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Topological coincidence principles

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Abstract

In this paper a number of general coincidence principles are presented for set valued maps defined on subsets of completely regular topological spaces.

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1. Introduction

In this paper we present coincidence principles for multimaps. We present two approaches. The first approach is based on the new notion of Φ-essential and d-Φ-essential maps (see [1, 3–5]) and the second approach is based on the notion of extendability (see [2]). The arguments presented are based on a Urysohn type lemma and homotopy type arguments.

2. Continuation principles

Let E be a completely regular topological space and U an open subset of E.

We consider classes **A** and **B** of maps.

Definition 2.1. We say $F \in A(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U}, E)$ (respectively $F \in \mathbf{B}(\overline{U}, E)$); here 2^E denotes the family of nonempty subsets of E.

In this section we fix a $\Phi \in B(\overline{U}, E)$.

Definition 2.2. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

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Definition 2.3. Let E be a completely regular (respectively normal) topological space, and U an open subset of E. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(., \eta(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and $\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively closed); here $H_t(x) = H(x,t)$.

Definition 2.4. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

Theorem 2.5. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, and let $F \in A_{\partial U}(\overline{U}, E)$ be Φ-essential in $A_{\partial U}(\overline{U}, E)$. Suppose there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(., \eta(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1]$, $H_0 = F$, and $\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively closed). In addition assume*

$$\begin{cases} \text{if } \mu : \overline{U} \to [0,1] \text{ is any continuous map with } \mu(\partial U) = 0, \text{ then} \\ \{x \in \overline{U} : \emptyset \neq \Phi(x) \cap H(x, t\mu(x)) \text{ for some } t \in [0,1]\} \text{ is closed.} \end{cases} \tag{2.1}$$

Then there exists $x \in U$ with $\Phi(x) \cap H_1(x) \neq \emptyset$; here $H_t(x) = H(x,t)$.

Proof. Let

$$D = \{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}.$$

Notice $D \neq \emptyset$ since F is Φ-essential in $A_{\partial U}(\overline{U}, E)$ (note from (2.1) that $F \cong F$ in $A_{\partial U}(\overline{U}, E)$). Also D is compact (respectively closed) if E is a completely regular (respectively normal) topological space. Note $D \cap \partial U = \emptyset$ (note $H_0 = F$ so for $t = 0$ we have $\Phi(x) \cap H_0(x) = \emptyset$ for $x \in \partial U$ since $F \in A_{\partial U}(\overline{U}, E)$). Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J : \overline{U} \to 2^E$ by $J(x) = H(x, \mu(x))$. Note $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ (note if $x \in \partial U$ then $J(x) = H_0(x) = F(x)$ and $J(x) \cap \Phi(x) = F(x) \cap \Phi(x) = \emptyset$). We now claim

$$J \cong F \text{ in } A_{\partial U}(\overline{U}, E). \tag{2.2}$$

If the claim is true then since F is Φ-essential in $A_{\partial U}(\overline{U}, E)$ then there exists a $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$ (i.e., $H_{\mu(x)}(x) \cap \Phi(x) \neq \emptyset$), and thus $x \in D$ so $\mu(x) = 1$ and as a result $H_1(x) \cap \Phi(x) \neq \emptyset$.

It remains to show (2.2). Let $Q : \overline{U} \times [0,1] \to 2^E$ be given by $Q(x,t) = H(x, t\mu(x))$. Note $Q(., \eta(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ and (see (2.1) and Definition 2.3)

$$\{x \in \overline{U} : \emptyset \neq \Phi(x) \cap Q(x,t) = \Phi(x) \cap H(x, t\mu(x)) \text{ for some } t \in [0,1]\}$$

is compact (respectively closed). Note $Q_0 = F$ and $Q_1 = J$. Finally if there exists a $t \in [0,1]$ and $x \in \partial U$ with $\Phi(x) \cap Q_t(x) \neq \emptyset$ then $\Phi(x) \cap H_{t\mu(x)}(x) \neq \emptyset$ so $x \in D$, and so $\mu(x) = 1$, i.e., $\Phi(x) \cap H_t(x) \neq \emptyset$, a contradiction. Thus (2.2) holds. □

Remark 2.6. Suppose we change Definition 2.4 as follows. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$. The argument above (note (2.2) is not needed) yields the following result. Let E be a completely regular (respectively normal) topological space, U an open subset of E and let $F \in A_{\partial U}(\overline{U}, E)$ be Φ-essential in $A_{\partial U}(\overline{U}, E)$. Suppose there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(., \eta(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1]$, $H_0 = F$ and $\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively closed). Then there exists $x \in U$ with $\Phi(x) \cap H_1(x) \neq \emptyset$; here $H_t(x) = H(x,t)$.

Again we consider the map $F : \overline{U} \to 2^E$. In our (quite abstract) result we will assume that we have a homotopy extension type property (i.e., a $H : E \times [0,1] \to 2^E$ with $H_1|_{\overline{U}} = F$).

Definition 2.7. We say $F \in A(E,E)$ if $F : E \to 2^E$ and $F \in \mathbf{A}(E,E)$.

Definition 2.8. Let E be a completely regular (respectively normal) topological space. If $F, G \in A(E,E)$, then we say $F \cong G$ in $A(E,E)$ if there exists a map $\Lambda : E \times [0,1] \to 2^E$ with $\Lambda(.,\eta(.)) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$, $\Lambda_1 = G$, $\Lambda_0 = F$ (here $\Lambda_t(x) = \Lambda(x,t)$) and $\{x \in E : \Phi(x) \cap \Lambda(x,t) \neq \emptyset$ for some $t \in [0,1]\}$ is compact (respectively closed).

We now <u>fix</u> a $\Phi \in B(E,E)$.

Theorem 2.9. *Let E be a completely regular (respectively normal) topological space and U an open subset of E. Suppose there exists a map $H : E \times [0,1] \to 2^E$ with $H(.,\eta(.)) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$ and with $\Phi(x) \cap H(x,0) = \emptyset$ for $x \in E \backslash U$, and $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, and $\{x \in E : \Phi(x) \cap H(x,t) \neq \emptyset$ for some $t \in [0,1]\}$ is compact (respectively closed). In addition assume the following hold:*

$$\text{for any } J \in A(E,E) \text{ with } J \cong H_0 \text{ in } A(E,E) \text{ there exists } x \in E \text{ with } \Phi(x) \cap J(x) \neq \emptyset, \tag{2.3}$$

$$\{x \in E \backslash U : H_t(x) \cap \Phi(x) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is closed,} \tag{2.4}$$

and

$$\begin{cases} \text{if } \mu : E \to [0,1] \text{ is any continuous map with } \mu(\overline{U}) = 1, \text{ then} \\ \{x \in E : \emptyset \neq \Phi(x) \cap H(x, t\mu(x)) \text{ for some } t \in [0,1]\} \text{ is closed.} \end{cases} \tag{2.5}$$

Then there exists $x \in U$ with $\Phi(x) \cap H_1(x) \neq \emptyset$; here $H_t(x) = H(x,t)$.

Proof. Let

$$D = \{x \in E \backslash U : \Phi(x) \cap H_t(x) \neq \emptyset \text{ for some } t \in [0,1]\}.$$

We consider two cases, as $D \neq \emptyset$ and $D = \emptyset$.

Case (i). $D = \emptyset$.

Then for every $t \in [0,1]$ we have $\Phi(x) \cap H_t(x) = \emptyset$ for $x \in E \backslash U$. Also from $H_1 \cong H_0$ in $A(E,E)$ and (2.3) we know there exists $y \in E$ with $\Phi(y) \cap H_1(y) \neq \emptyset$. Since $\Phi(x) \cap H_1(x) = \emptyset$ for $x \in E \backslash U$ we deduce that $y \in U$, and we are finished.

Case (ii). $D \neq \emptyset$.

Now (note $H_1 \cong H_0$ in $A(E,E)$ and (2.4)) D is compact (respectively closed) and $D \cap \overline{U} \neq \emptyset$ (since $\Phi(x) \cap H_t(x) = \emptyset$ for $x \in \partial U$ and $t \in [0,1]$). Then there exists a continuous map $\mu : E \to [0,1]$ with $\mu(D) = 0$ and $\mu(\overline{U}) = 1$. Define a map $R : E \to 2^E$ by

$$R(x) = H(x, \mu(x)).$$

Now $R \in A(E,E)$. In fact $R \cong H_0$ in $A(E,E)$. To see this let $\Omega : E \times [0,1] \to 2^E$ be given by

$$\Omega(x,t) = H(x, t\mu(x)).$$

Note $\Omega(.,\eta(.)) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$, and (note (2.5) and $H_1 \cong H_0$ in $A(E,E)$),

$$\{x \in E : \Phi(x) \cap \Omega(x,t) = \Phi(x) \cap H(x, t\mu(x)) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact (respectively closed). Also $\Omega_1 = R$ and $\Omega_0 = H_0$.

Now (2.3) guarantees that there exists $x \in E$ with $\Phi(x) \cap R(x) = \Phi(x) \cap H_{\mu(x)}(x) \neq \emptyset$. If $x \in E \backslash U$ then since $x \in D$ we have $\emptyset \neq \Phi(x) \cap H(x, \mu(x)) = \Phi(x) \cap H(x,0)$, a contradiction. Thus $x \in U$ and so $\emptyset \neq \Phi(x) \cap H(x, \mu(x)) = \Phi(x) \cap H(x,1)$. □

Remark 2.10. In Definition 2.8 and in the statement of Theorem 2.9 we could replace, any continuous map $\eta : E \to [0,1]$, with any continuous map $\eta : E \to [0,1]$ with $\eta(\overline{U}) = 1$.

We now show that the ideas in this section can be applied to other natural situations. Let E be a Hausdorff topological vector space (so automatically a completely regular space), Y a topological vector space, and U an open subset of E. Also let $L : \mathrm{dom}(L) \subseteq E \to Y$ be a linear single valued map; here $\mathrm{dom}(L)$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear single valued map with $L + T : \mathrm{dom}(L) \to Y$ a bijection; for convenience we say $T \in H_L(E, Y)$.

Definition 2.11. We say $F \in A(\overline{U}, Y; L, T)$ (respectively $F \in B(\overline{U}, Y; L, T)$) if $F : \overline{U} \to 2^Y$ and $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$ (respectively $(L+T)^{-1}(F+T) \in B(\overline{U}, E)$).

We now fix a $\Phi \in B(\overline{U}, Y; L, T)$.

Definition 2.12. We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Phi + T)(x) = \emptyset$ for $x \in \partial U$.

Definition 2.13. Let $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(.,\eta(.)) + T(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t + T)(x) \cap (L+T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and

$$\left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact; here $H_t(x) = H(x,t)$.

Definition 2.14. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$. We say F is L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ there exists $x \in U$ with $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$.

Theorem 2.15. *Let E be a topological vector space (so automatically completely regular), Y a topological vector space, U an open subset of E, $L : \mathrm{dom}(L) \subseteq E \to Y$ a linear single valued map, and $T \in H_L(E, Y)$. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ be L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$. Suppose there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(.,\eta(.)) + T(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1]$, $H_0 = F$ (here $H_t(x) = H(x,t)$) and $\left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}$ is compact. In addition assume*

$$\begin{cases} \text{if } \mu : \overline{U} \to [0,1] \text{ is any continuous map with } \mu(\partial U) = 0, \text{ then} \\ \{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_{t\mu(x)}+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is closed.} \end{cases}$$

Then there exists $x \in U$ with $(L+T)^{-1}(H_1+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$.

Proof. Let

$$D = \left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

Note $D \neq \emptyset$ (note F is L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$) and D is compact, $D \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J : \overline{U} \to 2^Y$ by $J(x) = H(x, \mu(x))$. Note $J \in A_{\partial U}(\overline{U}, Y; L, T)$ and $J|_{\partial U} = F|_{\partial U}$. Also note $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ (to see this let $Q : \overline{U} \times [0,1] \to 2^Y$ be given by $Q(x,t) = H(x, t\mu(x))$). Now since F is L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ there exists $x \in U$ with $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$ (i.e., $(L+T)^{-1}(H_{\mu(x)}+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$), and thus $x \in D$ so $\mu(x) = 1$ and we are finished. □

Remark 2.16. Suppose we change Definition 2.14 as follows. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$. We say F is L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$. The argument above yields the following result. Let E be a topological vector space, Y a topological vector space, U an open subset of E, $L : \mathrm{dom}(L) \subseteq E \to Y$ a linear single valued map, and $T \in H_L(E, Y)$. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ be L-Φ-essential in $A_{\partial U}(\overline{U}, Y; L, T)$. Suppose there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(.,\eta(.)) + T(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1]$, $H_0 = F$ (here $H_t(x) = H(x,t)$) and $\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset$ for some $t \in [0,1]\}$ is compact. Then there exists $x \in U$ with $(L+T)^{-1}(H_1+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$.

Remark 2.17. If E is a normal topological vector space then the assumption that D (in the proof of Theorem 2.15) is compact, can be replaced by D is closed, in the statement (and proof) of Theorem 2.15 and also the assumption that

$$\left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact, can be replaced by

$$\left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is closed, in Definition 2.13.

Definition 2.18. Let $F : E \to 2^Y$. We say $F \in A(E,Y;L,T)$ if $(L+T)^{-1}(F+T) \in A(E,E)$.

Definition 2.19. If $F, G \in A(E,Y;L,T)$ then we say $F \cong G$ in $A(E,Y;L,T)$ if there exists a map $\Lambda : E \times [0,1] \to 2^Y$ with $(L+T)^{-1}(\Lambda(.,\eta(.))+T) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$, $\Lambda_1 = F$, $\Lambda_0 = G$ (here $\Lambda_t(x) = \Lambda(x,t)$) and

$$\left\{x \in E : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(\Lambda_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact.

We now <u>fix</u> a $\Phi \in B(E,Y;L,T)$.

Theorem 2.20. *Let E be a completely regular topological vector space, Y a topological vector space, U an open subset of E, $L : \mathrm{dom}(L) \subseteq E \to Y$ a linear single valued map, and $T \in H_L(E,Y)$. Suppose there exists a map $H : E \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(.,\eta(.))+T) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$, $(L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_0+T)(x) = \emptyset$ for $x \in E\backslash U$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, and*

$$\left\{x \in E : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact. In addition assume the following conditions holds:

$$\begin{cases} \text{for any } J \in A(E,Y;L,T) \text{ with } J \cong H_0 \text{ in } A(E,Y;L,T) \text{ there exists } x \in E \text{ with} \\ (L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset, \end{cases} \tag{2.6}$$

$$\{x \in E\backslash U : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is closed,} \tag{2.7}$$

and

$$\begin{cases} \text{if } \mu : E \to [0,1] \text{ is any continuous map with } \mu(\overline{U}) = 1, \text{ then} \\ \{x \in E : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_{t\mu(x)}+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is closed.} \end{cases}$$

Then there exists $x \in U$ with $(L+T)^{-1}(H_1+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$; here $H_t(x) = H(x,t)$.

Proof. Let

$$D = \left\{x \in E\backslash U : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

We consider two cases, as $D \neq \emptyset$ and $D = \emptyset$.

Case (i). $D = \emptyset$.

Then for every $t \in [0,1]$ we have $(L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) = \emptyset$ for $x \in E \backslash U$. Also from $H_1 \cong H_0$ in $A(E,Y;L,T)$ and (2.6) we see there exists $y \in E$ with $(L+T)^{-1}(\Phi+T)(y) \cap (L+T)^{-1}(H_1+T)(y) \neq \emptyset$. Since $D = \emptyset$ we see that $y \in U$, and we are finished.

Case (ii). $D \neq \emptyset$.

Now (note $H_1 \cong H_0$ in $A(E,Y;L,T)$ and (2.7)) D is compact, and $D \cap \overline{U} \neq \emptyset$ (since $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for $x \in \partial U$ and $t \in [0,1]$). Then there exists a continuous map $\mu : E \to [0,1]$ with $\mu(D) = 0$ and $\mu(\overline{U}) = 1$. Define a map $R : E \to 2^Y$ by $R(x) = H(x, \mu(x))$. Note $R \in A(E,Y;L,T)$. Also note $R \cong H_0$ in $A(E,Y;L,T)$ (to see this let $\Omega : E \times [0,1] \to 2^E$ be given by $\Omega(x,t) = H(x, t\mu(x))$). Also $\Omega_1 = R$ and $\Omega_0 = H_0$.

Now (2.6) guarantees that there exists $x \in E$ with $\emptyset \neq (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(R+T)(x) = (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_{\mu(x)}+T)(x)$. If $x \in E \backslash U$ then since $x \in D$ we have $\emptyset \neq (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_{\mu(x)}+T)(x) = (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_0+T)(x)$, a contradiction. Thus $x \in U$ and so $\emptyset \neq (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_{\mu(x)}+T)(x) = (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_1+T)(x)$. □

Remark 2.21. In Definition 2.19 and in the statement of Theorem 2.20 we could replace any continuous map $\eta : E \to [0,1]$ with any continuous map $\eta : E \to [0,1]$ with $\eta(\overline{U}) = 1$.

Remark 2.22. There is an analogue of Remark 2.17 (for normal topological vector spaces) in the statement of Theorem 2.20 and in Definition 2.19.

3. Generalized continuation principles

Let E be a completely regular topological space and U an open subset of E. Again we consider classes **A** and **B** of maps.

In this section we fix a $\Phi \in B(\overline{U}, E)$.

For any map $F \in A(\overline{U}, E)$ let $F^\star = I \times F : \overline{U} \to 2^{\overline{U} \times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1}(B) \right\} \cup \{\emptyset\} \to \Omega \tag{3.1}$$

be any map with values in the nonempty set Ω; here $B = \left\{ (x, \Phi(x)) : x \in \overline{U} \right\}$.

Definition 3.1. Let E be a completely regular (respectively normal) topological space, and U an open subset of E. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(.,\eta(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and $\left\{ x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1] \right\}$ is compact (respectively closed); here $H^\star(x,t) = (x, H(x,t))$ and $H_t(x) = H(x,t)$.

Definition 3.2. Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d-Φ-essential if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ we have that $d\left((F^\star)^{-1}(B) \right) = d\left((J^\star)^{-1}(B) \right) \neq d(\emptyset)$.

Remark 3.3. If $F^\star$ is d-Φ-essential then

$$\emptyset \neq (F^\star)^{-1}(B) = \{x \in \overline{U} : F^\star(x) \cap B \neq \emptyset\} = \{x \in \overline{U} : (x, F(x)) \cap (x, \Phi(x)) \neq \emptyset\},$$

and this together with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$ implies that there exists $x \in U$ with $(x, \Phi(x)) \cap F^\star(x) \neq \emptyset$ (i.e., $\Phi(x) \cap F(x) \neq \emptyset$).

Theorem 3.4. *Let* E *be a completely regular (respectively normal) topological space,* U *an open subset of* E*,* $B = \{(x, \Phi(x)) : x \in \overline{U}\}$*,* d *a map defined in* (3.1) *and let* $F \in A_{\partial U}(\overline{U}, E)$ *and* $F^\star$ *be* d-Φ*-essential (here* $F^\star = I \times F$*). Suppose there exists a map* $H : \overline{U} \times [0,1] \to 2^E$ *with* $H(.,\eta(.)) \in A(\overline{U}, E)$ *for any continuous function* $\eta : \overline{U} \to [0,1]$ *with* $\eta(\partial U) = 0$*,* $H_t(x) \cap \Phi(x) = \emptyset$ *for any* $x \in \partial U$ *and* $t \in (0,1]$*,* $H_0 = F$ *and*

$\{x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset$ *for some* $t \in [0,1]\}$ *is compact (respectively closed); here* $H^\star(x,t) = (x, H(x,t))$ *and* $H_t(x) = H(x,t)$. *In addition assume*

$$\begin{cases} \textit{if } \mu : \overline{U} \to [0,1] \textit{ is any continuous map with } \mu(\partial U) = 0, \textit{ then} \\ \{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, t\mu(x))) \neq \emptyset \textit{ for some } t \in [0,1]\} \textit{ is closed.} \end{cases}$$

Let $H_1^\star = I \times H_1$. *Then*

$$d\left((H_1^\star)^{-1}(B)\right) = d\left((F^\star)^{-1}(B)\right) \neq d(\emptyset).$$

Proof. Let

$$D = \{x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\},$$

where $H^\star(x,t) = (x, H(x,t))$. Notice $D \neq \emptyset$ since $F^\star$ is d-Φ-essential. Also D is compact (respectively closed) if E is a completely regular (respectively normal) topological space and $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $R_\mu : \overline{U} \to 2^E$ by $R_\mu(x) = H(x, \mu(x))$ and let $R_\mu^\star = I \times R_\mu$. Note $R_\mu \in A_{\partial U}(\overline{U}, E)$ with $R_\mu|_{\partial U} = F|_{\partial U}$ (note if $x \in \partial U$ then $R_\mu(x) = H_0(x) = F(x)$ and $R_\mu(x) \cap \Phi(x) = F(x) \cap \Phi(x) = \emptyset$).

Next we note since $\mu(D) = 1$ that

$$\left(R_\mu^\star\right)^{-1}(B) = \{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, \mu(x)) \neq \emptyset\} = \{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x,1) \neq \emptyset\} = (H_1^\star)^{-1}(B),$$

so

$$d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d\left((H_1^\star)^{-1}(B)\right). \tag{3.2}$$

Also note $R_\mu \cong F$ in $A_{\partial U}(\overline{U}, E)$ (to see this let $Q : \overline{U} \times [0,1] \to 2^E$ be given by $Q(x,t) = H(x, t\mu(x))$). As a result since $F^\star$ is d-Φ-essential we have $d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d\left((F^\star)^{-1}(B)\right) \neq d(\emptyset)$. This together with (3.2) yields $d\left((H_1^\star)^{-1}(B)\right) = d\left((F^\star)^{-1}(B)\right) \neq d(\emptyset)$. □

Remark 3.5. Suppose we change Definition 3.2 as follows. Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d-Φ-essential if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ we have that $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. The argument above yields the following result. Let E be a completely regular (respectively normal) topological space, U an open subset of E, $B = \{(x, \Phi(x)) : x \in \overline{U}\}$, d a map defined in (3.1) and let $F \in A_{\partial U}(\overline{U}, E)$ and $F^\star$ be d-Φ-essential (here $F^\star = I \times F$). Suppose there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(., \eta(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1]$, $H_0 = F$ and $\{x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively closed); here $H^\star(x,t) = (x, H(x,t))$ and $H_t(x) = H(x,t)$. Then $d\left((H_1^\star)^{-1}(B)\right) = d\left((F^\star)^{-1}(B)\right) \neq d(\emptyset)$.

Remark 3.6. Suppose the following conditions holds (which is common in the literature on topological degree):

$$\begin{cases} \text{if } F, G \in A_{\partial U}(\overline{U}, E) \text{ with } F|_{\partial U} = G|_{\partial U} \text{ and } F \cong G \\ \text{in } A_{\partial U}(\overline{U}, E), \text{ then } d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right). \end{cases} \tag{3.3}$$

Then Definition 3.2 reduces to the following. Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d-Φ-essential if $d\left((F^\star)^{-1}(B)\right) \neq d(\emptyset)$.

Next in this paper we use the notion of extendability to establish new continuation theorems.

Definition 3.7. We say $F \in A(E,E)$ if $F : E \to 2^E$ and $F \in \mathbf{A}(E,E)$.

We now fix a $\Phi \in B(E,E)$.

For any map $F \in A(E,E)$ let $F^\star = I \times F : E \to 2^{E\times E}$, with $I : E \to E$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1}(B) \right\} \cup \{\emptyset\} \to \Omega \tag{3.4}$$

be any map with values in the nonempty set Ω; here $B = \{(x, \Phi(x)) : x \in E\}$. In our applications we will be interested in maps $F : \overline{U} \to 2^E$ so $F^\star = I \times F : \overline{U} \to 2^{\overline{U}\times E}$ and in this case we consider

$$d : \left\{ (F^\star)^{-1}(B_U) \right\} \cup \{\emptyset\} \to \Omega,$$

where $B_U = \{(x, \Phi(x)) : x \in U\}$.

Definition 3.8. If $F, G \in A(E,E)$, then we say $F \cong G$ in $A(E,E)$ if there exists a map $\Lambda : E \times [0,1] \to 2^E$ with $\Lambda(.,\eta(.)) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$, $\Lambda_1 = F$, $\Lambda_0 = G$ (here $\Lambda_t(x) = \Lambda(x,t)$) and $\{x \in E : (x,\Phi(x)) \cap \Lambda^\star(x,t) \neq \emptyset$ for some $t \in [0,1]\}$ is compact (respectively closed); here $\Lambda^\star(x,t) = (x, \Lambda(x,t))$.

Theorem 3.9. *Let E be a completely regular (respectively normal) topological space, U an open subset of E and d a map defined in (3.4). Suppose there exists a map $H : E \times [0,1] \to 2^E$ with $H(.,\eta(.)) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$ and with $\Phi(x) \cap H(x,0) = \emptyset$ for $x \in E\backslash U$, and $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, and*

$$\{x \in E : (x,\Phi(x)) \cap (x, H(x,t)) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact (respectively closed). In addition assume the following hold:

$$\begin{cases} \text{for any } J \in A(E,E) \text{ with } J^\star = I \times J \text{ and } J \cong H_0 \text{ in } A(E,E) \text{ we have that} \\ d\left((J^\star)^{-1}(B)\right) = d\left((H_0^\star)^{-1}(B)\right) \neq d(\emptyset), \end{cases} \tag{3.5}$$

$$\{x \in E\backslash U : (x,\Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is closed,}$$

and

$$\begin{cases} \text{if } \mu : E \to [0,1] \text{ is any continuous map with } \mu(\overline{U}) = 1, \text{ then} \\ \{x \in E : \emptyset \neq (x,\Phi(x)) \cap (x, H(x, t\mu(x))) \text{ for some } t \in [0,1]\} \text{ is closed;} \end{cases} \tag{3.6}$$

here $H_0^\star = I \times H_0$ and $H^\star(x,t) = (x, H(x,t))$. Let $H_1^\star = I \times H_1$. Then we have

$$d\left((H_1^\star)^{-1}(B_U)\right) = d\left((H_0^\star)^{-1}(B_U)\right) \neq d(\emptyset).$$

Proof. Let

$$D = \{x \in E\backslash U : (x,\Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\},$$

where $H^\star(x,t) = (x, H(x,t))$.

We consider two cases, as $D \neq \emptyset$ and $D = \emptyset$.

Case (i). $D = \emptyset$.

Then for every $t \in [0,1]$ we have $\Phi(x) \cap H_t(x) = \emptyset$ for $x \in E\backslash U$. Also from $H_1 \cong H_0$ in $A(E,E)$ and (3.5) we have

$$d\left((H_1^\star)^{-1}(B)\right) = d\left((H_0^\star)^{-1}(B)\right) \neq d(\emptyset). \tag{3.7}$$

Note $\left(H_1^\star\right)^{-1}(B) = \{x \in E : (x,\Phi(x)) \cap (x, H_1(x)) \neq \emptyset\}$. Consider $y \in E$ and $(y,\Phi(y)) \cap H_1^\star(y) \neq \emptyset$. Then $y \in E$ and $\Phi(y) \cap H_1(y) \neq \emptyset$. Now since $D = \emptyset$ we have $y \in U$ and $\Phi(y) \cap H_1(y) \neq \emptyset$ i.e., $y \in U$ and $(y,\Phi(y)) \cap H_1^\star(y) \neq \emptyset$. Consequently $\left(H_1^\star\right)^{-1}(B) \subseteq \left(H_1^\star\right)^{-1}(B_U)$ and on the other hand it is immediate that $\left(H_1^\star\right)^{-1}(B_U) \subseteq \left(H_1^\star\right)^{-1}(B)$. Thus $\left(H_1^\star\right)^{-1}(B) = \left(H_1^\star\right)^{-1}(B_U)$. It is also immediate that $\left(H_0^\star\right)^{-1}(B) = \left(H_0^\star\right)^{-1}(B_U)$.

Thus (3.7) implies $d\left(\left(H_1^\star\right)^{-1}(B_U) = d\left(\left(H_0^\star\right)^{-1}(B_U)\right)\right) \neq d(\emptyset)$, and we are finished.

Case (ii). $D \neq \emptyset$.

Note D is compact (respectively closed) and also note $D \cap \overline{U} \neq \emptyset$ (since $\Phi(x) \cap H_t(x) = \emptyset$ for $x \in \partial U$ and $t \in [0,1]$). Then there exists a continuous map $\mu : E \to [0,1]$ with $\mu(D) = 0$ and $\mu(\overline{U}) = 1$. Define a map $R : E \to 2^E$ by $R(x) = H(x, \mu(x))$. Note $R \in A(E,E)$. In fact $R \cong H_0$ in $A(E,E)$. To see this let $\Lambda : E \times [0,1] \to 2^E$ be given by $\Lambda(x,t) = H(x, t\mu(x))$. Note $\Lambda(., \eta(.)) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$, and (note (3.6) and $H_1 \cong H_0$ in $A(E,E)$),

$$\{x \in E : (x, \Phi(x)) \cap (x, H(x, t\mu(x))) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact (respectively closed). Also $\Lambda_1 = R$ and $\Lambda_0 = H_0$.

Let $R^\star = I \times R$. Now (3.5) guarantees that

$$d\left((R^\star)^{-1}(B)\right) = d\left((H_0^\star)^{-1}(B)\right) \neq d(\emptyset). \tag{3.8}$$

Note $(R^\star)^{-1}(B) = \{x \in E : (x, \Phi(x)) \cap (x, R(x)) \neq \emptyset\}$. Consider $x \in E$ and $(x, \Phi(x)) \cap R^\star(x) \neq \emptyset$. Then $x \in E$ and $\emptyset \neq \Phi(x) \cap R(x) = \Phi(x) \cap H_{\mu(x)}(x)$. If $x \in E \backslash U$ then since $x \in D$ we have $\emptyset \neq \Phi(x) \cap H_{\mu(x)}(x) = \Phi(x) \cap H(x, 0)$, which is a contradiction. Thus $x \in U$ and $\emptyset \neq \Phi(x) \cap R(x)$. Consequently $(R^\star)^{-1}(B) \subseteq (R^\star)^{-1}(B_U)$ and on the other hand it is immediate that $(R^\star)^{-1}(B_U) \subseteq (R^\star)^{-1}(B)$. Thus $(R^\star)^{-1}(B) = (R^\star)^{-1}(B_U)$. Also $\left(H_0^\star\right)^{-1}(B) = \left(H_0^\star\right)^{-1}(B_U)$. Thus (3.8) implies

$$d\left((R^\star)^{-1}(B_U)\right) = d\left((H_0^\star)^{-1}(B_U)\right) \neq d(\emptyset). \tag{3.9}$$

Finally notice (note $\mu(\overline{U}) = 1$) that

$$\begin{aligned} (R^\star)^{-1}(B_U) &= \{x \in U : (x, \Phi(x)) \cap (x, H(x, \mu(x))) \neq \emptyset\} \\ &= \{x \in U : (x, \Phi(x)) \cap (x, H(x, 1)) \neq \emptyset\} = (H_1^\star)^{-1}(B_U), \end{aligned}$$

so from (3.9) we have $d\left((H_1^\star)^{-1}(B_U)\right) = d\left((H_0^\star)^{-1}(B_U)\right) \neq d(\emptyset)$. □

Remark 3.10. In Definition 3.8 and in the statement of Theorem 3.9 we could replace, any continuous map $\eta : E \to [0,1]$, with, any continuous map $\eta : E \to [0,1]$ with $\eta(\overline{U}) = 1$.

Let E be a topological vector space, Y a topological vector space, U an open subset of E, $L : \mathrm{dom}(L) \subseteq E \to Y$ a linear single valued map, and $T \in H_L(E,Y)$.

We now <u>fix</u> a $\Phi \in B(\overline{U}, Y; L, T)$.

For any map $F \in A(\overline{U}, Y; L, T)$ let $F^\star = I \times (L+T)^{-1}(F+T) : \overline{U} \to 2^{\overline{U} \times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{(F^\star)^{-1}(B)\right\} \cup \{\emptyset\} \to \Omega \tag{3.10}$$

be any map with values in the nonempty set Ω; here $B = \left\{(x, (L+T)^{-1}(\Phi+T)(x)) : x \in \overline{U}\right\}$.

Definition 3.11. Let $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(., \eta(.)) + T(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and

$$\left\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact; here $H_t(x) = H(x,t)$ and $H^\star(x, \lambda) = (x, (L+T)^{-1}(H+T)(x, \lambda))$.

Definition 3.12. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d-L-Φ-essential if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^\star = I \times (L+T)^{-1}(J+T)$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ we have that $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$.

Theorem 3.13. *Let* E *be a Hausdorff topological vector space,* Y *a topological vector space,* U *an open subset of* E, $B = \{(x,(L+T)^{-1}(\Phi+T)(x)) : x \in \overline{U}\}$, $L : \mathrm{dom}(L) \subseteq E \to Y$ *a linear single valued map,* $T \in H_L(E,Y)$, d *a map defined in* (3.10)*, and let* $F \in A_{\partial U}(\overline{U},Y;L,T)$ *and* $F^\star$ *be* d-L-Φ*-essential (here* $F^\star = I \times (L+T)^{-1}(F+T)$*). Suppose here exists a map* $H : \overline{U} \times [0,1] \to 2^Y$ *with* $(L+T)^{-1}(H(.,\eta(.))+T(.)) \in A(\overline{U},E)$ *for any continuous function* $\eta : \overline{U} \to [0,1]$ *with* $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ *for any* $x \in \partial U$ *and* $t \in (0,1]$, $H_0 = F$*, and*

$$\{x \in \overline{U} : (x,(L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact; here $H_t(x) = H(x,t)$ *and* $H^\star(x,\lambda) = (x,(L+T)^{-1}(H+T)(x,\lambda))$*. In addition assume*

$$\begin{cases} \text{if } \mu : \overline{U} \to [0,1] \text{ is any continuous map with } \mu(\partial U) = 0, \text{ then} \\ \{x \in \overline{U} : (x,(L+T)^{-1}(\Phi+T)(x)) \cap (x,(L+T)^{-1}(H_{t\mu(x)}+T)(x)) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is closed.} \end{cases}$$

Let $H_1^\star = I \times (L+T)^{-1}(H_1+T)$*. Then*

$$d\left((H_1^\star)^{-1}(B)\right) = d\left((F^\star)^{-1}(B)\right) \neq d(\emptyset).$$

Proof. Let

$$D = \{x \in \overline{U} : (x,(L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\},$$

where $H^\star(x,\lambda) = (x,(L+T)^{-1}(H+T)(x,\lambda))$. Notice $D \neq \emptyset$, D is compact, and $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $R_\mu : \overline{U} \to 2^Y$ by $R_\mu(x) = H(x,\mu(x)) = H_{\mu(x)}(x)$ and let $R_\mu^\star = I \times (L+T)^{-1}(R_\mu+T)$. Note $R_\mu \in A_{\partial U}(\overline{U},Y;L,T)$ with $R_\mu|_{\partial U} = F|_{\partial U}$ (note if $x \in \partial U$ then $R_\mu(x) = H_0(x) = F(x)$ and $R_\mu(x) \cap \Phi(x) = F(x) \cap \Phi(x) = \emptyset$).

Next we note, since $\mu(D) = 1$, that

$$\begin{aligned}(R_\mu^\star)^{-1}(B) &= \{x \in \overline{U} : (x,(L+T)^{-1}(\Phi+T)(x)) \cap (x,(L+T)^{-1}(H_{\mu(x)}+T)(x)) \neq \emptyset\} \\ &= \{x \in \overline{U} : (x,(L+T)^{-1}(\Phi+T)(x)) \cap x,(L+T)^{-1}(H_1+T)(x)) \neq \emptyset\} = (H_1^\star)^{-1}(B)\end{aligned}$$

and so

$$d\left((R_\mu^\star)^{-1}(B)\right) = d\left((H_1^\star)^{-1}(B)\right). \tag{3.11}$$

Also note $R_\mu \cong F$ in $A_{\partial U}(\overline{U},Y;L,T)$ (to see this let $Q : \overline{U} \times [0,1] \to 2^Y$ be given by $Q(x,t) = H(x,t\mu(x))$). As a result since $F^\star$ is d-L-Φ-essential we have $d\left((R_\mu^\star)^{-1}(B)\right) = d\left((F^\star)^{-1}(B)\right) \neq d(\emptyset)$. This together with (3.11) yields $d\left((H_1^\star)^{-1}(B)\right) = d\left((F^\star)^{-1}(B)\right) \neq d(\emptyset)$. □

Remark 3.14. Suppose we change Definition 3.12 as follows. Let $F \in A_{\partial U}(\overline{U},Y;L,T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d-L-Φ-essential if for every map $J \in A_{\partial U}(\overline{U},Y;L,T)$ with $J^\star = I \times (L+T)^{-1}(J+T)$ and $J|_{\partial U} = F|_{\partial U}$ we have that $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. The argument above yields the following result. Let E be a Hausdorff topological vector space, Y a topological vector space, U an open subset of E, $B = \{(x,(L+T)^{-1}(\Phi+T)(x)) : x \in \overline{U}\}$, $L : \mathrm{dom}(L) \subseteq E \to Y$ a linear single valued map, $T \in H_L(E,Y)$, d a map defined in (3.10) and let $F \in A_{\partial U}(\overline{U},Y;L,T)$ and $F^\star$ be d-L-Φ-essential (here $F^\star = I \times (L+T)^{-1}(F+T)$). Suppose here exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(.,\eta(.))+T(.)) \in A(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and

$$\{x \in \overline{U} : (x,(L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact; here $H_t(x) = H(x,t)$ and $H^\star(x,\lambda) = (x,(L+T)^{-1}(H+T)(x,\lambda))$. Let $H_1^\star = I \times (L+T)^{-1}(H_1+T)$. Then $d\left((H_1^\star)^{-1}(B)\right) = d\left((F^\star)^{-1}(B)\right) \neq d(\emptyset)$.

Remark 3.15. Suppose the following condition holds:

$$\begin{cases} \text{if } F, G \in A_{\partial U}(\overline{U}, Y; L, T) \text{ with } F|_{\partial U} = G|_{\partial U} \text{ and } F \cong G \\ \text{in } A_{\partial U}(\overline{U}, Y; L, T) \text{ then } d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right). \end{cases}$$

Then Definition 3.12 reduces to the following. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d-L-Φ-essential if $d\left((F^\star)^{-1}(B)\right) \neq d(\emptyset)$.

Remark 3.16. If E is a normal topological vector space then the assumption that D (in the proof of Theorem 2.15) is compact, can be replaced by D is closed, in the statement (and proof) of Theorem 3.13 and also the assumption that

$$\left\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact, can be replaced by

$$\left\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is closed, in Definition 3.11; here $H^\star(x,\lambda) = (x, (L+T)^{-1}(H+T)(x,\lambda))$.

We now fix a $\Phi \in B(E, Y; L, T)$.

Definition 3.17. Let $F : E \to 2^Y$. We say $F \in A(E, Y; L, T)$ if $(L+T)^{-1}(F+T) \in A(E,E)$.

For any map $F \in A(E, Y; L, T)$ let $F^\star = I \times (L+T)^{-1}(F+T) : E \to 2^{E\times E}$, with $I : E \to E$ given by $I(x) = x$, and let

$$d : \left\{(F^\star)^{-1}(B)\right\} \cup \{\emptyset\} \to \Omega \tag{3.12}$$

be any map with values in the nonempty set Ω; here $B = \left\{(x, (L+T)^{-1}(\Phi+T)(x)) : x \in E\right\}$. In our applications we will be interested in maps $F : \overline{U} \to 2^Y$ so $F^\star = I \times (L+T)^{-1}[F+T] : \overline{U} \to 2^{\overline{U}\times E}$ and in this case we consider

$$d : \left\{(F^\star)^{-1}(B_U)\right\} \cup \{\emptyset\} \to \Omega,$$

where $B_U = \left\{(x, (L+T)^{-1}(\Phi+T)(x)) : x \in U\right\}$.

Definition 3.18. If $F, G \in A(E, Y; L, T)$ then we say $F \cong G$ in $A(E, Y; L, T)$ if there exists a map $\Lambda : E \times [0,1] \to 2^Y$ with $(L+T)^{-1}(\Lambda(.,\eta(.)) + T) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$, $\Lambda_1 = F$ and $\Lambda_0 = G$ (here $\Lambda_t(x) = \Lambda(x,t)$) and

$$\left\{x \in E : (x, (L+T)^{-1}(\Phi+T)(x)) \cap \Lambda^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact; here $\Lambda^\star(x,\lambda) = (x, (L+T)^{-1}(\Lambda+T)(x,\lambda))$.

Theorem 3.19. *Let E be a Hausdorff topological vector space, Y a topological vector space, U an open subset of E, $L : \text{dom}(L) \subseteq E \to Y$ a linear single valued map, $T \in H_L(E,Y)$ and d a map defined in (3.12). Suppose there exists a map $H : E \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(.,\eta(.)) + T) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$, $(L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_0+T)(x) = \emptyset$ for $x \in E\backslash U$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, and*

$$\left\{x \in E : (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \textit{ for some } t \in [0,1]\right\}$$

is compact; here $H^\star(x,\lambda) = (x, (L+T)^{-1}(H+T)(x,\lambda))$. In addition assume the following hold:

$$\begin{cases} \textit{for any } J \in A(E, Y; L, T) \textit{ with } J^\star = I \times (L+T)^{-1}(J+T) \\ \textit{and } \Phi \cong H_0 \textit{ in } A(E, Y; L, T) \textit{ we have that } d\left((J^\star)^{-1}(B)\right) = d\left((H_0^\star)^{-1}(B)\right) \neq d(\emptyset), \end{cases} \tag{3.13}$$

$$\left\{\begin{array}{l}\left\{x \in E\backslash U : (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}\\ \text{is closed}\end{array}\right.$$

and

$$\left\{\begin{array}{l}\text{if } \mu : E \to [0,1] \text{ is any continuous map with } \mu(\overline{U}) = 1, \text{ then}\\ \{x \in \overline{U} : (x, (L+T)^{-1}(\Phi+T)(x)) \cap (x, (L+T)^{-1}(H_{t\mu(x)}+T)(x)) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is closed.}\end{array}\right.$$

Here $H_0^\star = I \times (L+T)^{-1}(H_0+T)$. *Let* $H_1^\star = I \times (L+T)^{-1}(H_1+T)$. *Then we have* $d\left(\left(H_1^\star\right)^{-1}(B_U)\right) = d\left(\left(H_0^\star\right)^{-1}(B_U)\right) \neq d(\emptyset)$.

Proof. Let

$$D = \left\{x \in E\backslash U : (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

We consider two cases, as $D \neq \emptyset$ and $D = \emptyset$.

Case (i). $D = \emptyset$.

Then for every $t \in [0,1]$ we have $(x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset$. Also from $H_1 \cong H_0$ in $A(E,Y;L,T)$ and (3.13) we have

$$d\left(\left(H_1^\star\right)^{-1}(B)\right) = d\left(\left(H_0^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{3.14}$$

Note $\left(H_1^\star\right)^{-1}(B) = \{x \in E : (x, (L+T)^{-1}(\Phi+T)(x)) \cap (x, (L+T)^{-1}(H_1+T)(x)) \neq \emptyset\}$. Consider $y \in E$ and $(y, (L+T)^{-1}(\Phi+T)(y)) \cap H_1^\star(y) \neq \emptyset$. Then $y \in E$ and $\Phi(y) \cap (L+T)^{-1}(H_1+T)(y) \neq \emptyset$. Now since $D = \emptyset$ we have $y \in U$ and $\Phi(y) \cap (L+T)^{-1}(H_1+T)(y) \neq \emptyset$ i.e., $y \in U$ and $(y, (L+T)^{-1}(\Phi+T)(y)) \cap H_1^\star(y) \neq \emptyset$. Consequently $\left(H_1^\star\right)^{-1}(B) \subseteq \left(H_1^\star\right)^{-1}(B_U)$ and on the other hand it is immediate that $\left(H_1^\star\right)^{-1}(B_U) \subseteq \left(H_1^\star\right)^{-1}(B)$. Thus $\left(H_1^\star\right)^{-1}(B) = \left(H_1^\star\right)^{-1}(B_U)$. It is also immediate that $\left(H_0^\star\right)^{-1}(B) = \left(H_0^\star\right)^{-1}(B_U)$.

Thus (3.14) implies $d\left(\left(H_1^\star\right)^{-1}(B_U)\right) = d\left(\left(H_0^\star\right)^{-1}(B_U)\right) \neq d(\emptyset)$, and we are finished.

Case (ii). $D \neq \emptyset$.

Note D is compact and also note $D \cap \overline{U} \neq \emptyset$. Then there exists a continuous map $\mu : E \to [0,1]$ with $\mu(D) = 0$ and $\mu(\overline{U}) = 1$. Define a map $R : E \to 2^Y$ by

$$R(x) = H(x, \mu(x)).$$

Note $R \in A(E,Y;L,T)$. In fact $R \cong H_0$ in $A(E,Y;L,T)$ (to see this let $\Lambda : E \times [0,1] \to 2^Y$ be given by $\Lambda(x,t) = H(x, t\mu(x))$).

Let $R^\star = I \times (L+T)^{-1}(R+T)$. Now (3.13) guarantees that

$$d\left(\left(R^\star\right)^{-1}(B)\right) = d\left(\left(H_0^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{3.15}$$

Note $(R^\star)^{-1}(B) = \{x \in E : (x, (L+T)^{-1}(\Phi+T)(x)) \cap (x, (L+T)^{-1}(R+T)(x)) \neq \emptyset\}$. Consider $x \in E$ and $(x, (L+T)^{-1}(\Phi+T)(x)) \cap R^\star(x) \neq \emptyset$. Then $x \in E$ and $\emptyset \neq (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(R+T)(x) = (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_{\mu(x)}+T)(x)$. If $x \in E\backslash U$ then since $x \in D$ we have $\emptyset \neq (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_{\mu(x)}+T)(x) = (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_0+T)(x)$ which is a contradiction. Thus $x \in U$ and $\emptyset \neq (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(R+T)(x)$. Consequently $(R^\star)^{-1}(B) \subseteq (R^\star)^{-1}(B_U)$ and on the other hand it is immediate that $(R^\star)^{-1}(B_U) \subseteq (R^\star)^{-1}(B)$. Thus $(R^\star)^{-1}(B) = (R^\star)^{-1}(B_U)$. Also $\left(H_0^\star\right)^{-1}(B) = \left(H_0^\star\right)^{-1}(B_U)$. Thus (3.15) implies

$$d\left(\left(R^\star\right)^{-1}(B_U)\right) = d\left(\left(H_0^\star\right)^{-1}(B_U)\right) \neq d(\emptyset). \tag{3.16}$$

Finally notice

$$\begin{aligned}(R^\star)^{-1}(B_U) &= \left\{x \in U : (x,(L+T)^{-1}(\Phi+T)(x)) \cap (x,(L+T)^{-1}(H_{\mu(x)}+T)(x)) \neq \emptyset\right\}\\ &= \left\{x \in U : (x,(L+T)^{-1}(\Phi+T)(x)) \cap (x,(L+T)^{-1}(H_1+T)(x)) \neq \emptyset\right\} = (H_1^\star)^{-1}(B_U),\end{aligned}$$

so from (3.16) we have $d\left((H_1^\star)^{-1}(B_U)\right) = d\left((H_0^\star)^{-1}(B_U)\right) \neq d(\emptyset)$. □

Remark 3.20. In Definition 3.18 and in the statement of Theorem 3.19 we could replace, any continuous map $\eta : E \to [0,1]$, with, any continuous map $\eta : E \to [0,1]$ with $\eta(\overline{U}) = 1$.

Remark 3.21. There is an analogue of Remark 3.16 (for normal topological vector spaces) in the statement of Theorem 3.19 and in Definition 3.18.

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