function $\beta(r)$ which tends to infinity as $r \to \infty$.

However, it is very difficult for obtaining the existence of an ω-periodic Markov process by using Lemma 3.4. In the present paper, by using some inequalities lemmas (Lemma 2.3 and 2.4) and some mathematic analysis techniques we can easily obtain existence results of periodic solutions for neural networks system.

When the initial values in (3.1) are unbounded, we can formulate conditions for the unique existence of periodic solution of the form

$$\begin{cases} dx_i(t) = [-a_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_i(x_i(t)) + \sum_{j=1}^n d_{ij}(t)g_i(x_i(t-\gamma))]dt \\ \qquad\qquad +\sigma_i(t, x_i(t), x_i(t-\gamma))d\xi_i(t), \quad t \geqslant 0, \ \gamma \in (-\infty, 0], \\ x_i(\theta) = \phi_i(\theta), \quad \theta \in (-\infty, 0], \ i = 1, 2, \dots, n. \end{cases} \tag{3.9}$$

We shall say that the T-periodic solution of (3.9) is unique if for any two T-periodic solution of this system $x(t) = (x_1(t), \dots, x_n(t))^\top$ and $y(t) = (y_1(t), \dots, y_n(t))^\top$ such that $\mathbb{E}|x(t)|^2 \leqslant C < \infty$ and $\mathbb{E}|y(t)|^2 \leqslant C < \infty$, and for all t the following relations holds

$$P_i(x_i(t) = y_i(t)) = 1, \quad i = 1, \dots, n.$$

We need the following assumptions.

(H_3) For $i = 1, \dots, n$, assume $a_i(t)$ such that the solution $z_i(t, s)$ of homogeneous equation $x_i'(t) = -a_i(t)x_i(t)$ satisfies the following estimate

$$|z_i(t, s)| \leqslant e^{-\lambda(t-s)}, \quad \lambda > 0, \quad i = 1, \dots, n.$$

(H_4) There exist $l_i \geqslant 0$ such that

$$|f_j(x_j(t)) - f_j(y_j(t))| \leqslant l_i, \quad i = 1, \dots, n.$$

(H_5) For $i = 1, 2, \dots, n$,

$$|g_i(y_i(t-\tau)) - g_i(x_i(t-\tau))|^2 \leqslant \int_{-\infty}^0 |\phi_i(s) - \psi_i(s)|^2 dk_{1i}(s) - l_i,$$

$$|\sigma_i(t, y_i(t), y_i(t-\tau)) - \sigma_i(t, x_i(t), x_i(t-\tau))|^2 \leqslant \int_{-\infty}^0 |\phi_i(s) - \psi_i(s)|^2 dk_{2i}(s),$$

where l_i is defined by (H_3), and k_{1i} and k_{2i} are scalar non-decreasing bounded on $(-\infty, 0]$ functions.

Theorem 3.5. *Let the coefficients of (3.9) satisfy the conditions of Theorem 3.1 and assumptions (H_3)-(H_5) hold. Then there exists unique T-periodic solution of (3.9).*

Proof. Assume that (3.9) has two T-periodic solution $x(t)$ and $y(t)$ satisfying initial condition $\phi(t) = (\phi_1(t), \dots, \phi_n(t))^\top$ and $\psi(t) = (\psi_1(t), \dots, \psi_n(t))^\top$, respectively. Assume further that $x(t)$ and $y(t)$ satisfy $\mathbb{E}|x(t)|^2 \leqslant C < \infty$ and $\mathbb{E}|y(t)|^2 \leqslant C < \infty$. Let $u_i(t) = y_i(t) - x_i(t)$, $i = 1, \dots, n$ satisfies the relation

$$\begin{aligned} u_i(t) = z_i(t,s)u_i(s) &+ \int_s^t z_i(t,\tau)\Big[\sum_{j=1}^n a_{ij}(\tau)f_i(y_i(\tau)) + \sum_{j=1}^n d_{ij}(\tau)g_i(y_i(\tau-\gamma)) \\ &- \sum_{j=1}^n a_{ij}(\tau)f_i(x_i(\tau)) - \sum_{j=1}^n d_{ij}(\tau)g_i(x_i(\tau-\gamma))\Big]d\tau \\ &+ \int_s^t z_i(t,\tau)[\sigma_i(\tau, y_i(t), y_i(\tau-\gamma)) - \sigma_i(\tau, x_i(\tau), x_i(\tau-\gamma))]d\xi_i(\tau). \end{aligned}$$

Then for any positive constants ε and ε_1, we have

$$\begin{aligned} \mathbb{E}|u_i(t)|^2 \leqslant (1+\varepsilon^{-1})|z_i(t,s)|^2\mathbb{E}|u_i(s)|^2 &+ (1+\varepsilon)\Big[(1+\varepsilon_1)\int_s^t |z_i(t,\tau_1)|d\tau_1 \int_s^t |z_i(t,\tau)| \\ &\times \mathbb{E}|\sum_{j=1}^n a_{ij}(\tau)f_i(y_i(\tau)) + \sum_{j=1}^n d_{ij}(\tau)g_i(y_i(\tau-\gamma)) \\ &- \sum_{j=1}^n a_{ij}(\tau)f_i(x_i(\tau)) - \sum_{j=1}^n d_{ij}(\tau)g_i(x_i(\tau-\gamma))|^2 d\tau \\ &+ (1+\varepsilon_1^{-1})\int_s^t |z_i(t,\tau)|^2\mathbb{E}|\sigma_i(\tau, y_i(t), y_i(\tau-\gamma)) - \sigma_i(\tau, x_i(\tau), x_i(\tau-\gamma))|^2 d\tau\Big]. \end{aligned}$$

Let $s \to -\infty$ in the above inequality, by (H_3)-(H_5), we get

$$\mathbb{E}|u_i(t)|^2 \leqslant (1+\varepsilon)[(1+\varepsilon_1)\lambda^{-2}\alpha_{01} + (1+\varepsilon_1^{-1})(2\lambda)^{-1}\alpha_{02}]f_i(t), \tag{3.10}$$

where $f_i(t) = \sup_{s \leqslant t} \mathbb{E}|u_i(s)|^2$. Obviously, when $\varepsilon_1 = (\lambda\alpha_{02})^{1/2}(2\alpha_{01})^{11/2}$, we have the minimum of the righthand side of (3.10). Thus,

$$f_i(t) \leqslant (1+\varepsilon)[\lambda^{-1}\alpha_{01}^{1/2} + (2\lambda)^{-1}\alpha_{02}^{1/2}]^2 f(t), \quad i = 1, \dots, n. \tag{3.11}$$

Let

$$\nu = \lambda^{-1}\alpha_{01}^{1/2} + (2\lambda)^{-1}\alpha_{02}^{1/2} < 1.$$

Then we choose $\varepsilon > 0$ such that $(1+\varepsilon)\nu < 1$. By (3.11) we have $P(x_i(t) = y_i(t)) = 1$ for all t and $i = 1, 2, \dots, n$. Hence $P(x(t) = y(t)) = 1$. The proof is completed. □

Remark 3.6. In [7], the authors studied the uniqueness of periodic solution for equations containing linear terms $A(t)x(t)$ of the form

$$\begin{cases} dx(t) = [A(t)x(t) + a(t, x_t)]dt + b(t, x_t)d\xi(t), & t \geqslant 0, \\ x_t(\theta) = x(t+\theta), \ \theta \leqslant 0, \ x(t) \in \mathbb{R}^n, \ \xi(t) \in\in \mathbb{R}^l, \end{cases} \tag{3.12}$$

where $A(t)$ is a matrix with continuous T-periodic entries such that the fundamental matrix $Z(t,s)$ of the homogeneous equation $x'(t) = A(t)x(t)$ satisfies for some constant $\lambda > 0$ the estimate

$$Z(t,s) \leqslant e^{-\lambda(t-s)}.$$

Under the conditions

$$|a(t,\phi) - a(t,\psi)|^2 \leqslant \int_{-\infty}^0 |\phi(s) - \psi(s)|^2 dk_1(s), \quad |b(t,\phi) - b(t,\psi)|^2 \leqslant \int_{-\infty}^0 |\phi(s) - \psi(s)|^2 dk_2(s),$$

then (3.12) has unique T-periodic solution. The proof of Theorem 3.5 is similar to the proof in [7]. It is worth pointing out that assumptions (H_4) and (H_5) are different from the conditions in [7] in order to obtain unique periodic solution.

4. Numerical example

In order to verify the feasibility of our results, we examine the existence of periodic solutions for the following stochastic nonlinear system:

$$\begin{cases} dx_1(t) = -a_1(t) + \sum_{j=1}^{2} a_{1j}(t) f(x_1(t)) + \sum_{j=1}^{n} d_{1j}(t) g_1(x_1(t-\gamma)) dt \\ \qquad + \sigma_1(t, x_1(t), x_1(t-\gamma)) d\xi_1(t), \\ dx_2(t) = -a_2(t) + \sum_{j=1}^{2} a_{2j}(t) f(x_2(t)) + \sum_{j=1}^{n} d_{2j}(t) g_2(x_2(t-\gamma)) dt \\ \qquad + \sigma_2(t, x_2(t), x_2(t-\gamma)) d\xi_2(t), \quad t \geqslant 0, \ \gamma \in [-1, 0], \\ x_i(\theta) = \phi_i(\theta), \quad \theta \in [-1, 0], \quad i = 1, 2, \end{cases} \tag{4.1}$$

where

$$a_i(t) = 20 + \sin t, \qquad a_{ij}(t) = d_{ij}(t) = 2 + \sin t,$$

$$f_i(x_i(t)) = -x_i^3(t) - x_i(t), \qquad g_i(x_i(t-\gamma)) = \sin(x_i(t-\gamma)), \quad \sigma_i(t, x_i(t), x_i(t-\gamma)) = \frac{1}{2}\cos t.$$

Obviously, $\sum_{j=1}^{2} a_{1j}(t) = \sum_{j=1}^{2} a_{2j}(t) = 4 + 2\sin t > 0$ and $x_i f_i(x_i) = -x_i^4 - x_i^2 \leqslant 0$, hence assumption ($H_2$) holds. Choosing $k_i(s) = s, \ i = 1, 2$, we have

$$\alpha_{01} = \alpha_{02} = \int_{-1}^{0} dk_1(s) = 1$$

and

$$\gamma_1 = \inf_{t \geqslant 0} \left[a_1(t) - \alpha_{01}^{1/2} \sum_{j=1}^{n} |d_{ij}(t)|/2 - \alpha_{01}/2 - \alpha_{02}/2 \right] = 18 - \frac{\sqrt{6}}{2} > 0.$$

Then we can choose proper q_1, q_2 and initial function ϕ in order to other conditions of Theorem 3.1 satisfy. It follows from Theorem 3.1 that system (4.1) exists a periodic solution.

The numerical simulations of system (4.1) are shown in Fig. 1. Fig. 1 shows the state trajectories of periodic solution $x(t) = (x_1(t), x_2(t))^\top$ for system (4.1).

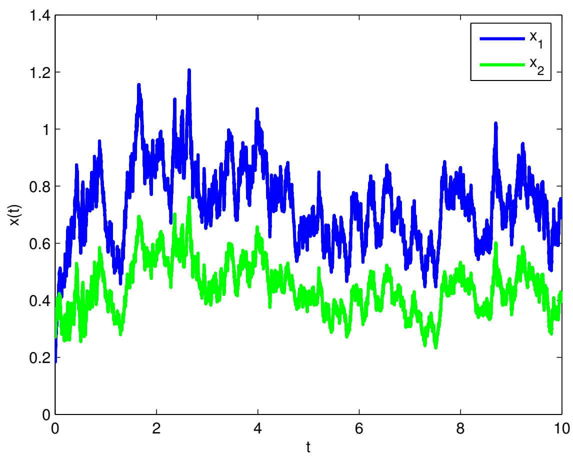

Figure 1: The States' evolution of the system (4.1).

5. Conclusion

In this paper, novel results for existence and uniqueness of periodic solutions for a class of stochastic nonlinear system are obtained. As Kolmanovskii and Myshkis [7] pointed out that the proof of existence theorems of periodic solutions of stochastic equations with delays is similar to the proof of existence theorems of stationary solutions, the modifications arising are due to the fact that the distribution of probabilities of periodic solutions must be invariant with respect to shifts of argument by quantities multiple to the period while for the stationary solutions it is invariant with respect to arbitrary shifts of the argument. That is why in this paper we shall give some coefficient conditions for the existence of periodic solutions. At last, we provide a numerical example to illustrate the effectiveness of the theoretical results.

Acknowledgment

The authors would like to thank the editors and the anonymous referees for their valuable comments and suggestions to improve the presentation of this paper. The first author is supported by author Natural Science Foundation of Jiangsu High Education Institutions of China (Grant No. 17KJB110001). The second author is supported by the National Science Foundation (DMS # 1737861).

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