



Ground states solutions for modified fourth-order elliptic systems with steep well potential



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Communicated by D. Baleanu

Abstract

In this paper, we study the following modified quasilinear fourth-order elliptic systems

$$\begin{cases} \Delta^2 u - \Delta u + (\lambda\alpha(x) + 1)u - \frac{1}{2}\Delta(u^2)u = \frac{p}{p+q}|u|^{p-2}|v|^q u, & \text{in } \mathbb{R}^N, \\ \Delta^2 v - \Delta v + (\lambda\beta(x) + 1)v - \frac{1}{2}\Delta(v^2)v = \frac{q}{p+q}|u|^p|v|^{q-2}v, & \text{in } \mathbb{R}^N, \end{cases}$$

where $\Delta^2 = \Delta(\Delta)$ is the biharmonic operator, $\lambda > 0$, and $2 < p, 2 < q, 4 < p + q < 22^{**}, 2^{**} = \frac{2N}{N-4}$ ($N \leq 5$) (if $N \leq 4, 2^{**} = \infty$) is the critical Sobolev exponent for the embedding $W^{2,2}(\mathbb{R}^N) \hookrightarrow L^{2^{**}}(\mathbb{R}^N)$. Under some appropriate assumptions on $\alpha(x)$ and $\beta(x)$, we obtain that the above problem has nontrivial ground state solutions via the variational methods. We also explore the phenomenon of concentration of solutions.

Keywords: Fourth-order elliptic, variational methods, ground state solutions, concentration.

2010 MSC: 35B09, 35J20.

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1. Introduction and main results

In this paper, we consider the following modified quasilinear fourth-order elliptic systems

$$\begin{cases} \Delta^2 u - \Delta u + (\lambda\alpha(x) + 1)u - \frac{1}{2}\Delta(u^2)u = \frac{p}{p+q}|u|^{p-2}|v|^q u, & \text{in } \mathbb{R}^N, \\ \Delta^2 v - \Delta v + (\lambda\beta(x) + 1)v - \frac{1}{2}\Delta(v^2)v = \frac{q}{p+q}|u|^p|v|^{q-2}v, & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $\Delta^2 = \Delta(\Delta)$ is the biharmonic operator. $\lambda > 0$, and $2 < p, 2 < q, 4 < p + q < 22^{**}, 2^{**} = \frac{2N}{N-4}$ ($N \leq 5$) (if $N \leq 4, 2^{**} = \infty$) is the critical Sobolev exponent for the embedding $W^{2,2}(\mathbb{R}^N) \hookrightarrow L^{2^{**}}(\mathbb{R}^N)$. In the

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doi: [10.22436/jnsa.011.03.01](https://doi.org/10.22436/jnsa.011.03.01)

Received: 2017-11-29 Revised: 2017-12-31 Accepted: 2018-01-07

last several decades, many researchers considered the problem

$$\begin{cases} \Delta^2 u + c\Delta u + V(x)u = f(x, u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \Omega, \end{cases}$$

where Ω is a smooth bounded domain of \mathbb{R}^N . Such equation arises in applications from mathematical physics. For instance, in [12], Lazer and McKenna studied the problem of nonlinear oscillation in a suspension bridge. They presented a mathematical model for the bridge that takes account of the fact that the coupling provided by the stays connecting the suspension cable to the deck of the road bed is fundamentally nonlinear. For more details, we refer the readers to [6, 13, 17] and the references therein.

In recently years, with the aid of variational methods, the existence, nonexistence, and multiplicity results of various solutions for (1.1) have been extensively investigated (see [1–5, 7–11, 14–16, 18–23, 26–32]). For example, in [19], Pimenta and Soares considered the superlinear fourth-order elliptic equation (1.1) and by concentrating at a point in the limit by using a weak version of the Ambrosetti-Rabinowitz condition proved the existence of a family of solutions. Furthermore, in [33], Zhang et al. proved the existence of nontrivial solutions and investigated the concentration phenomenon of solutions on the set $V^{-1}(0)$. But to the best of our knowledge, little researches have been done for (1.1) with the quasilinear term $\int_{\mathbb{R}^N} u\Delta(u^2)dx$, because the appearance of the quasilinear term $\int_{\mathbb{R}^N} u\Delta(u^2)dx$, makes the studies more difficult and interesting.

Motivated by the above facts, the goal of this paper is to consider the ground state solutions for problem (1.1) with steep well potential. Under some natural assumptions for $\alpha(x)$ and $\beta(x)$, we obtain that the problem (1.1) has a nontrivial ground state solutions via the variation methods. Furthermore, we also explore the phenomenon of concentration of solutions.

Before stating our main results, we give the following assumption on $\alpha(x)$ and $\beta(x)$.

- (F₁) $\alpha(x), \beta(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfying $\alpha(x) \geq 0, \beta(x) \geq 0$, and $\Lambda = \text{int}\{x \in \mathbb{R}^N : \alpha(x) = 0, \beta(x) = 0\}$, is nonempty and has smooth boundary with $\bar{\Lambda} = \{x \in \mathbb{R}^N : \alpha(x) = 0, \beta(x) = 0\}$;
(F₂) there exist $M_1, M_2 > 0$ such that

$$m(\{x \in \mathbb{R}^N | \alpha(x) \leq M_1\}) < \infty, \quad \text{and} \quad m(\{x \in \mathbb{R}^N | \beta(x) \leq M_2\}) < \infty,$$

where m denotes the Lebesgue measure in \mathbb{R}^N .

For simplicity, in the rest of the paper, we denote $V_1(x) = \lambda\alpha(x) + 1$, and $V_2(x) = \lambda\beta(x) + 1$. We also make the following assumption

- (F₀) $V_i \in C(\mathbb{R}^N, \mathbb{R})$, and $\inf V_i(x) \geq 1 > 0$, moreover, there is a constant $d_0 > 0$ such that

$$\lim_{|y| \rightarrow \infty} \text{meas}\{x \in \mathbb{R}^N : |x - y| \leq d_0, V_i(x) \leq M\} = 0, \forall M > 0,$$

where $i = 1, 2$ and $\text{meas}(\cdot)$ denotes the Lebesgue measure in \mathbb{R}^N .

This kind of hypotheses was first introduced by Bartsch and Wang [2, 24] in the study of a nonlinear Schrödinger equation. The hypotheses (F₁)-(F₂) imply that $V_i(x)$ represents a potential well whose depth is controlled by λ and $V_i(x)$ is called a steep potential well if λ is large, where $(i = 1, 2)$.

Define

$$N_\lambda := \{(u, v) \in E_\lambda \setminus \{(0, 0)\} : \langle I'_\lambda(u, v), (u, v) \rangle = 0\}.$$

Then, N_λ is a Nehari manifold associated to I_λ . We point out that the assumption (F₂) is very weak even in dealing with the semilinear operator $-\Delta + (\lambda\alpha(x) + 1)I$ on \mathbb{R}^N .

Now we state our main results.

Theorem 1.1. *Under assumptions (F₁)-(F₂), problem (1.1) has a least energy solution, i.e., a nontrivial solution (u_0, v_0) satisfying $I_\lambda(u_0, v_0) = \inf_{N_\lambda} I_\lambda(u, v)$.*

Theorem 1.2. Assume that (u_λ, v_λ) is the ground state solution obtained by Theorem 1.1. Then $(u_\lambda, v_\lambda) \rightarrow (u_0, v_0)$ as $\lambda \rightarrow \infty$, and (u_0, v_0) is the nontrivial ground state solution of

$$\begin{cases} \Delta^2 u - \Delta u + u - \frac{1}{2} \Delta(u^2)u = \frac{p}{p+q} |u|^{p-2} |v|^q u, & \text{in } \mathbb{R}^N, \\ \Delta^2 v - \Delta v + v - \frac{1}{2} \Delta(v^2)v = \frac{q}{p+q} |u|^p |v|^{q-2} v, & \text{in } \mathbb{R}^N. \end{cases} \quad (1.2)$$

Corollary 1.3. Under assumptions (F₁)-(F₂), system (1.1) has at least one nontrivial solution. Moreover if $(u_\lambda, v_\lambda) \rightarrow (u_0, v_0)$ as $\lambda \rightarrow \infty$, and (u_0, v_0) is the nontrivial solution of

$$\begin{cases} \Delta^2 u - \Delta u + u - \frac{1}{2} \Delta(u^2)u = \frac{p}{p+q} |u|^{p-2} |v|^q u, & \text{in } \mathbb{R}^N, \\ \Delta^2 v - \Delta v + v - \frac{1}{2} \Delta(v^2)v = \frac{q}{p+q} |u|^p |v|^{q-2} v, & \text{in } \mathbb{R}^N. \end{cases}$$

In this paper we make use of the following notations: C will denote various positive constants; the strong (respectively weak) convergence is denoted by \rightarrow (respectively \rightharpoonup); $o(1)$ denotes $o(1) \rightarrow 0$ as $n \rightarrow \infty$, $B_\rho(0)$ denotes a ball centered at the origin with radius $\rho > 0$.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of our main results.

2. Variational setting and preliminaries

In this section, we give the variational setting for (1.1) and establish the compactness conditions. We set

$$E = \left\{ (u, v) \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \alpha(x) u^2 dx < \infty, \int_{\mathbb{R}^N} \beta(x) u^2 dx < \infty \right\} \quad (2.1)$$

be equipped with the inner product and norm

$$\langle u, w \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta w + \nabla u \nabla w + uw) dx, \quad \|u\| = \langle u, u \rangle^{\frac{1}{2}}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$\begin{aligned} \langle u, w \rangle_\lambda &= \int_{\mathbb{R}^N} (\Delta u \Delta w + \nabla u \nabla w + V_1(x)uw) dx, & \|u\|_\lambda &= \langle u, u \rangle_\lambda^{\frac{1}{2}}, \\ \langle v, h \rangle_\lambda &= \int_{\mathbb{R}^N} (\Delta v \Delta h + \nabla v \nabla h + V_2(x)v h) dx, & \|v\|_\lambda &= \langle v, v \rangle_\lambda^{\frac{1}{2}}, \end{aligned}$$

with norm

$$\|(u, v)\|_\lambda^2 = \|u\|_\lambda^2 + \|v\|_\lambda^2.$$

It follows from (F₁)-(F₂) and Poincaré's inequality that the embedding $E \hookrightarrow H^2(\mathbb{R}^N)$ is continuous.

Now we will work on Hilbert space E . A weak solution for problem (1.1) is a critical point of the following energy functional I_λ defined on E by

$$\begin{aligned} I_\lambda(u, v) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V_1(x)u^2) dx + \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta v|^2 + |\nabla v|^2 + V_2(x)v^2) dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} (u^2 |\nabla u|^2 + v^2 |\nabla v|^2) dx - \frac{1}{p+q} \int_{\mathbb{R}^N} |u|^p |v|^q dx \end{aligned}$$

for all $(u, v) \in E$. By standard arguments, we see that $I(u, v)$ is a C^1 functional on the space E . Moreover $I'_\lambda(u, v) = 0$ if and only if (u, v) satisfies, for all $(\phi, \varphi) \in E$,

$$\begin{cases} \int_{\mathbb{R}^N} (\Delta u \Delta \phi + |\nabla u| |\nabla \phi| + V_1(x)u\phi) dx + \int_{\mathbb{R}^N} (u\phi |\nabla u|^2 + u^2 |\nabla u| |\nabla \phi|) dx = \frac{p}{p+q} \int_{\mathbb{R}^N} |u|^{p-1} |v|^q \phi dx, \\ \int_{\mathbb{R}^N} (\Delta v \Delta \varphi + |\nabla v| |\nabla \varphi| + V_2(x)v\varphi) dx + \int_{\mathbb{R}^N} (v\varphi |\nabla v|^2 + v^2 |\nabla v| |\nabla \varphi|) dx = \frac{q}{p+q} \int_{\mathbb{R}^N} |u|^p |v|^{q-1} \varphi dx. \end{cases}$$

Recall that I satisfies the (PS) condition at the level $c \in \mathbb{R}$ ((PS) $_c$ condition for short) if any sequence $\{u_n\} \subset E_\lambda$ along with $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ possesses a convergent subsequence. If I satisfies (PS) $_c$ condition for each $c \in \mathbb{R}$, then we say that I satisfies the (PS) condition. In this paper, we will take $I_\lambda = I_\lambda(u, v)$ and $E_\lambda = (E, \|(u, v)\|_\lambda)$.

Definition 2.1. A function u is a ground state solution of problem (1.1), we mean that u is such a solution which has the least energy among all nontrivial solutions of problem (1.1).

Lemma 2.2. Under the assumptions (F $_0$), the embedding $E_\lambda \hookrightarrow L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$ is compact for $2 \leq s < 2^{**}$.

Proof. Using Lemma 3.4 in [34], the proofs is similar to Lemma 2.1 in [5], so we omit it here. □

3. Proof of main results

In the following we show that the functional I_λ satisfies the mountain pass geometry.

Lemma 3.1. The functional I_λ satisfies the following conditions.

- (i) There exists ρ such that $I_\lambda(u, v) \geq \alpha$ with $\|(u, v)\|_\lambda = \rho$.
- (ii) There exists $(u_0, v_0) \in B_\rho(0)$ with $I_\lambda(u_0, v_0) < 0$.

Proof. Using the Hölder inequality, we obtain

$$\int_{\mathbb{R}^N} |u|^p |v|^q dx \leq \left(\int_{\mathbb{R}^N} |u|^{p+q} dx \right)^{\frac{p}{p+q}} \left(\int_{\mathbb{R}^N} |v|^{p+q} dx \right)^{\frac{q}{p+q}}.$$

According to the Hölder and Nirenberg inequalities for any $p + q \in [4, 2^{**})$ there exists $C > 0$, such that

$$\begin{aligned} \int_{\mathbb{R}^N} |u|^{p+q} dx &\leq \left(\int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{[2N-(p+q)(N-4)]}{8}} \left(\int_{\mathbb{R}^N} |u|^{2^{**}} dx \right)^{\frac{(p+q-2)(N-4)}{8}} \\ &\leq C_1^{\frac{N(p+q-2)}{4}} \left(\int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{[2N-(p+q)(N-4)]}{8}} \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx \right)^{\frac{N(p+q-2)}{8}} \leq C \|(u, v)\|_\lambda^{p+q}. \end{aligned} \tag{3.1}$$

We denote $C = C_1^{\frac{N(p+q-2)}{4}}$. Similarly, we have

$$\int_{\mathbb{R}^N} |u|^{p+q} dx \leq C \|(u, v)\|_\lambda^{p+q}.$$

By (3.1), we obtain

$$\begin{aligned} I_\lambda(u, v) &= \frac{1}{2} \|(u, v)\|_\lambda^2 + \frac{1}{2} \int_{\mathbb{R}^N} (u^2 |\nabla u|^2 + v^2 |\nabla v|^2) dx - \frac{1}{p+q} \int_{\mathbb{R}^N} |u|^p |v|^q dx \\ &\geq \frac{1}{2} \|(u, v)\|_\lambda^2 - \frac{1}{p+q} \int_{\mathbb{R}^N} |u|^p |v|^q dx \geq \frac{1}{2} \|(u, v)\|_\lambda^2 - C \|(u, v)\|_\lambda^{p+q}. \end{aligned}$$

Hence, we can choose small enough $\rho > 0$, so there exists $\alpha > 0$ such that $I_\lambda(u, v) \geq \alpha$ with $\|(u, v)\|_\lambda = \rho$.

(ii) We see that for all $t \in \mathbb{R}$. We have

$$I_\lambda(tu, tv) = \frac{t^2}{2} \|(u, v)\|_\lambda^2 + \frac{t^4}{2} \int_{\mathbb{R}^N} (u^2 |\nabla u|^2 + v^2 |\nabla v|^2) dx - \frac{t^{p+q}}{p+q} \int_{\mathbb{R}^N} |u|^p |v|^q dx < 0$$

for $t > 0$ large enough. Hence, we take $e = t^*(u_0, v_0)$ with some $t^* > 0$ and (ii) follows. □

Lemma 3.2. For any $(u, v) \in E_\lambda \setminus \{0\}$, there exists a unique $t > 0$ such that $(tu, tv) \in N_\lambda$. Moreover, $I_\lambda(tu, tv) = \max_{t \geq 0} I_\lambda(tu, tv)$.

Proof. Let $(u, v) \in E_\lambda \setminus \{0\}$ be fixed and define the function $g(t) = I_\lambda(tu, tv)$ for $t \geq 0$. It is easy to obtain that $g(0) = 0$, by Lemma 3.1, we have $g(t) > 0$ for small $t > 0$. Moreover, applying a similar argument as in the proof of Lemma 3.1, it is easy to see that $g(t) < 0$ for large enough t . Hence, $\max_{t \geq 0} I_\lambda(tu, tv)$

exists and is achieved at $t > 0$. Thus, we get $g'(t) = 0$, that is,

$$I'_\lambda(tu, tv) = t\|(u, v)\|_\lambda^2 - t^{p+q-1} \int_{\mathbb{R}^N} |u|^p |v|^q dx + 2t^3 \int_{\mathbb{R}^N} (u^2 |\nabla u|^2 + v^2 |\nabla v|^2) dx = 0, \quad (3.2)$$

which implies $(tu, tv) \in N_\lambda$. So we obtain

$$\int_{\mathbb{R}^N} (u^2 |\nabla u|^2 + v^2 |\nabla v|^2) dx = -\frac{\|(u, v)\|_\lambda^2}{2t^2} + \frac{t^{p+q}}{2t^4} \int_{\mathbb{R}^N} |u|^p |v|^q dx. \quad (3.3)$$

By (3.2) and $4 < p + q < 2^{**}$, we obtain that the right is strictly increasing about $t \geq 0$. It follows from (3.3) that there exists a unique $t_0 > 0$, such that $(t_0 u, t_0 v) \in N_\lambda$ and $I_\lambda(t_0 u, t_0 v) = \max_{t \geq 0} I_\lambda(t_0 u, t_0 v)$. \square

Lemma 3.3. Assume that (F_1) and (F_2) hold, then $I_\lambda(u, v)$ satisfies (PS) condition.

Proof. We first prove that $\{(u_n, v_n)\}$ is bounded in E_λ . We take $(u, v) \in E_\lambda$. So we obtain

$$\begin{aligned} c + o(1) &\geq I_\lambda(u_n, v_n) - \frac{1}{4} \langle I'_\lambda(u_n, v_n), (u_n, v_n) \rangle \\ &= \frac{1}{4} \|(u_n, v_n)\|_\lambda^2 + \left(\frac{1}{4} - \frac{1}{p+q}\right) \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \geq \frac{1}{4} \|(u_n, v_n)\|_\lambda^2, \end{aligned}$$

which implies that $\{(u_n, v_n)\}$ is bounded.

Finally, we show that $\{(u_n, v_n)\}$ possesses a strong convergent subsequence in E_λ . In fact, in view of the boundedness of $\{(u_n, v_n)\}$, without loss of generality, we may assume $(u_n, v_n) \rightarrow (u_0, v_0)$ as $n \rightarrow \infty$ for some $(u_0, v_0) \in E_\lambda$. Since $E_\lambda \hookrightarrow L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$ is compact for $s \in [2, 2^{**})$, $(u_n, v_n) \rightarrow (u_0, v_0)$ in $L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$ for any $s \in [2, 2^{**})$. We easily obtain

$$\begin{aligned} &\langle (I'_\lambda(u_n, v_n) - I'_\lambda(u, v)), ((u_n - u, v_n - v)) \rangle \\ &= \|(u_n - u, v_n - v)\|_\lambda^2 + \int_{\mathbb{R}^N} (u_n^2 \nabla u_n - u^2 \nabla u) \nabla (u_n - u) dx + \int_{\mathbb{R}^N} (v_n |\nabla v_n|^2 - v |\nabla v|^2) (v_n - v) dx \\ &\quad + \int_{\mathbb{R}^N} (v_n^2 \nabla v_n - v^2 \nabla v) \nabla (v_n - v) dx + \int_{\mathbb{R}^N} (|u_n|^{p-1} |v_n|^q - |u|^{p-1} |v|^q) |u_n - u| dx \\ &\quad + \int_{\mathbb{R}^N} (u_n |\nabla u_n|^2 - u |\nabla u|^2) (u_n - u) dx + \int_{\mathbb{R}^N} (|v_n|^{q-1} |u_n|^p - |v|^{q-1} |u|^p) |v_n - v| dx. \end{aligned}$$

It is clear that

$$\langle (I'_\lambda(u_n, v_n) - I'_\lambda(u, v)), ((u_n - u, v_n - v)) \rangle \rightarrow 0.$$

By the Hölder's inequality we can get that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|u_n|^{p-1} |v_n|^q - |u|^{p-1} |v|^q) |u_n - u| dx \\ &\leq C \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n - u|^p |v_n - v|^q dx \\ &\leq C \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |u_n - u|^{p+q} dx \right)^{\frac{p}{p+q}} \left(\int_{\mathbb{R}^N} |v_n - v|^{p+q} dx \right)^{\frac{q}{p+q}}. \end{aligned} \quad (3.4)$$

Note that $u_n \rightarrow u$ in $L^r(\mathbb{R}^N)$ for $2 < q < 2^{**}$. Since $4 < p + q < 22^{**}$, we can choose $\kappa \in (0, 1)$ such that $p + q = \kappa r + (1 - \kappa)22^{**}$, then

$$\int_{\mathbb{R}^N} |u_n - u|^{p+q} dx \leq \left(\int_{\mathbb{R}^N} |u_n - u|^q dx \right)^{\frac{1}{\kappa}} \left(\int_{\mathbb{R}^N} |u_n - u|^{22^{**}} dx \right)^{\frac{1}{1-\kappa}} \rightarrow 0.$$

Similarly, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|v_n|^{q-1}|u_n|^p - |v|^{q-1}|u|^p)|v_n - v|dx \\ & \leq \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |u_n - u|^{p+q}dx \right)^{\frac{p}{p+q}} \left(\int_{\mathbb{R}^N} |v_n - v|^{p+q}dx \right)^{\frac{q}{p+q}} \rightarrow 0. \end{aligned}$$

Note that $n \leq 5$ and $6 < 2^{**}$. Using Hölder’s inequality and $u_n \rightarrow u$ in $L^s(\mathbb{R}^N)$ for any $s \in [2, 2^{**})$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (u_n|\nabla u_n|^2 - u|\nabla u|^2)(u_n - u)dx \right| & \leq \int_{\mathbb{R}^N} |u_n||\nabla u_n|^2|u_n - u|dx + \int_{\mathbb{R}^N} |u||\nabla u|^2|u_n - u|dx \\ & \leq \left(\int_{\mathbb{R}^N} |u_n|^6dx \right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^3dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^N} |u_n - u|^6dx \right)^{\frac{1}{6}} \\ & \quad + \left(\int_{\mathbb{R}^N} |u|^6dx \right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^N} |\nabla u|^3dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^N} |u_n - u|^6dx \right)^{\frac{1}{6}} \\ & \leq C\|u_n - u\|_{L^6} \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Here again

$$\left| \int_{\mathbb{R}^N} (v_n|\nabla v_n|^2 - v|\nabla v|^2)(v_n - v)dx \right| \rightarrow 0.$$

Similarly we can also obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} (u_n^2 - u^2)\nabla u_n \nabla(u_n - u)dx \right| \\ & \leq \int_{\mathbb{R}^N} |u_n - u||u_n + u||\nabla u_n||\nabla(u_n - u)|dx \\ & \leq \left(\int_{\mathbb{R}^N} |u_n - u|^6 \right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^N} |u_n + u|^6 \right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^3 \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^N} |\nabla(u_n - u)|^3 \right)^{\frac{1}{3}} \\ & \leq C\|u_n - u\|_{L^6} \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

We also get

$$\left| \int_{\mathbb{R}^N} (v_n^2 - v^2)\nabla v_n \nabla(v_n - v)dx \right| \rightarrow 0.$$

Thus we obtain that $\|(u_n - u), (v_n - v)\|_{\lambda}^2 \rightarrow 0$. This completes the proof. □

We define

$$c = \inf_{\eta \in \Gamma} \sup_{t \in [0,1]} I_{\lambda}(\eta(t)), \quad c^* = \inf_{(u,v) \in N_{\lambda}} I_{\lambda}(u,v), \quad c^{**} = \inf_{(u,v) \in E_{\lambda} \setminus \{0\}} \sup_{t \geq 0} I_{\lambda}(tu, tv),$$

where u is given by Lemma 3.2. We also define

$$\begin{aligned} & \widetilde{\gamma}_1(t) := tu \text{ for } 0 \leq t \leq 1, \quad \widetilde{\gamma}_2(s) := sv \text{ for } 0 \leq s \leq 1, \quad \text{and } \widetilde{\gamma}(t,s) := (\widetilde{\gamma}_1(t), \widetilde{\gamma}_2(s)), \\ & \Gamma = \{\eta \in C([0,1], X) | \widetilde{\gamma}(0,0) = 0, \quad I_{\lambda}(\widetilde{\gamma}(1,1)) \leq 0, \quad \widetilde{\gamma}(1,1) \neq 0\}. \end{aligned}$$

Lemma 3.4. $c = c^* = c^{**}$.

Proof. The proof is standard. We sketch it here for the completeness of the paper. It follows from Lemma 3.2 that $c^* = c^{**}$. Note that for any $(u,v) \in E_{\lambda} \setminus \{0\}$, there exists $T > 0$ large enough such that $I_{\lambda}(tu, sv) < 0$. We may define a path $\widetilde{\gamma}(t,s) = (tu, sv)$, clearly, $\widetilde{\gamma} \in \Gamma$ and consequently, $c \leq c^* = c^{**}$. Now, we show that

$c^* \leq c$. By (2.1) and Lemma 3.1,

$$\begin{aligned} \langle I'_\lambda(u, v), (u, v) \rangle &= \|(u, v)\|_\lambda^2 + 2 \int_{\mathbb{R}^N} (u^2 |\nabla u|^2 + v^2 |\nabla v|^2) dx - \int_{\mathbb{R}^N} |u|^p |v|^q dx \\ &\geq \|(u, v)\|_\lambda^2 - C \|(u, v)\|_\lambda^{p+q}. \end{aligned} \tag{3.5}$$

Since, $\|u\|_\lambda^2 + \|v\|_\lambda^2 \rightarrow 0$, if $\|(u, v)\|_\lambda \rightarrow 0$, it follows from (3.5) that for nonzero (u, v) with small enough $\|(u, v)\|_\lambda$, $\langle I'_\lambda(u, v), (u, v) \rangle > 0$. Thus from Lemma 3.1, for small enough $\|(u, v)\|_\lambda$, we get

$$\|u\|_\lambda^2 + \|v\|_\lambda^2 + 2 \int_{\mathbb{R}^N} (u^2 |\nabla u|^2 + v^2 |\nabla v|^2) dx > \frac{1}{2} \int_{\mathbb{R}^N} |u|^p |v|^q dx. \tag{3.6}$$

In fact, any path $\tilde{\gamma}(t, s)$ in Γ crosses N_λ . Otherwise, by the continuity of $\tilde{\gamma}(t, s)$, inequality (3.6) still holds true for each nonzero $\tilde{\gamma}(t, s)$, $(t, s) \in [0, 1] \times [0, 1]$. Hence

$$\begin{aligned} I_\lambda(\tilde{\gamma}(1, 1)) &= \frac{1}{2} \|(\tilde{\gamma}_1(1), \tilde{\gamma}_2(1))\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (\tilde{\gamma}_1(1)^2 |\nabla \tilde{\gamma}_1(1)|^2 + \tilde{\gamma}_2(1)^2 |\nabla \tilde{\gamma}_2(1)|^2) dx \\ &\quad - \frac{1}{p+q} \int_{\mathbb{R}^N} |\tilde{\gamma}_1(1)|^p |\tilde{\gamma}_2(1)|^q dx > \left(\frac{1}{4} - \frac{1}{p+q}\right) \int_{\mathbb{R}^N} |\tilde{\gamma}_1(1)|^p |\tilde{\gamma}_2(1)|^q dx > 0, \end{aligned}$$

which contradict the definition of $\tilde{\gamma}(1, 1)$. Thus, every path $\tilde{\gamma}(t, s) \in \Gamma$ crosses N_λ and consequently, $c^* \leq c$. This completes the proof. \square

Lemma 3.5. *The critical points of the functional I_λ on N_λ are critical points of I_λ in E_λ .*

Proof. Let $\{(u_n, v_n)\} \subset N_\lambda$ be such that $I_\lambda(u_n, v_n) \rightarrow c$ and $I'_\lambda(u_n, v_n) \rightarrow 0$. Then there exists $\mu_n \subset \mathbb{R}$ such that $I'_\lambda(u_n, v_n) = \mu_n \tilde{I}'_\lambda(u_n, v_n) + o(1)$, where

$$\tilde{I}_\lambda(u, v) := \|(u, v)\|_\lambda^2 + 2 \int_{\mathbb{R}^N} (u^2 |\nabla u|^2 + v^2 |\nabla v|^2) dx - \int_{\mathbb{R}^N} |u|^p |v|^q dx.$$

We denote that $\tilde{I}_\lambda(u, v) = \langle \tilde{I}'_\lambda(u, v), (u, v) \rangle$, thus

$$\begin{aligned} \langle \tilde{I}'_\lambda(u, v), (u, v) \rangle &= 2\|(u, v)\|_\lambda^2 - (p+q) \int_{\mathbb{R}^N} |u|^p |v|^q dx + 4 \int_{\mathbb{R}^N} (u^2 |\nabla u|^2 + v^2 |\nabla v|^2) dx \\ &\leq -(p+q-2) \int_{\mathbb{R}^N} |u|^p |v|^q dx. \end{aligned}$$

If $(u, v) \in N_\lambda$, since $(u, v) \neq 0$, then $\langle \tilde{I}'_\lambda(u, v), (u, v) \rangle < 0$ and hence $\tilde{I}'_\lambda(u, v) \neq 0$ on N_λ . Let $J_\lambda := I_\lambda|_{N_\lambda}$, then we have

$$J'_\lambda(u, v) = I'_\lambda(u, v) - \frac{\langle I'_\lambda(u, v), (u, v) \rangle}{\|\tilde{I}'_\lambda(u, v)\|^2} \tilde{I}'_\lambda(u, v),$$

and

$$\langle J'_\lambda(u, v), (u, v) \rangle = \langle I'_\lambda(u, v), (u, v) \rangle \left(1 - \frac{\langle \tilde{I}'_\lambda(u, v), (u, v) \rangle}{\|\tilde{I}'_\lambda(u, v)\|^2}\right).$$

So $J'_\lambda(u, v) = 0$ if and only if $I'_\lambda(u, v) = 0$. \square

Proposition 3.6. *Assume that C_0 is fixed, then for all $\varepsilon > 0$, there exist $\Gamma_\varepsilon, R_\varepsilon > 0$ such that if $\{(u_n, v_n)\}$ is a $(PS)_c$ sequence of $I_\lambda(u, v)$ with $\lambda > \Gamma_\varepsilon, c \leq C_0$, then*

$$\int_{\overline{\Omega}_{R_\varepsilon}^c} |u_n|^p |v_n|^q dx \leq \varepsilon \text{ for } 4 < p+q < 22^{**},$$

where $\overline{\Omega}_{R_\varepsilon}^c = \{x \in \mathbb{R}^N \mid |x| \geq R_\varepsilon\}$.

Proof. For $\forall R > 0$, we set

$$\Omega(R) := \{x \in \mathbb{R}^N \mid |x| \geq R, \alpha(x) \geq M_1\}, \quad \bar{\Omega}(R) := \{x \in \mathbb{R}^N \mid |x| \geq R, \alpha(x) \leq M_1\}.$$

Then

$$\begin{aligned} \int_{\Omega(R)} u_n^2 dx &\leq \frac{1}{\lambda M_1} \int_{\Omega(R)} \lambda \alpha(x) u_n^2 dx \\ &\leq \frac{1}{\lambda M_1} \int_{\Omega(R)} (|\Delta u_n|^2 + |\nabla u_n|^2 + \lambda \alpha(x) u_n^2) dx \\ &\leq \frac{1}{\lambda M_1} \| (u_n, v_n) \|_{\lambda'}^2, \quad (\| (u_n, v_n) \|_{\lambda} \text{ is bounded}) \\ &\rightarrow 0, \quad \text{as } \lambda \rightarrow \infty, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \int_{\bar{\Omega}(R)} u_n^2 dx &\leq \left[\int_{\mathbb{R}^N} (u_n^2)^p dx \right]^{\frac{1}{p}} m(\bar{\Omega}(R))^{\frac{p-1}{p}}, \quad (2 < p < 2^{**}) \\ &\leq \left(\int_{\mathbb{R}^N} u_n^2 dx \right)^{\frac{\theta}{2}} \left(\int_{\mathbb{R}^N} u_n^{**} dx \right)^{\frac{1-\theta}{2^{**}}} m(\bar{\Omega}(R))^{\frac{p-1}{p}}, \quad (0 < \theta < 1) \\ &\leq C \| (u_n, v_n) \|_{\lambda}^{\theta} \| (u_n, v_n) \|_{\lambda}^{1-\theta} m(\bar{\Omega}(R))^{\frac{p-1}{p}} \\ &\leq C m(\bar{\Omega}(R))^{\frac{p-1}{p}} \rightarrow 0, \quad \text{as } R \rightarrow \infty. \end{aligned} \tag{3.8}$$

Applying the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned} \int_{\bar{\Omega}_R^c} |u_n|^{p+q} dx &\leq C \left(\int_{\bar{\Omega}_R^c} |\Delta u_n|^2 dx \right)^{\frac{(p+q)\theta}{2}} \left(\int_{\bar{\Omega}_R^c} |u_n|^2 dx \right)^{\frac{(p+q)(1-\theta)}{2}}, \quad \left(\theta = \frac{N(p+q-2)}{2(p+q)} \right) \\ &\leq C \| (u_n, v_n) \|_{\lambda}^{(p+q)\theta} \left(\int_{\Omega(R)} |u_n|^2 dx + \int_{\bar{\Omega}_R^c} |u_n|^2 dx \right)^{\frac{(p+q)(1-\theta)}{2}} \\ &\leq C \left(\int_{\Omega(R)} |u_n|^2 dx + \int_{\bar{\Omega}_R^c} |u_n|^2 dx \right)^{\frac{(p+q)(1-\theta)}{2}} \\ &\rightarrow 0 \quad \text{as } \lambda, R \rightarrow \infty \text{ (by (3.7) and (3.8)).} \end{aligned}$$

Similarly, we obtain $\int_{\bar{\Omega}_R^c} |v_n|^{p+q} dx \leq \varepsilon$ for $\lambda, R \rightarrow \infty$. Finally, using the Hölder Inequality, we obtain

$$\limsup_{n \rightarrow \infty} \int_{\bar{\Omega}_{R\varepsilon}^c} |u_n|^p |v_n|^q dx \leq \left(\int_{\bar{\Omega}_{R\varepsilon}^c} |u_n|^{p+q} dx \right)^{\frac{p}{p+q}} \left(\int_{\bar{\Omega}_{R\varepsilon}^c} |v_n|^{p+q} dx \right)^{\frac{q}{p+q}} \leq \varepsilon.$$

This completes the proof. □

Proof of Theorem 1.1. We only need to prove that c is achieved at some $u \in N_{\lambda}$ from Lemmas 3.4 and 3.5. Letting $\{(u_n, v_n)\} \subset E_{\lambda} \cap N_{\lambda}$ be a minimizing sequence of I_{λ} , then we have

$$c + 1 \geq I_{\lambda}(u_n, v_n) - \frac{1}{4} \langle I'_{\lambda}(u_n, v_n), (u_n, v_n) \rangle = \frac{1}{4} \| (u, v) \|_{\lambda}^2 + \left(\frac{1}{4} - \frac{1}{p+q} \right) \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \geq \frac{1}{4} \| (u, v) \|_{\lambda}^2.$$

Therefore, $\{(u_n, v_n)\}$ is bounded in $E_{\lambda} \cap N_{\lambda}$.

Recall the compact embedding

$$E_{\lambda} \hookrightarrow L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N), \quad s \in [2, 2^{**}).$$

Thus, going if necessary to a subsequence, we may assume that there exists a function $u \in E_\lambda$ such that

$$\begin{cases} (u_n, v_n) \rightharpoonup (u, v) \text{ in } E_\lambda, \\ (u_n, v_n) \rightarrow (u, v) \text{ in } L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N), \forall s \in [2, 2^{**}), \\ (u_n, v_n) \rightarrow (u, v) \text{ a.e. on } \mathbb{R}^N \times \mathbb{R}^N. \end{cases}$$

By Fatou Lemma,

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} (I_\lambda(u_n, v_n) - \frac{1}{4} \langle I'(u_n, v_n), (u_n, v_n) \rangle) \\ &\geq \liminf_{n \rightarrow \infty} (I_\lambda(u_n, v_n) - \frac{1}{4} \langle I'(u_n, v_n), (u_n, v_n) \rangle) \\ &= \frac{1}{4} \|(u, v)\|_\lambda^2 + \left(\frac{1}{4} - \frac{1}{p+q}\right) \int_{\mathbb{R}^N} (|u|^p |v|^q) dx = I_\lambda(u, v) - \frac{1}{4} \langle I'(u, v), (u, v) \rangle = c. \end{aligned}$$

This implies that $I_\lambda(u, v) = c$. Hence u is the problem of ground state solution.

Now, we prove that $(u, v) \neq (0, 0)$.

$$\begin{aligned} I_\lambda(u_n, v_n) - \frac{1}{2} \langle I'_\lambda(u_n, v_n), (u_n, v_n) \rangle &= -\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 u_n^2 + |\nabla v_n|^2 v_n^2) dx + \left(\frac{1}{2} - \frac{1}{p+q}\right) \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \\ &= c + o(1). \end{aligned}$$

It deduces that

$$\left(\frac{1}{2} - \frac{1}{p+q}\right) \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \geq c + o(1).$$

Let $\varepsilon_0 = \frac{1}{4}c_0 \leq \frac{1}{4}c$, since $v_n \rightarrow v$ strongly in $L^r(\Omega_R)$, for any $R > 0$ and $2 < r < 2^{**}$, by Proposition 3.6, there exists $\Gamma_0 > 0, R_0 > 0$ such that for $\lambda \geq \Gamma_0, R \geq R_0$,

$$\left(\frac{1}{2} - \frac{1}{p+q}\right) \int_{\Omega_R^c} |u_n|^p |v_n|^q dx < \varepsilon_0,$$

thus

$$\int_{\Omega_R} |u|^p |v|^q dx \geq \frac{1}{4}c > 0,$$

which implies that $(u, v) \neq (0, 0)$. This completes the proof. □

Proof of Theorem 1.2. For any sequence $0 < \lambda_n \rightarrow +\infty$, as $n \rightarrow +\infty$, let $(u_n, v_n) := (u_{\lambda_n}, v_{\lambda_n})$ be the critical points of $I_{\lambda_n}(u, v)$ in Theorem 1.1. First, we show that $\{(u_n, v_n)\}$ is bounded. Since (u_n, v_n) is the critical points of I_{λ_n} , we have

$$\begin{aligned} c + o(1) &= I_{\lambda_n}(u_n, v_n) \\ &= \frac{1}{2} \|(u_n, v_n)\|_{\lambda_n}^2 + \frac{1}{2} \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |v_n|^2 |\nabla v_n|^2 dx - \frac{1}{p+q} \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \quad (3.9) \\ &\geq \frac{1}{4} \|(u_n, v_n)\|_{\lambda_n}^2, \end{aligned}$$

which deduces that $\|(u_n, v_n)\|_{\lambda_n} \leq 2c^{\frac{1}{2}}$, that is, $\|(u_n, v_n)\|_{\lambda_n}$ is bounded in E_{λ_n} . Therefore, we may assume the $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E_{λ_n} and $(u_n, v_n) \rightarrow (u_0, v_0)$ strongly in $L^r_{loc}(\mathbb{R}^N) \times L^r_{loc}(\mathbb{R}^N)$ for $2 \leq r < 2^{**}$. By Fatou’s Lemma, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \alpha(x) u_0^2 dx + \int_{\mathbb{R}^N} \beta(x) v_0^2 dx &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \alpha(x) u_n^2 dx + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \beta(x) v_n^2 dx \\ &\leq \liminf_{n \rightarrow \infty} \frac{\|(u_n, v_n)\|_{\lambda_n}^2}{\lambda_n} = 0, \end{aligned}$$

which implies that $(u_0, v_0) = (0, 0)$, a.e. in $\mathbb{R}^N \setminus \{\alpha^{-1}(0) \cup \beta^{-1}(0)\}$ and $(u_0, v_0) \in H^2(\Omega) \cap H_0^1(\Omega)$. Now for any $(\varphi, \phi) \in C_0^\infty(\mathbb{R}^N)$, since $\langle I'_{\lambda_n}(u_n, v_n), (\varphi, \phi) \rangle = 0$, it is easy to check that

$$\begin{aligned} & \int_{\mathbb{R}^N} (\Delta u_0 \Delta \varphi + \Delta v_0 \Delta \phi + \nabla u_0 \nabla \varphi + \nabla v_0 \nabla \phi + V_1(x)u_0 \varphi + V_2(x)v_0 \phi) dx \\ & + \int_{\mathbb{R}^N} (u_0 \varphi |\nabla u_0|^2 + u_0^2 |\nabla u_0| |\nabla \varphi|) dx + \int_{\mathbb{R}^N} (v_0 \phi |\nabla v_0|^2 + v_0^2 |\nabla v_0| |\nabla \phi|) dx \\ & = \frac{q}{p+q} \int_{\mathbb{R}^N} |v_0|^{q-1} |u_0|^p \phi dx + \frac{p}{p+q} \int_{\mathbb{R}^N} |u_0|^{p-1} v_0^q \varphi dx, \end{aligned}$$

that is, (u_0, v_0) is a weak of solution of (1.2) by that density of $C_0^\infty(\Omega)$ in $H^2(\Omega) \cap H_0^1(\Omega)$. Now we show that $u_n \rightarrow u_0, v_n \rightarrow v_0$ in $L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ for $2 \leq r < 22^{**}$. Otherwise, by Lions vanishing Lemma [18, 25], there exist $\delta > 0, R_0 > 0$ and $x_n, \bar{x}_n \in \mathbb{R}^N$, such that

$$\int_{B(x_n, R_0)} (u_n - u_0)^2 dx \geq \delta, \quad \int_{B(\bar{x}_n, R_0)} (v_n - v_0)^2 dx \geq \delta.$$

Moreover, $x_n \rightarrow \infty$, hence $\text{meas}(B(x_n, R_0) \cap \{x \in \mathbb{R}^N, \alpha(x) < M_1\}) \rightarrow 0$ and $\text{meas}(B(\bar{x}_n, R_0) \cap \{x \in \mathbb{R}^N, \beta(x) < M_2\}) \rightarrow 0$. By the Hölder inequality, we have

$$\int_{B(x_n, R_0) \cap \{\alpha(x) < M_1\}} (u_n - u_0)^2 dx \rightarrow 0, \quad \int_{B(\bar{x}_n, R_0) \cap \{\beta(x) < M_2\}} (v_n - v_0)^2 dx \rightarrow 0.$$

Consequently

$$\begin{aligned} & \int_{\mathbb{R}^N} (|\Delta u_n|^2 + |\Delta v_n|^2 + |\nabla u_n|^2 + |\nabla v_n|^2 + \lambda_n \alpha(x)(x)u_n + \lambda_n \beta(x)v_n) dx \\ & \geq \lambda_n M_1 \int_{B(x_n, R_0) \cap \{\alpha(x) \geq M_1\}} |u_n|^2 dx + \lambda_n M_2 \int_{B(\bar{x}_n, R_0) \cap \{\beta(x) \geq M_2\}} |v_n|^2 dx \\ & \geq \lambda_n M_1 \int_{B(x_n, R_0) \cap \{\alpha(x) \geq M_1\}} (u_n - u_0)^2 dx + \lambda_n M_2 \int_{B(\bar{x}_n, R_0) \cap \{\beta(x) \geq M_2\}} (v_n - v_0)^2 dx \\ & = \lambda_n M_1 \int_{B(x_n, R_0)} (u_n - u_0)^2 dx - \lambda_n M_1 \int_{B(x_n, R_0) \cap \{\alpha(x) < M_1\}} (u_n - u_0)^2 dx + o(1) \\ & \quad + \lambda_n M_2 \int_{B(\bar{x}_n, R_0)} (v_n - v_0)^2 dx - \lambda_n M_2 \int_{B(\bar{x}_n, R_0) \cap \{\beta(x) < M_2\}} (v_n - v_0)^2 dx + o(1) \\ & \rightarrow \infty, \text{ as } n \rightarrow \infty, \end{aligned}$$

which contradicts to (3.9). In the above proof, we have used the facts that $|B(x_n, R_0) \cap \{\alpha(x) < M_1\}| \rightarrow 0, |B(\bar{x}_n, R_0) \cap \{\beta(x) < M_2\}| \rightarrow 0$ as $n \rightarrow \infty$ and also that the L^2 -norm of $\{(u_n, v_n)\}$ is bounded. Therefore, $u_n \rightarrow v_0, v_n \rightarrow v_0$ in $L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ for $2 \leq r < 22^{**}$.

In the following, we show that $(u_n, v_n) \rightarrow (u_0, v_0)$ in $H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$. Since

$$\langle I'_{\lambda_n}(u_n, v_n), (u_n, v_n) \rangle = \langle I'_{\lambda_n}(u_n, v_n), (u_0, v_0) \rangle = 0,$$

we have

$$\begin{aligned} & \|(u_n, v_n)\|_{\lambda_n}^2 + 2 \int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2 dx + 2 \int_{\mathbb{R}^N} v_n^2 |\nabla v_n|^2 dx \\ & = \frac{p}{p+q} \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx + \frac{q}{p+q} \int_{\mathbb{R}^N} |v_n|^q |u_n|^p dx, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} & \langle (u_n, v_n), (u_0, v_0) \rangle_{\lambda_n} + \int_{\mathbb{R}^N} (u_n u_0 |\nabla u_n|^2 + u_n^2 |\nabla u_n| |\nabla u_0|) dx + \int_{\mathbb{R}^N} (v_n v_0 |\nabla v_n|^2 + v_n^2 |\nabla v| |\nabla v_0|) dx \\ &= \frac{p}{p+q} \int_{\mathbb{R}^N} |u_n|^{p-1} |v_n|^q u_0 dx + \frac{q}{p+q} \int_{\mathbb{R}^N} |v_n|^{q-1} |u_n|^p v_0 dx. \end{aligned} \quad (3.11)$$

It is easy to verify that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} (|u_n|^{p-1} |v_n|^q u_n - |u_n|^{p-1} |v_n|^q u_0) dx \right| \rightarrow 0, \\ & \left| \int_{\mathbb{R}^N} (|v_n|^{q-1} |u_n|^p v_n - |v_n|^{q-1} |u_n|^p v_0) dx \right| \rightarrow 0. \end{aligned} \quad (3.12)$$

In fact for (3.12) similar to the proof of (3.4). Since

$$\begin{aligned} \left| \int_{\mathbb{R}^N} |u_n| |\nabla u_n|^2 (u_n - u_0) dx \right| &\leq \left(\int_{\mathbb{R}^N} |u_n|^6 dx \right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^3 dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^N} |u_n - u_0|^6 dx \right)^{\frac{1}{6}} \\ &\leq C \|u_n - u_0\|_{L^6} \rightarrow 0, \end{aligned}$$

similarly, we have

$$\left| \int_{\mathbb{R}^N} |v_n| |\nabla v_n|^2 (v_n - v_0) dx \right| \rightarrow 0.$$

By Hölder inequality, we obtain

$$\left| \int_{\mathbb{R}^N} u_n^2 |\nabla u_n| (|\nabla u_n| - |\nabla u_0|) dx \right| \leq \left(\int_{\mathbb{R}^N} |u_n|^3 dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^6 dx \right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_0|^6 dx \right)^{\frac{1}{6}} \rightarrow 0.$$

Analogously, we get

$$\left| \int_{\mathbb{R}^N} v_n^2 |\nabla v_n| (|\nabla v_n| - |\nabla v_0|) dx \right| \rightarrow 0.$$

By (3.10)-(3.11) we have

$$\lim_{n \rightarrow \infty} \|(u_n, v_n)\|_{\lambda_n}^2 = \lim_{n \rightarrow \infty} \langle (u_n, v_n), (u_0, v_0) \rangle_{\lambda_n} = \lim_{n \rightarrow \infty} \langle (u_n, v_n), (u_0, v_0) \rangle = \|(u_0, v_0)\|^2.$$

On the other hand, weakly lower semi-continuity of norm yields that

$$\|(u_0, v_0)\|^2 \leq \liminf_{n \rightarrow \infty} \|(u_n, v_n)\|^2 \leq \limsup_{n \rightarrow \infty} \|(u_n, v_n)\|^2 \leq \lim_{n \rightarrow \infty} \|(u_n, v_n)\|_{\lambda_n}^2.$$

This shows that $(u_n, v_n) \rightarrow (u_0, v_0)$ in $H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$. Finally, we can get (u_0, v_0) is a nontrivial, and the proof is the similar to Theorem 1.1. This completes the proof. \square

Acknowledgment

This work is partially supported by Natural Science Foundation of China 11671403 and by the Fundamental Research Funds for the Central Universities of Central South University 2017zzts056.

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