Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 274–287
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Existence of nonoscillatory solutions to third-order neutral functional dynamic equations on time scales

Yang-Cong Qiu[a], Haixia Wang[b], Cuimei Jiang[c,d], Tongxing Li[e,f,*]

[a]School of Humanities and Social Science, Shunde Polytechnic, Foshan, Guangdong 528333, P. R. China.

[b]School of Economics, Ocean University of China, Qingdao, Shandong 266100, P. R. China.

[c]School of Science, Qilu University of Technology, Jinan, Shandong 250353, P. R. China.

[d]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, P. R. China.

[e]School of Information Science and Engineering, Linyi University, Linyi, Shandong 276005, P. R. China.

[f]School of Control Science and Engineering, Shandong University, Jinan, Shandong 250061, P. R. China.

Communicated by M. Bohner

Abstract

By employing Krasnoselskii's fixed point theorem, we establish the existence of nonoscillatory solutions to a class of third-order neutral functional dynamic equations on time scales. In addition, the significance of the results is illustrated by three examples.

Keywords: Nonoscillatory solution, neutral dynamic equation, third-order, time scale.

2010 MSC: 34K11, 34N05, 39A10, 39A13.

1. Introduction

Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{T}$ be a time scale which is a nonempty closed subset of $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. More details on time scale theory can be found in [1, 4, 7, 8, 15, 16]. Analysis of the oscillatory and nonoscillatory behavior of solutions to various classes of third-order dynamic equations has always attracted interest of researchers; see, for instance, [2, 3, 5, 6, 13, 14, 18, 22]. We remark that there has been some research achievement about the existence of oscillatory and nonoscillatory solutions to neutral dynamic equations on time scales; see the papers [10, 12, 17, 19–21, 23, 24] and the references cited therein.

Some significative results for existence of nonoscillatory solutions to neutral functional differential equations were given in [9, 11]. Afterwards, some open problems were presented in the paper by Mathsen et al. [19]. Zhu and Wang [24] discussed the existence of nonoscillatory solutions to a first-order nonlinear

*Corresponding author
Email address: litongx2007@163.com (Tongxing Li)

doi: 10.22436/jnsa.011.02.09

Received: 2017-09-18 *Revised:* 2017-10-19 *Accepted:* 2018-01-11

neutral dynamic equation

$$[x(t)+p(t)x(g(t))]^{\Delta}+f(t,x(h(t)))=0$$

on a time scale $\mathbb{T}$. Gao and Wang [12] investigated a second-order nonlinear neutral dynamic equation

$$\left[r(t)(x(t)+p(t)x(g(t)))^{\Delta}\right]^{\Delta}+f(t,x(h(t)))=0 \tag{1.1}$$

under the condition $\int_{t_0}^{\infty} 1/r(t)\Delta t<\infty$, whereas Deng and Wang [10] studied the same problem of (1.1) under another condition $\int_{t_0}^{\infty} 1/r(t)\Delta t=\infty$. Inspired by [10], Qiu [20] considered a third-order nonlinear neutral functional dynamic equation

$$\left(r_1(t)\left(r_2(t)\left(x(t)+p(t)x(g(t))\right)^{\Delta}\right)^{\Delta}\right)^{\Delta}+f(t,x(h(t)))=0 \tag{1.2}$$

assuming that $\int_{t_0}^{\infty} 1/r_1(t)\Delta t=\int_{t_0}^{\infty} 1/r_2(t)\Delta t=\infty$. Qiu and Wang [21] studied (1.2) in the case where

$$\int_{t_0}^{\infty}\frac{\Delta t}{r_1(t)}<\infty \quad \text{and} \quad \int_{t_0}^{\infty}\frac{\Delta t}{r_2(t)}<\infty.$$

As a matter of fact, the study of dynamic behaviors of the nonoscillatory solutions to (1.2) is difficult because of the diverse cases of convergence or divergence of the integrals $\int_{t_0}^{\infty} 1/r_1(t)\Delta t$ and $\int_{t_0}^{\infty} 1/r_2(t)\Delta t$. Qiu et al. [23] analyzed (1.2) under the assumptions that

$$\int_{t_0}^{\infty}\frac{\Delta t}{r_1(t)}=\infty \quad \text{and} \quad \int_{t_0}^{\infty}\frac{\Delta t}{r_2(t)}=M_0<\infty.$$

In this paper, we further consider (1.2) on a time scale $\mathbb{T}$ satisfying $\sup\mathbb{T}=\infty$, where $t\in[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\cap\mathbb{T}$ with $t_0\in\mathbb{T}$. The motivation originates from [10, 12, 20, 23, 24]. We shall establish the existence of nonoscillatory solutions to (1.2) by employing Krasnoselskii's fixed point theorem, and we will give three examples to show the versatility of the results. Throughout this paper, we assume that

(C1) $r_1, r_2\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and there exists a constant $M_0>0$ such that

$$\int_{t_0}^{\infty}\frac{\Delta t}{r_1(t)}=M_0<\infty \quad \text{and} \quad \int_{t_0}^{\infty}\frac{\Delta t}{r_2(t)}=\infty;$$

(C2) $p\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and there exists a constant p_0 with $|p_0|<1$ such that $\lim_{t\to\infty}p(t)=p_0$;

(C3) $g,h\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $g(t)\leqslant t$, $\lim_{t\to\infty}g(t)=\lim_{t\to\infty}h(t)=\infty$, and

$$\lim_{t\to\infty}\frac{R(g(t))}{R(t)}=\eta\in(0,1],$$

where

$$R(t)=1+\int_{t_0}^{t}\frac{\Delta s}{r_2(s)};$$

if $p_0\in(-1,0]$, then there exists a sequence $\{c_k\}_{k\geqslant 0}$ such that $\lim_{k\to\infty}c_k=\infty$ and $g(c_{k+1})=c_k$;

(C4) $f\in C([t_0,\infty)_{\mathbb{T}}\times\mathbb{R},\mathbb{R})$, $f(t,x)$ is nondecreasing in x, and $xf(t,x)>0$ for $x\neq 0$.

Definition 1.1. A solution x of (1.2) is termed eventually positive (or eventually negative) if there exists a $c\in\mathbb{T}$ such that $x(t)>0$ (or $x(t)<0$) for all $t\geqslant c$ in $\mathbb{T}$. A solution x of (1.2) is called nonoscillatory if it is either eventually positive or eventually negative; otherwise, it is oscillatory.

2. Auxiliary results

Let $C([T_0,\infty)_{\mathbb{T}},\mathbb{R})$ denote all continuous functions mapping $[T_0,\infty)_{\mathbb{T}}$ into $\mathbb{R}$. For $\lambda=0,1$, define

$$BC_\lambda[T_0,\infty)_{\mathbb{T}}=\left\{x : x\in C([T_0,\infty)_{\mathbb{T}},\mathbb{R}) \text{ and } \sup_{t\in[T_0,\infty)_{\mathbb{T}}}\left|\frac{x(t)}{R^{2\lambda}(t)}\right|<\infty\right\}. \tag{2.1}$$

Endowing $BC_\lambda[T_0,\infty)_{\mathbb{T}}$ with the norm $\|x\|_\lambda=\sup_{t\in[T_0,\infty)_{\mathbb{T}}}\left|x(t)/R^{2\lambda}(t)\right|$, it is clear that $(BC_\lambda[T_0,\infty)_{\mathbb{T}},\|\cdot\|_\lambda)$ is a Banach space.

Lemma 2.1 ([24, Lemma 4]). *Assume that $X\subseteq BC_\lambda[T_0,\infty)_{\mathbb{T}}$ is bounded and uniformly Cauchy. Furthermore, suppose that X is equi-continuous on $[T_0,T_1]_{\mathbb{T}}$ for any $T_1\in[T_0,\infty)_{\mathbb{T}}$. Then X is relatively compact.*

Lemma 2.2 ([9, Krasnoselskii's fixed point theorem]). *Suppose that X is a Banach space and Ω is a bounded, convex, and closed subset of X. Assume further that there exist two operators $U,S:\Omega\to X$ such that*

(i) *$Ux+Sy\in\Omega$ for all $x,y\in\Omega$;*
(ii) *U is a contraction mapping;*
(iii) *S is completely continuous.*

Then $U+S$ has a fixed point in Ω.

Without loss of generality, we mainly consider eventually positive solutions of (1.2) in the sequel. Letting

$$z(t)=x(t)+p(t)x(g(t)), \tag{2.2}$$

we have the following lemma; its proof is similar to that of [10, Lemma 2.3], and so is omitted.

Lemma 2.3. *Assume that x is an eventually positive solution of (1.2) and $\lim_{t\to\infty}z(t)/R^\lambda(t)=a$ for $\lambda=0,1$.*

(i) *If a is finite, then*

$$\lim_{t\to\infty}\frac{x(t)}{R^\lambda(t)}=\frac{a}{1+p_0\eta^\lambda}.$$

(ii) *If a is infinite, then x/R^λ is unbounded, or*

$$\limsup_{t\to\infty}\frac{x(t)}{R^\lambda(t)}=\infty.$$

Let S denote the set of all eventually positive solutions of (1.2) and

$$A(\alpha,\beta)=\left\{x\in S:\lim_{t\to\infty}x(t)=\alpha,\quad \lim_{t\to\infty}\frac{x(t)}{R(t)}=\beta\right\}.$$

The following theorem is established for a classification scheme of eventually positive solutions to (1.2).

Theorem 2.4. *If x is an eventually positive solution of (1.2), then x belongs to $A(0,0)$, $A(b,0)$, $A(\infty,b)$ for some positive constant b, or x is infinite with $\lim_{t\to\infty}x(t)/R(t)=0$.*

Proof. Suppose that x is an eventually positive solution of (1.2). From (C2) and (C3), there exist a $t_1\in[t_0,\infty)_{\mathbb{T}}$ and $|p_0|<p_1<1$ such that $x(t)>0$, $x(g(t))>0$, $x(h(t))>0$, and $|p(t)|\leqslant p_1$ for $t\in[t_1,\infty)_{\mathbb{T}}$. By (1.2) and (C4), it follows that for $t\in[t_1,\infty)_{\mathbb{T}}$,

$$\left(r_1(t)\left(r_2(t)z^\Delta(t)\right)^\Delta\right)^\Delta=-f(t,x(h(t)))<0.$$

Hence, $r_1(t)\left(r_2(t)z^\Delta(t)\right)^\Delta$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. Then there are two cases to be considered.

Case 1. $r_1\left(r_2 z^{\Delta}\right)^{\Delta}$ and $\left(r_2 z^{\Delta}\right)^{\Delta}$ are eventually negative. It implies that there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $r_2(t)z^{\Delta}(t)$ is strictly decreasing on $[t_2, \infty)_{\mathbb{T}}$. We claim that

$$\lim_{t\to\infty} r_2(t)z^{\Delta}(t) = L_1, \tag{2.3}$$

where $0 \leqslant L_1 < \infty$. If not, then there exist a constant $c < 0$ and a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $r_2(t)z^{\Delta}(t) \leqslant c$ for $t \in [t_3, \infty)_{\mathbb{T}}$, which means that

$$z^{\Delta}(t) \leqslant \frac{c}{r_2(t)}, \quad t \in [t_3, \infty)_{\mathbb{T}}. \tag{2.4}$$

Letting t be replaced by s and integrating (2.4) from t_3 to t, $t \in [\sigma(t_3), \infty)_{\mathbb{T}}$, we obtain

$$z(t) \leqslant z(t_3) + c\int_{t_3}^{t} \frac{\Delta s}{r_2(s)}.$$

Letting $t \to \infty$, by (C1) we have $z(t) \to -\infty$. From (2.2), it follows that $p_0 \in (-1, 0]$, and there exists a $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $z(t) < 0$ or

$$x(t) < -p(t)x(g(t)) \leqslant p_1 x(g(t)), \quad t \in [t_4, \infty)_{\mathbb{T}}.$$

By (C3), we can choose some positive integer k_0 such that $c_k \in [t_4, \infty)_{\mathbb{T}}$ for all $k \geqslant k_0$. Then for any $k \geqslant k_0 + 1$, we have

$$x(c_k) < p_1 x(g(c_k)) = p_1 x(c_{k-1}) < p_1^2 x(g(c_{k-1})) = p_1^2 x(c_{k-2}) < \cdots < p_1^{k-k_0} x(g(c_{k_0+1})) = p_1^{k-k_0} x(c_{k_0}).$$

This inequality implies that $\lim_{k\to\infty} x(c_k) = 0$. It follows from (2.2) that $\lim_{k\to\infty} z(c_k) = 0$, which contradicts $\lim_{t\to\infty} z(t) = -\infty$. So (2.3) holds, and we conclude that $r_2 z^{\Delta}$ and z^{Δ} are eventually positive.

If $\lim_{t\to\infty} r_2(t)z^{\Delta}(t) = b$ for some positive constant b, then for $t \in [t_2, \infty)_{\mathbb{T}}$, we have $r_2(t)z^{\Delta}(t) > b$, or

$$z^{\Delta}(t) > \frac{b}{r_2(t)}. \tag{2.5}$$

Substituting s for t and integrating (2.5) from t_2 to t, $t \in [\sigma(t_2), \infty)_{\mathbb{T}}$, we obtain

$$z(t) \geqslant z(t_2) + b\int_{t_2}^{t} \frac{\Delta s}{r_2(s)}.$$

Letting $t \to \infty$, by (C1) we have $z(t) \to \infty$.

Suppose that $\lim_{t\to\infty} r_2(t)z^{\Delta}(t) = 0$. Since z^{Δ} is eventually positive, there exists a $t_5 \in [t_2, \infty)_{\mathbb{T}}$ such that $z(t)$ is strictly increasing on $[t_5, \infty)_{\mathbb{T}}$. Assume that $\lim_{t\to\infty} z(t) < 0$. Similarly, it will cause a contradiction as before. Therefore,

$$\lim_{t\to\infty} z(t) = L_0,$$

where $0 \leqslant L_0 \leqslant \infty$.

Case 2. $r_1\left(r_2 z^{\Delta}\right)^{\Delta}$ and $\left(r_2 z^{\Delta}\right)^{\Delta}$ are eventually positive. It implies that there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $r_2(t)z^{\Delta}(t)$ is strictly increasing on $[t_2, \infty)_{\mathbb{T}}$. Then we also have two cases: $r_2 z^{\Delta}$ is either eventually positive or eventually negative.

Assume first that $r_2 z^{\Delta}$ is eventually positive. Since $r_1(t)\left(r_2(t)z^{\Delta}(t)\right)^{\Delta}$ is eventually positive and strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$, there exist a constant $M > 0$ and a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $r_1(t)\left(r_2(t)z^{\Delta}(t)\right)^{\Delta} \leqslant M$ or

$$\left(r_2(t)z^{\Delta}(t)\right)^{\Delta} \leqslant \frac{M}{r_1(t)} \tag{2.6}$$

for $t \in [t_3, \infty)_{\mathbb{T}}$. Letting t be replaced by s and integrating (2.6) from t_3 to t, $t \in [\sigma(t_3), \infty)_{\mathbb{T}}$, we obtain

$$r_2(t)z^{\Delta}(t) \leqslant r_2(t_3)z^{\Delta}(t_3) + M\int_{t_3}^{t} \frac{\Delta s}{r_1(s)} < r_2(t_3)z^{\Delta}(t_3) + M \cdot M_0,$$

which implies that $r_2 z^{\Delta}$ is upper bounded. Hence, $\lim_{t\to\infty} r_2(t)z^{\Delta}(t) = b$ for some positive constant b. Then, there exists a $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $r_2(t)z^{\Delta}(t) \geqslant b/2$ or

$$z^{\Delta}(t) \geqslant \frac{b}{2r_2(t)} \tag{2.7}$$

for $t \in [t_4, \infty)_{\mathbb{T}}$. Letting t be replaced by s and integrating (2.7) from t_4 to t, $t \in [\sigma(t_4), \infty)_{\mathbb{T}}$, we get

$$z(t) \geqslant z(t_4) + \frac{b}{2}\int_{t_4}^{t} \frac{\Delta s}{r_2(s)}.$$

Letting $t \to \infty$, by (C1) we have $z(t) \to \infty$.

Assume now that $r_2 z^{\Delta}$ is eventually negative. Since $r_2(t)z^{\Delta}(t)$ is strictly increasing on $[t_2, \infty)_{\mathbb{T}}$, we have

$$-\infty < \lim_{t\to\infty} r_2(t)z^{\Delta}(t) \leqslant 0.$$

Suppose that there exists a constant $d < 0$ such that $\lim_{t\to\infty} r_2(t)z^{\Delta}(t) = d$. Then we have $r_2(t)z^{\Delta}(t) < d$ for $t \in [t_2, \infty)_{\mathbb{T}}$. Similarly, it will cause a contradiction as before. Hence, $\lim_{t\to\infty} r_2(t)z^{\Delta}(t) = 0$ and we have $r_2(t)z^{\Delta}(t) < 0$ and $z^{\Delta}(t) < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. We claim that $0 \leqslant \lim_{t\to\infty} z(t) = L_0 < \infty$. If $-\infty \leqslant L_0 < 0$, then it will also cause a contradiction as before. Therefore, $L_0 = b$ for some positive constant b, or $L_0 = 0$.

It follows from L'Hôpital's rule (see [7, Theorem 1.120]) and (2.3) that

$$\lim_{t\to\infty} r_2(t)z^{\Delta}(t) = \lim_{t\to\infty} \frac{z(t)}{R(t)} = L_1.$$

To sum up, by Lemma 2.3, we see that x belongs to $A(0,0)$, $A(b,0)$, $A(\infty, b)$ for some positive constant b, or x is infinite with $\lim_{t\to\infty} x(t)/R(t) = 0$. The proof is complete. □

3. Main results

In this section, by employing Krasnoselskii's fixed point theorem, we establish the existence criteria for each type of eventually positive solutions to (1.2).

Theorem 3.1. *Equation* (1.2) *has an eventually positive solution in* $A(\infty, b)$ *if and only if there exists some constant* $K > 0$ *such that*

$$\int_{t_0}^{\infty}\int_{t_0}^{s} \frac{f(u, KR(h(u)))}{r_1(s)} \Delta u \Delta s < \infty, \tag{3.1}$$

where b *is a positive constant.*

Proof. Suppose that x is an eventually positive solution to (1.2) in $A(\infty, b)$, that is,

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{R(t)} = b. \tag{3.2}$$

Assume that $\lim_{t\to\infty} z(t) < \infty$. By Lemma 2.3, we have $\lim_{t\to\infty} x(t) < \infty$, which contradicts (3.2). Then we have

$$\lim_{t\to\infty} z(t) = \infty, \quad \lim_{t\to\infty} \frac{z(t)}{R(t)} = (1 + p_0\eta)b,$$

and there exists a $T_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, and $x(h(t)) \geqslant bR(h(t))/2$ for $t \in [T_1, \infty)_{\mathbb{T}}$. Integrating (1.2) from T_1 to s, $s \in [\sigma(T_1), \infty)_{\mathbb{T}}$, we obtain

$$r_1(s)\left(r_2(s)z^{\Delta}(s)\right)^{\Delta} - r_1(T_1)\left(r_2(T_1)z^{\Delta}(T_1)\right)^{\Delta} = -\int_{T_1}^{s} f(u, x(h(u)))\Delta u$$

or

$$\left(r_2(s)z^{\Delta}(s)\right)^{\Delta} = \frac{r_1(T_1)\left(r_2(T_1)z^{\Delta}(T_1)\right)^{\Delta}}{r_1(s)} - \frac{\int_{T_1}^{s} f(u, x(h(u)))\Delta u}{r_1(s)}. \tag{3.3}$$

Integrating (3.3) from T_1 to ν, $\nu \in [\sigma(T_1), \infty)_{\mathbb{T}}$, we obtain

$$r_2(\nu)z^{\Delta}(\nu) - r_2(T_1)z^{\Delta}(T_1) = r_1(T_1)\left(r_2(T_1)z^{\Delta}(T_1)\right)^{\Delta}\int_{T_1}^{\nu}\frac{1}{r_1(s)}\Delta s - \int_{T_1}^{\nu}\int_{T_1}^{s}\frac{f(u, x(h(u)))}{r_1(s)}\Delta u\Delta s.$$

Letting $\nu \to \infty$, we have

$$\int_{T_1}^{\infty}\int_{T_1}^{s}\frac{f(u, x(h(u)))}{r_1(s)}\Delta u\Delta s < \infty.$$

In view of (C4), it follows that

$$\int_{T_1}^{\infty}\int_{T_1}^{s}\frac{f(u, bR(h(u))/2)}{r_1(s)}\Delta u\Delta s \leqslant \int_{T_1}^{\infty}\int_{T_1}^{s}\frac{f(u, x(h(u)))}{r_1(s)}\Delta u\Delta s < \infty,$$

which means that (3.1) holds.

On the other hand, suppose that there exists some constant $K > 0$ such that (3.1) holds. There will be two cases to be considered.

Case (i). $0 \leqslant p_0 < 1$. When $p_0 > 0$, take p_1 such that $p_0 < p_1 < (1+4p_0)/5 < 1$. Choose a sufficiently large $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$p(t) > 0, \quad \frac{5p_1 - 1}{4} \leqslant p(t) \leqslant p_1 < 1, \quad p(t)\frac{R(g(t))}{R(t)} \geqslant \frac{5p_1-1}{4}\eta, \quad t \in [T_0, \infty)_{\mathbb{T}},$$

$$\int_{T_0}^{\infty}\int_{T_0}^{s}\frac{f(u, KR(h(u)))}{r_1(s)}\Delta u\Delta s \leqslant \frac{(1-p_1\eta)K}{8}. \tag{3.4}$$

When $p_0 = 0$, choose p_1 such that $|p(t)| \leqslant p_1 \leqslant 1/13$ for $t \in [T_0, \infty)_{\mathbb{T}}$. From (C3), there exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \geqslant T_0$ and $h(t) \geqslant T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define $BC_1[T_0, \infty)_{\mathbb{T}}$ as in (2.1) with $\lambda = 1$, and let

$$\Omega_1 = \left\{x \in BC_1[T_0, \infty)_{\mathbb{T}} : \frac{K}{2}R(t) \leqslant x(t) \leqslant KR(t)\right\}.$$

It is easy to prove that Ω_1 is a bounded, convex, and closed subset of $BC_1[T_0, \infty)_{\mathbb{T}}$. Define two operators U_1 and S_1: $\Omega_1 \to BC_1[T_0, \infty)_{\mathbb{T}}$ as follows

$$(U_1x)(t) = \begin{cases} 3Kp_1\eta R(t)/4 - p(T_1)x(g(T_1))R(t)/R(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ 3Kp_1\eta R(t)/4 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases}$$

$$(S_1x)(t) = \begin{cases} 3KR(t)/4, & t \in [T_0, T_1)_{\mathbb{T}}, \\ 3KR(t)/4 + \int_{T_1}^{t}\int_{\nu}^{\infty}\int_{T_1}^{s} f(u, x(h(u)))/(r_1(s)r_2(\nu))\Delta u\Delta s\Delta\nu, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases} \tag{3.5}$$

Similarly to the proofs of [10, Theorem 2.5] and [20, Theorem 3.1], it is not difficult to prove that U_1 and S_1 satisfy the conditions in Lemma 2.2. Hence, there exists an $x \in \Omega_1$ such that $(U_1 + S_1)x = x$. For $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$x(t) = \frac{3(1+p_1\eta)K}{4}R(t) - p(t)x(g(t)) + \int_{T_1}^{t}\int_{\nu}^{\infty}\int_{T_1}^{s}\frac{f(u, x(h(u)))}{r_1(s)r_2(\nu)}\Delta u\Delta s\Delta\nu.$$

Since
$$\int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v \leqslant \int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s}\frac{f(u,KR(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v$$
and
$$\lim_{t\to\infty}\frac{1}{R(t)}\int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s}\frac{f(u,KR(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v = \lim_{t\to\infty}\int_{t}^{\infty}\int_{T_1}^{s}\frac{f(u,KR(h(u)))}{r_1(s)}\Delta u\Delta s = 0,$$
we have
$$\lim_{t\to\infty}\frac{z(t)}{R(t)} = \frac{3(1+p_1\eta)K}{4} \quad\text{and}\quad \lim_{t\to\infty}\frac{x(t)}{R(t)} = \frac{3(1+p_1\eta)K}{4(1+p_0\eta)} > 0.$$
Moreover, it is obvious that $\lim_{t\to\infty} x(t) = \infty$.

Case (ii). $-1 < p_0 < 0$. Take p_1 such that $-p_0 < p_1 < (1-4p_0)/5 < 1$. Choose a sufficiently large $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that (3.4) holds and
$$p(t) < 0,\quad \frac{5p_1-1}{4} \leqslant -p(t) \leqslant p_1 < 1,\quad -p(t)\frac{R(g(t))}{R(t)} \geqslant \frac{5p_1-1}{4}\eta,\quad t\in[T_0,\infty)_{\mathbb{T}}.$$
There also exists a $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that $g(t) \geqslant T_0$ and $h(t) \geqslant T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Define S_1 as in (3.5) and U_1' on Ω_1 as follows
$$(U_1'x)(t) = \begin{cases} -3Kp_1\eta R(t)/4 - p(T_1)x(g(T_1))R(t)/R(T_1), & t\in[T_0,T_1)_{\mathbb{T}},\\ -3Kp_1\eta R(t)/4 - p(t)x(g(t)), & t\in[T_1,\infty)_{\mathbb{T}}.\end{cases}$$
The remainder of the proof is similar to that of the case where $0 \leqslant p_0 < 1$ and we omit it here. By Lemma 2.2, there exists an $x \in \Omega_1$ such that $(U_1'+S_1)x = x$. For $t \in [T_1,\infty)_{\mathbb{T}}$, we have
$$x(t) = \frac{3(1-p_1\eta)K}{4}R(t) - p(t)x(g(t)) + \int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v.$$
Letting $t\to\infty$, we conclude that
$$\lim_{t\to\infty}\frac{z(t)}{R(t)} = \frac{3(1-p_1\eta)K}{4},\quad \lim_{t\to\infty}\frac{x(t)}{R(t)} = \frac{3(1-p_1\eta)K}{4(1+p_0\eta)} > 0,\quad\text{and}\quad \lim_{t\to\infty}x(t) = \infty.$$
The proof is complete. □

Next, assume that the condition
$$\int_{t_0}^{\infty}\int_{v}^{\infty}\frac{1}{r_1(s)r_2(v)}\Delta s\Delta v < \infty \tag{3.6}$$
holds. Then we will have Theorems 3.2 and 3.3.

Theorem 3.2. *Assume that (3.6) holds. Then (1.2) has an eventually positive solution in $A(b,0)$ if and only if there exists some constant $K>0$ such that*
$$\int_{t_0}^{\infty}\int_{v}^{\infty}\int_{t_0}^{s}\frac{f(u,K)}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v < \infty, \tag{3.7}$$
where b is a positive constant.

Proof. Suppose that x is an eventually positive solution to (1.2) in $A(b,0)$. Then
$$\lim_{t\to\infty}z(t) = (1+p_0)b,\quad \lim_{t\to\infty}r_2(t)z^{\Delta}(t) = \lim_{t\to\infty}\frac{z(t)}{R(t)} = 0,$$
and there exists a $T_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) \geqslant b/2$, $x(g(t)) \geqslant b/2$, and $x(h(t)) \geqslant b/2$ for $t \in [T_1,\infty)_{\mathbb{T}}$.

Integrating (1.2) from T_1 to s, $s \in [\sigma(T_1), \infty)_{\mathbb{T}}$, we arrive at (3.3). Integrating (3.3) from T_2 ($T_2 \in [T_1, \infty)_{\mathbb{T}}$) to ν, $\nu \in [\sigma(T_2), \infty)_{\mathbb{T}}$, we obtain

$$r_2(\nu)z^{\Delta}(\nu) - r_2(T_2)z^{\Delta}(T_2) = r_1(T_1)\left(r_2(T_1)z^{\Delta}(T_1)\right)^{\Delta}\int_{T_2}^{\nu}\frac{1}{r_1(s)}\Delta s - \int_{T_2}^{\nu}\int_{T_1}^{s}\frac{f(u, x(h(u)))}{r_1(s)}\Delta u\Delta s.$$

Letting $\nu \to \infty$, we have

$$-r_2(T_2)z^{\Delta}(T_2) = r_1(T_1)\left(r_2(T_1)z^{\Delta}(T_1)\right)^{\Delta}\int_{T_2}^{\infty}\frac{1}{r_1(s)}\Delta s - \int_{T_2}^{\infty}\int_{T_1}^{s}\frac{f(u, x(h(u)))}{r_1(s)}\Delta u\Delta s, \tag{3.8}$$

which implies that

$$\int_{T_2}^{\infty}\int_{T_1}^{s}\frac{f(u, x(h(u)))}{r_1(s)}\Delta u\Delta s < \infty.$$

In view of (C4), since $x(h(t)) \geqslant b/2$ for $t \in [T_1, \infty)_{\mathbb{T}}$, it follows that

$$\int_{T_2}^{\infty}\int_{T_1}^{s}\frac{f(u, b/2)}{r_1(s)}\Delta u\Delta s < \infty,$$

which means that there exists a constant $K > 0$ such that

$$\int_{t_0}^{\infty}\int_{t_0}^{s}\frac{f(u, K)}{r_1(s)}\Delta u\Delta s < \infty. \tag{3.9}$$

Substituting ν for T_2 in (3.8), we have

$$z^{\Delta}(\nu) = -\frac{r_1(T_1)\left(r_2(T_1)z^{\Delta}(T_1)\right)^{\Delta}}{r_2(\nu)}\int_{\nu}^{\infty}\frac{1}{r_1(s)}\Delta s + \frac{1}{r_2(\nu)}\int_{\nu}^{\infty}\int_{T_1}^{s}\frac{f(u, x(h(u)))}{r_1(s)}\Delta u\Delta s. \tag{3.10}$$

Integrating (3.10) from T_3 ($T_3 \in [T_2, \infty)_{\mathbb{T}}$) to t, $t \in [\sigma(T_3), \infty)_{\mathbb{T}}$, we deduce that

$$z(t) - z(T_3) = -r_1(T_1)\left(r_2(T_1)z^{\Delta}(T_1)\right)^{\Delta}\int_{T_3}^{t}\int_{\nu}^{\infty}\frac{1}{r_1(s)r_2(\nu)}\Delta s\Delta\nu + \int_{T_3}^{t}\int_{\nu}^{\infty}\int_{T_1}^{s}\frac{f(u, x(h(u)))}{r_1(s)r_2(\nu)}\Delta u\Delta s\Delta\nu. \tag{3.11}$$

Letting $t \to \infty$, by (3.6) we get

$$\int_{T_3}^{\infty}\int_{\nu}^{\infty}\int_{T_1}^{s}\frac{f(u, x(h(u)))}{r_1(s)r_2(\nu)}\Delta u\Delta s\Delta\nu < \infty.$$

By (C4), it follows that

$$\int_{T_3}^{\infty}\int_{\nu}^{\infty}\int_{T_1}^{s}\frac{f(u, b/2)}{r_1(s)r_2(\nu)}\Delta u\Delta s\Delta\nu < \infty,$$

which means that (3.7) holds. It is obvious that (3.7) yields (3.9).

Suppose that there exists some constant $K > 0$ such that (3.7) holds. Then, we also consider two cases.

Case (i). $0 \leqslant p_0 < 1$. When $p_0 > 0$, take p_1 as in the proof of Theorem 3.1. Choose a sufficiently large $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$p(t) > 0, \quad \frac{5p_1 - 1}{4} \leqslant p(t) \leqslant p_1 < 1, \quad t \in [T_0, \infty)_{\mathbb{T}}, \quad \int_{T_0}^{\infty}\int_{\nu}^{\infty}\int_{T_0}^{s}\frac{f(u, K)}{r_1(s)r_2(\nu)}\Delta u\Delta s\Delta\nu \leqslant \frac{(1 - p_1)K}{8}.$$

When $p_0 = 0$, choose p_1 such that $|p(t)| \leqslant p_1 \leqslant 1/13$ for $t \in [T_0, \infty)_{\mathbb{T}}$. From (C3), there exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \geqslant T_0$ and $h(t) \geqslant T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define $BC_0[T_0,\infty)_{\mathbb{T}}$ as in (2.1) with $\lambda = 0$, and let

$$\Omega_2 = \left\{ x \in BC_0[T_0,\infty)_{\mathbb{T}} : \frac{K}{2} \leqslant x(t) \leqslant K \right\}. \tag{3.12}$$

Similarly, Ω_2 is a bounded, convex, and closed subset of $BC_0[T_0,\infty)_{\mathbb{T}}$. Define the operators U_2 and S_2: $\Omega_2 \to BC_0[T_0,\infty)_{\mathbb{T}}$ as follows

$$(U_2x)(t) = \begin{cases} (U_2x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3Kp_1/4 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}, \end{cases}$$
$$(S_2x)(t) = \begin{cases} 3K/4, & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3K/4 + \int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s} f(u,x(h(u)))/(r_1(s)r_2(v))\Delta u\Delta s\Delta v, & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases} \tag{3.13}$$

Then, U_2 and S_2 satisfy the conditions in Lemma 2.2. The proof is similar to those of [10, Theorem 2.5] and [20, Theorem 3.1], so we also omit it here. By Lemma 2.2, there exists an $x \in \Omega_2$ such that $(U_2+S_2)x = x$. For $t \in [T_1,\infty)_{\mathbb{T}}$, we have

$$x(t) = \frac{3(1+p_1)K}{4} - p(t)x(g(t)) + \int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s} \frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v. \tag{3.14}$$

Since

$$0 < \int_{T_1}^{\infty}\int_{v}^{\infty}\int_{T_1}^{s} \frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v \leqslant \int_{T_1}^{\infty}\int_{v}^{\infty}\int_{T_1}^{s} \frac{f(u,K)}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v < \infty,$$

letting $t \to \infty$ in (3.14), we have $0 < \lim_{t\to\infty} z(t) < \infty$, which yields

$$\lim_{t\to\infty} x(t) = b \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{R(t)} = 0,$$

where b is a positive constant.

Case (ii). $-1 < p_0 < 0$. Introduce $BC_0[T_0,\infty)_{\mathbb{T}}$ and its subset Ω_2 as in (3.12). Define S_2 as in (3.13) and U_2' on Ω_2 as follows

$$(U_2'x)(t) = \begin{cases} (U_2'x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ -3Kp_1/4 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

The following proof is also similar to that of the case as above and so is omitted. By Lemma 2.2, there exists an $x \in \Omega_2$ such that $(U_2'+S_2)x = x$. For $t \in [T_1,\infty)_{\mathbb{T}}$, we have

$$x(t) = \frac{3(1-p_1)K}{4} - p(t)x(g(t)) + \int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s} \frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v.$$

Similarly, we conclude that

$$\lim_{t\to\infty} x(t) = b \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{R(t)} = 0,$$

where b is a positive constant.

The proof is complete. □

Theorem 3.3. *Assume that (3.6) holds. If (1.2) has an eventually positive solution in $A(\infty,0)$, then*

$$\int_{t_0}^{\infty}\int_{t_0}^{s} \frac{f(u,3/4)}{r_1(s)}\Delta u\Delta s < \infty, \quad \int_{t_0}^{\infty}\int_{v}^{\infty}\int_{t_0}^{s} \frac{f(u,R(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v = \infty. \tag{3.15}$$

If there exists a constant $M > 0$ such that $|p(t)R(t)| \leqslant M$ for $t \in [t_0,\infty)_{\mathbb{T}}$, and

$$\int_{t_0}^{\infty}\int_{t_0}^{s} \frac{f(u,R(h(u)))}{r_1(s)}\Delta u\Delta s < \infty, \quad \int_{t_0}^{\infty}\int_{v}^{\infty}\int_{t_0}^{s} \frac{f(u,M+3/4)}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v = \infty, \tag{3.16}$$

then (1.2) has an eventually positive solution in $A(\infty,0)$.

Proof. Suppose that x is an eventually positive solution to (1.2) in $A(\infty, 0)$. Similarly, we have

$$\lim_{t\to\infty} z(t) = \infty, \quad \lim_{t\to\infty} r_2(t)z^{\Delta}(t) = \lim_{t\to\infty} \frac{z(t)}{R(t)} = 0,$$

and there exists a $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that $3/4 \leqslant x(t) \leqslant R(t)$ for $t \in [T_0, \infty)_{\mathbb{T}}$. From (C3), there exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \geqslant T_0$ and $h(t) \geqslant T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 3.2, we arrive at (3.8), which implies that

$$\int_{T_2}^{\infty}\int_{T_1}^{s} \frac{f(u, 3/4)}{r_1(s)} \Delta u \Delta s < \infty$$

due to (C4) and $x(h(t)) \geqslant 3/4$ for $t \in [T_1, \infty)_{\mathbb{T}}$. Then, continuing the proof as in Theorem 3.2, we arrive at (3.11). Letting $t \to \infty$, we have

$$\int_{T_3}^{\infty}\int_{v}^{\infty}\int_{T_1}^{s} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v = \infty.$$

By (C4), since $x(h(t)) \leqslant R(h(t))$ for $t \in [T_1, \infty)_{\mathbb{T}}$, it follows that

$$\int_{T_3}^{\infty}\int_{v}^{\infty}\int_{T_1}^{s} \frac{f(u, R(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v = \infty,$$

which means that (3.15) holds.

Conversely, if there exists a constant $M > 0$ such that $|p(t)R(t)| \leqslant M$ for $t \in [t_0, \infty)_{\mathbb{T}}$, and (3.16) holds, then $\lim_{t\to\infty} p(t) = 0$. Choose a sufficiently large $T_0 \in [t_0, \infty)_{\mathbb{T}}$ and $0 < p_1 < 1$ such that

$$|p(t)| \leqslant p_1 < 1, \quad 2M + \frac{3}{2} \leqslant \frac{1}{4}R(t), \quad t \in [T_0, \infty)_{\mathbb{T}}, \quad \int_{T_0}^{\infty}\int_{T_0}^{s} \frac{f(u, R(h(u)))}{r_1(s)} \Delta u \Delta s \leqslant \frac{1-p_1}{8}.$$

From (C3), there exists a $T_1 \in (T_0, \infty)_{\mathbb{T}}$ such that $g(t) \geqslant T_0$ and $h(t) \geqslant T_0$ for $t \in [T_1, \infty)_{\mathbb{T}}$.

Define $BC_1[T_0, \infty)_{\mathbb{T}}$ as in (2.1) with $\lambda = 1$, and let

$$\Omega_3 = \left\{ x \in BC_1[T_0, \infty)_{\mathbb{T}} : M + \frac{3}{4} \leqslant x(t) \leqslant R(t) \right\}.$$

Then, Ω_3 is a bounded, convex, and closed subset of $BC_1[T_0, \infty)_{\mathbb{T}}$. Define the operators U_3 and S_3: $\Omega_3 \to BC_1[T_0, \infty)_{\mathbb{T}}$ as follows

$$(U_3x)(t) = \begin{cases} M + 3/4 - p(T_1)x(g(T_1))R(t)/R(T_1), & t \in [T_0, T_1)_{\mathbb{T}}, \\ M + 3/4 - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases}$$

$$(S_3x)(t) = \begin{cases} M + 3/4, & t \in [T_0, T_1)_{\mathbb{T}}, \\ M + 3/4 + \int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s} f(u, x(h(u)))/(r_1(s)r_2(v)) \Delta u \Delta s \Delta v, & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Similarly, U_3 and S_3 satisfy the conditions in Lemma 2.2. The proof is similar to those of Theorems 3.1 and 3.2, and thus is omitted. Then, there exists an $x \in \Omega_3$ such that $(U_3 + S_3)x = x$. For $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$x(t) = 2M + \frac{3}{2} - p(t)x(g(t)) + \int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s} \frac{f(u, x(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v.$$

Since

$$\lim_{t\to\infty} \frac{1}{R(t)} \int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s} \frac{f(u, R(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v = \lim_{t\to\infty} \int_{t}^{\infty}\int_{T_1}^{s} \frac{f(u, R(h(u)))}{r_1(s)} \Delta u \Delta s = 0$$

and

$$\lim_{t\to\infty}\int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s}\frac{f(u,x(h(u)))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v \geqslant \lim_{t\to\infty}\int_{T_1}^{t}\int_{v}^{\infty}\int_{T_1}^{s}\frac{f(u,M+3/4)}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v=\infty,$$

it follows that

$$\lim_{t\to\infty} z(t)=\infty, \quad \lim_{t\to\infty}\frac{z(t)}{R(t)}=0.$$

Since $|p(t)x(g(t))|\leqslant|p(t)R(t)|\leqslant M$, by Lemma 2.3, we deduce that

$$\lim_{t\to\infty} x(t)=\infty, \quad \lim_{t\to\infty}\frac{x(t)}{R(t)}=0.$$

The proof is complete. □

Remark 3.4. It is not difficult to see that assumption (3.6) can be deleted in the sufficiency of the proofs of Theorems 3.2 and 3.3. Therefore, we have the following corollaries, respectively.

Corollary 3.5. *If there exists some constant $K>0$ such that (3.7) holds, then (1.2) has an eventually positive solution in $A(b,0)$, where b is a positive constant.*

Corollary 3.6. *If there exists a constant $M>0$ such that $|p(t)R(t)|\leqslant M$ for $t\in[t_0,\infty)_{\mathbb{T}}$, and (3.16) holds, then (1.2) has an eventually positive solution in $A(\infty,0)$.*

When p is eventually nonnegative, we have the following theorem. The proof is similar to that of [20, Theorem 3.5] and hence is omitted.

Theorem 3.7. *If there exist a constant $K>0$ and a sufficiently large $T\in[t_0,\infty)_{\mathbb{T}}$ such that for $t\in[T,\infty)_{\mathbb{T}}$,*

$$0\leqslant p(t)\leqslant Kg(t)e^{-t}, \qquad \int_{t}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty}\frac{f\left(u,e^{-h(u)}\right)}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v\geqslant(K+1)e^{-t},$$

and

$$\int_{t}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty}\frac{f(u,1/h(u))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v\leqslant\frac{1}{t},$$

then (1.2) has an eventually positive solution in $A(0,0)$.

When p is eventually negative, we have another result. The proof is similar to that of [12, Theorem 3] and thus is omitted.

Theorem 3.8. *If there exists a sufficiently large $T\in[t_0,\infty)_{\mathbb{T}}$ such that for $t\in[T,\infty)_{\mathbb{T}}$,*

$$p(t)e^{-g(t)}\leqslant -e^{-t} \tag{3.17}$$

and

$$\int_{t}^{\infty}\int_{v}^{\infty}\int_{s}^{\infty}\frac{f(u,1/h(u))}{r_1(s)r_2(v)}\Delta u\Delta s\Delta v\leqslant\frac{1}{t}+\frac{p(t)}{g(t)}, \tag{3.18}$$

then (1.2) has an eventually positive solution in $A(0,0)$.

4. Examples

In this section, the applications of our results will be shown in three examples. The first example is given to demonstrate Theorems 3.1-3.3.

Example 4.1. Let $\mathbb{T}=\bigcup_{n=0}^{\infty}[3^n,2\cdot 3^n]$. For $t\in[3,\infty)_{\mathbb{T}}$, consider

$$\left(t^5\left(\frac{1}{t}\left(x(t)+\frac{1}{4t^2}x\left(\frac{t}{3}\right)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta}+\frac{x(t)}{1+t^2}=0. \tag{4.1}$$

Here, $r_1(t)=t^5$, $r_2(t)=1/t$, $p(t)=1/(4t^2)$, $g(t)=t/3$, $h(t)=t$, and $f(t,x)=x/(1+t^2)$. It is obvious that the coefficients of (4.1) satisfy (C1)-(C4) and (3.6), and by (C3) we have

$$R(t) = 1 + \int_3^t s\Delta s \leqslant 1 + \frac{1}{2}t^2 - \frac{9}{2} < t^2.$$

Therefore,

$$\int_3^\infty \int_3^s \frac{f(u, R(h(u)))}{r_1(s)} \Delta u \Delta s < \int_3^\infty \int_3^s \frac{u^2}{(1+u^2)s^5} \Delta u \Delta s < \int_3^\infty \int_3^s \frac{1}{s^5} \Delta u \Delta s < \int_3^\infty \frac{1}{s^4} \Delta s < \infty,$$

$$\int_3^\infty \int_v^\infty \int_3^s \frac{f(u,1)}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v = \int_3^\infty \int_v^\infty \int_3^s \frac{v}{(1+u^2)s^5} \Delta u \Delta s \Delta v < \int_3^\infty \int_v^\infty \int_3^s \frac{v}{s^5} \Delta u \Delta s \Delta v < \int_3^\infty \int_v^\infty \frac{v}{s^4} \Delta s \Delta v < \infty,$$

and

$$\begin{aligned} \int_3^\infty \int_v^\infty \int_3^s \frac{f(u, R(h(u)))}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v &\leqslant \int_3^\infty \int_v^\infty \int_3^s \frac{u^2 v}{(1+u^2)s^5} \Delta u \Delta s \Delta v \\ &< \int_3^\infty \int_v^\infty \int_3^s \frac{v}{s^5} \Delta u \Delta s \Delta v < \int_3^\infty \int_v^\infty \frac{v}{s^4} \Delta s \Delta v < \infty. \end{aligned}$$

By Theorems 3.1-3.3, we see that (4.1) has eventually positive solutions $x_1 \in A(\infty, b_1)$ and $x_2 \in A(b_2, 0)$, where b_1, b_2 are positive constants, but it has no eventually positive solutions in $A(\infty, 0)$.

Then, we give the second example to demonstrate Theorem 3.3 (or Corollary 3.6).

Example 4.2. For any time scale $\mathbb{T}$ which satisfies that $t/2$, $\sqrt[3]{t} \in \mathbb{T}$ for any $t \in [t_0, \infty)_\mathbb{T}$ with $t_0 \geqslant 1$ and $\int_{t_0}^\infty t^{-\lambda} \Delta t < \infty$ for $\lambda > 1$, consider

$$\left(t^3 \left(\frac{1}{t} \left(x(t) + \frac{1}{4t^2} x\left(\frac{t}{2}\right) \right)^\Delta \right)^\Delta \right)^\Delta + x(\sqrt[3]{t}) = 0. \tag{4.2}$$

Here, $r_1(t) = t^3$, $r_2(t) = 1/t$, $p(t) = 1/(4t^2)$, $g(t) = t/2$, $h(t) = \sqrt[3]{t}$, and $f(t,x) = x$. Obviously, the coefficients of (4.2) satisfy (C1)-(C4) and (3.6), and by (C3) we have

$$R(t) = 1 + \int_{t_0}^t s\Delta s \leqslant 1 + \frac{1}{2}t^2 - \frac{1}{2}t_0^2 \leqslant t^2.$$

Therefore,

$$|p(t)R(t)| \leqslant \frac{1}{4},$$

$$\int_{t_0}^\infty \int_{t_0}^s \frac{f(u, R(h(u)))}{r_1(s)} \Delta u \Delta s \leqslant \int_{t_0}^\infty \int_{t_0}^s \frac{u^{2/3}}{s^3} \Delta u \Delta s < \frac{3}{5} \int_{t_0}^\infty \frac{s^{5/3}}{s^3} \Delta s = \frac{3}{5} \int_{t_0}^\infty s^{-4/3} \Delta s < \infty,$$

and

$$\begin{aligned} \int_{t_0}^\infty \int_v^\infty \int_{t_0}^s \frac{f(u, 1/4 + 3/4)}{r_1(s)r_2(v)} \Delta u \Delta s \Delta v &= \int_{t_0}^\infty \int_v^\infty \int_{t_0}^s \frac{v}{s^3} \Delta u \Delta s \Delta v \\ &> \int_{2t_0}^\infty \int_v^\infty \frac{(s-t_0)v}{s^3} \Delta s \Delta v \geqslant \frac{1}{2} \int_{2t_0}^\infty \int_v^\infty \frac{v}{s^2} \Delta s \Delta v \geqslant \frac{1}{2} \int_{2t_0}^\infty \Delta v = \infty. \end{aligned}$$

It follows that (4.2) has an eventually positive solution $x_0 \in A(\infty, 0)$ in terms of Theorem 3.3 (or Corollary 3.6). Furthermore, by Theorem 2.4, we can see that $((x_0(t) + x_0(t/2)/(4t^2))^\Delta/t)^\Delta$ is eventually negative, and $(x_0(t) + x_0(t/2)/(4t^2))^\Delta/t$ is eventually strictly decreasing.

The third example illustrates Theorem 3.8.

Example 4.3. Let $\mathbb{T} = [0, \infty)_{\mathbb{R}}$. For $t \in [t_0, \infty)_{\mathbb{T}}$ with $t_0 \geqslant 1$, consider

$$\left(e^t\left(e^{-2t}\left(x(t) - \frac{t-1}{2t}x(t-1)\right)'\right)'\right)' + e^{-t}x(e^t) = 0. \tag{4.3}$$

Here, $r_1(t) = e^t$, $r_2(t) = e^{-2t}$, $p(t) = -(t-1)/(2t)$, $g(t) = t-1$, $h(t) = e^t$, and $f(t, x) = e^{-t}x$. Similarly, the coefficients of (4.3) satisfy (C1)-(C4), and we have

$$\frac{1}{t} + \frac{p(t)}{g(t)} = \frac{1}{t} - \frac{1}{2t} = \frac{1}{2t}, \quad p(t)e^{-g(t)} = -\frac{t-1}{2t}e^{-(t-1)} = -\left(\frac{1}{2} - \frac{1}{2t}\right)e \cdot e^{-t},$$

and

$$\begin{aligned}\int_t^\infty \int_v^\infty \int_s^\infty \frac{f(u, 1/h(u))}{r_1(s)r_2(v)}\Delta u \Delta s \Delta v &= \int_t^\infty \int_v^\infty \int_s^\infty \frac{e^{-2u}}{e^s \cdot e^{-2v}} du ds dv \\ &= \frac{1}{2}\int_t^\infty \int_v^\infty \frac{e^{-3s}}{e^{-2v}} ds dv = \frac{1}{6}\int_t^\infty e^{-v} dv = \frac{1}{6}e^{-t}.\end{aligned}$$

There exists a sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ such that for $t \in [T, \infty)_{\mathbb{T}}$, the conditions (3.17) and (3.18) hold. By Theorem 3.8, we see that (4.3) has an eventually positive solution $x \in A(0, 0)$.

Acknowledgment

The authors express their sincere gratitude to the editors and two anonymous referees for careful reading of the manuscript and valuable suggestions that helped to improve the paper. This project is supported by the National Natural Science Foundation of P. R. China (Grant Nos. 11671406, 61503171, and 61403061), China Postdoctoral Science Foundation (Grant No. 2015M582091), Program of Cultivation for Outstanding Young Scholars Sponsored by Guangdong Province (Grant No. ZX03240302), Natural Science Foundation of Shandong Province (Grant No. ZR2016JL021), Program of Cultivation for Young Scholars Sponsored by Shunde Polytechnic (Grant No. 2015-KJZX080), Doctoral Scientific Research Foundation of Linyi University (Grant No. LYDX2015BS001), and Applied Mathematics Enhancement Program of Linyi University.

References

[1] R. P. Agarwal, M. Bohner, *Basic calculus on time scales and some of its applications*, Results Math., **35** (1999), 3–22. 1

[2] R. P. Agarwal, M. Bohner, T.-X. Li, C.-H. Zhang, *Hille and Nehari type criteria for third-order delay dynamic equations*, J. Difference Equ. Appl., **19** (2013), 1563–1579. 1

[3] R. P. Agarwal, M. Bohner, T.-X. Li, C.-H. Zhang, *A Philos-type theorem for third-order nonlinear retarded dynamic equations*, Appl. Math. Comput., **249** (2014), 527–531. 1

[4] R. P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math., **141** (2002), 1–26. 1

[5] R. P. Agarwal, M. Bohner, S.-H. Tang, T.-X. Li, C.-H. Zhang, *Oscillation and asymptotic behavior of third-order nonlinear retarded dynamic equations*, Appl. Math. Comput., **219** (2012), 3600–3609. 1

[6] E. Akın, T. S. Hassan, *Comparison criteria for third order functional dynamic equations with mixed nonlinearities*, Appl. Math. Comput., **268** (2015), 169–185. 1

[7] M. Bohner, A. Peterson, *Dynamic equations on time scales, An introduction with applications*, Birkhäuser Boston, Boston, (2001). 1, 2

[8] M. Bohner, A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser Boston, Boston, (2003). 1

[9] Y.-S. Chen, *Existence of nonoscillatory solutions of nth order neutral delay differential equations*, Funkcial. Ekvac., **35** (1992), 557–570. 1, 2.2

[10] X.-H. Deng, Q.-R. Wang, *Nonoscillatory solutions to second-order neutral functional dynamic equations on time scales*, Commun. Appl. Anal., **18** (2014), 261–280. 1, 1, 1, 2, 3, 3

[11] L. H. Erbe, Q.-K. Kong, B. G. Zhang, *Oscillation theory for functional differential equations*, Marcel Dekker, New York, (1995). 1

[12] J. Gao, Q.-R. Wang, *Existence of nonoscillatory solutions to second-order nonlinear neutral dynamic equations on time scales*, Rocky Mountain J. Math., **43** (2013), 1521–1535. 1, 1, 3

[13] T. S. Hassan, *Oscillation of third order nonlinear delay dynamic equations on time scales*, Math. Comput. Modelling, **49** (2009), 1573–1586. 1

[14] T. S. Hassan, R. P. Agarwal, W. W. Mohammed, *Oscillation criteria for third-order functional half-linear dynamic equations*, Adv. Difference Equ., **2017** (2017), 28 pages. 1

[15] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. thesis, Universität Würzburg, Würzburg, Germany, (1988). 1

[16] S. Hilger, *Analysis on measure chains–a unified approach to continuous and discrete calculus*, Results Math., **18** (1990), 18–56. 1

[17] T.-X. Li, Z.-L. Han, S.-R. Sun, D.-W. Yang, *Existence of nonoscillatory solutions to second-order neutral delay dynamic equations on time scales*, Adv. Difference Equ., **2009** (2009), 10 pages. 1

[18] T.-X. Li, Z.-L. Han, S.-R. Sun, Y.-G. Zhao, *Oscillation results for third order nonlinear delay dynamic equations on time scales*, Bull. Malays. Math. Sci. Soc., **34** (2011), 639–648. 1

[19] R. M. Mathsen, Q.-R. Wang, H.-W. Wu, *Oscillation for neutral dynamic functional equations on time scales*, J. Difference Equ. Appl., **10** (2004), 651–659. 1

[20] Y.-C. Qiu, *Nonoscillatory solutions to third-order neutral dynamic equations on time scales*, Adv. Difference Equ., **2014** (2014), 25 pages. 1, 1, 3, 3, 3

[21] Y.-C. Qiu, Q.-R. Wang, *Existence of nonoscillatory solutions to higher-order nonlinear neutral dynamic equations on time scales*, Bull. Malays. Math. Sci. Soc., **2016** (2016), 18 pages. 1, 1

[22] Y.-C. Qiu, A. Zada, H.-Y. Qin, T.-X. Li, *Oscillation criteria for nonlinear third-order neutral dynamic equations with damping on time scales*, J. Funct. Spaces, **2017** (2017), 8 pages. 1

[23] Y.-C. Qiu, A. Zada, S.-H. Tang, T.-X. Li, *Existence of nonoscillatory solutions to nonlinear third-order neutral dynamic equations on time scales*, J. Nonlinear Sci. Appl., **10** (2017), 4352–4363. 1, 1

[24] Z.-Q. Zhu, Q.-R. Wang, *Existence of nonoscillatory solutions to neutral dynamic equations on time scales*, J. Math. Anal. Appl., **335** (2007), 751–762. 1, 1, 2.1