Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 335–341
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Some additive mappings on Banach $*$-algebras with derivations

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Communicated by R. Saadati

Abstract

We take into account some additive mappings in Banach $*$-algebras with derivations. We will first study the conditions for additive mappings with derivations on Banach $*$-algebras. Then we prove some theorems involving linear mappings on Banach $*$-algebras with derivations. So derivations on C^*-algebra are characterized.

Keywords: Banach $*$-algebra, C^*-algebra, additive mapping with involution, derivation.

2010 MSC: 16N60, 16W80, 39B72, 39B82, 46H40, 46L57.

1. Introduction and preliminaries

Let $\mathcal{A}$ be an algebra over the real or complex field $\mathbb{F}$. An additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a derivation (resp., left derivation) if the functional equation

$$\delta(xy) = \delta(x)y + x\delta(y), \ (\text{resp., } \delta(xy) = y\delta(x) + x\delta(y))$$

holds for all $x, y \in \mathcal{A}$. In addition, if $\delta(tx) = t\delta(x)$ is fulfilled for all $x \in \mathcal{A}$ and $t \in \mathbb{F}$, then δ is said to be a linear derivation (resp., linear left derivation). An additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a Jordan derivation, if

$$\delta(x^2) = \delta(x)x + x\delta(x), \quad \forall\, x \in \mathcal{A}.$$

Furthermore, if $\delta(tx) = t\delta(x)$ holds for all $x \in \mathcal{A}$ and $t \in \mathbb{F}$, then δ is said to be a linear Jordan derivation.

Let us introduce the background of our investigation. Singer and Wermer [21] obtained a fundamental result which started investigation into the ranges of linear derivations on Banach algebras. The result,

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doi: 10.22436/jnsa.011.03.02

Received: 2017-06-26 *Revised:* 2017-11-25 *Accepted:* 2017-12-01

which is called the Singer-Wermer theorem, states that every continuous linear derivation on a commutative Banach algebra maps into the radical. In the same paper, they made a very insightful conjecture that the assumption of continuity is unnecessary. This is called the Singer-Wermer conjecture. Thomas [22] proved this conjecture. Hence linear derivations on Banach algebras (if everywhere defined) genuinely belong to the noncommutative setting.

The stability problem of functional equations originated from a famous talk given by Ulam [23]:
"Under what condition does there exist a homomorphism near an approximate homomorphism?"

Hyers [14] had answered affirmatively the question of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [2] and for approximately linear mappings was presented by Rassias [19]. Bourgin proved the superstability of homomorphism in [7]. The stability result, i.e., superstability concerning derivations between operator algebras was first obtained by Šemrl [20]. Badora [4] gave a generalization of the Bourgin's result [7]. As well, he dealt with the stability and the superstability of Bourgin-type for derivations in [5]. Since then, many interesting results of the stability problems to a number of functional equations and inequalities (or involving derivations) have been investigated. The reader is referred to the references [1, 3, 10, 16–18] for many information of stability problem with a large variety of applications.

In this work, we consider some additive mappings with involution related to derivations or a sort of additive mappings introduced in [8, 11], and then prove some theorems concerning additive mappings on complex Banach $*$-algebras with derivations.

2. Main results

In this work, we assume that $\mathbb{T}_\varepsilon := \{e^{i\theta} : 0 \leqslant \theta \leqslant \varepsilon\}$ and we write the unit element by e.

Theorem 2.1. *Let $\mathcal{A}$ be a complex Banach $*$-algebra. Assume that mappings $\Phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ and $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfy the assumptions*

1. $\sigma(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \Phi(2^j x, 2^j y) < \infty, \ (x, y \in \mathcal{A})$;
2. $\lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, y) = 0, \ (x, y \in \mathcal{A})$.

Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is mapping such that

$$\|\delta(x + y) - \delta(x) - \delta(y)\| \leqslant \Phi(x, y) \tag{2.1}$$

for all $x, y \in \mathcal{A}$ and

$$\|\delta(xy^* + yx^*) - \delta(x)y^* - x\delta(y^*) - \delta(y)x^* - y\delta(x^*)\| \leqslant \varphi(x, y), \quad (x, y \in \mathcal{A}). \tag{2.2}$$

Then there exists a unique additive mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ with

$$\mathcal{L}(xy^* + yx^*) = \mathcal{L}(x)y^* + x\mathcal{L}(y^*) + \mathcal{L}(y)x^* + y\mathcal{L}(x^*), \quad \forall x, y \in \mathcal{A}, \tag{2.3}$$

and

$$\|\mathcal{L}(x) - \delta(x)\| \leqslant \frac{1}{2}\sigma(x, x), \quad \forall\, x \in \mathcal{A}. \tag{2.4}$$

Moreover, the following equation

$$x\{\mathcal{L}(y) - \delta(y)\} = 0 \tag{2.5}$$

holds for all $x, y \in \mathcal{A}$.

Proof. It follows from the Găvruta theorem [12] that there exists a unique additive mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{L}(x) = \lim_{n\to\infty} \frac{1}{2^n}\delta(2^n x) \tag{2.6}$$

for all $x \in \mathcal{A}$ satisfying (2.4).

We first prove (2.3). We obtain from (2.2) and (2.6) that

$$\begin{aligned}
&\|\mathcal{L}(xy^*+yx^*) - \mathcal{L}(x)y^* - x\delta(y^*) - \delta(y)x^* - y\mathcal{L}(x^*)\| \\
&\quad = \lim_{n\to\infty} \frac{1}{2^n}\|\delta(2^n(xy^*+yx^*)) - \delta(2^n x)y^* - 2^n x\delta(y^*) - 2^n\delta(y)x^* - y\delta(2^n x^*)\| \\
&\quad \leqslant \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, y) = 0,
\end{aligned}$$

which implies that

$$\mathcal{L}(xy^*+yx^*) = \mathcal{L}(x)y^* + x\delta(y^*) + \delta(y)x^* + y\mathcal{L}(x^*) \tag{2.7}$$

for all $x, y \in \mathcal{A}$. In view of (2.7), we see that

$$\begin{aligned}
2^n\mathcal{L}(x)y^* + 2^n x\delta(y^*) + 2^n\delta(y)x^* + 2^n y\mathcal{L}(x^*) &= \mathcal{L}(2^n x\cdot y^* + y\cdot 2^n x^*) = \mathcal{L}(x\cdot 2^n y^* + 2^n y\cdot x^*) \\
&= 2^n\mathcal{L}(x)y^* + x\delta(2^n y^*) + \delta(2^n y)x^* + 2^n y\mathcal{L}(x^*).
\end{aligned} \tag{2.8}$$

It follows by (2.8) that

$$x\mathcal{L}(y^*) + \mathcal{L}(y)x^* = \lim_{n\to\infty}\left[x\frac{\delta(2^n y^*)}{2^n} + \frac{\delta(2^n y)}{2^n}x^*\right] = x\delta(y^*) + \delta(y)x^* \tag{2.9}$$

for all $x, y \in \mathcal{A}$. Therefore, we get (2.3).

Finally, it is sufficient to show that the property (2.5) holds. Multiplying by i on both sides in (2.9), we obtain that

$$ix\mathcal{L}(y^*) + i\mathcal{L}(y)x^* = ix\delta(y^*) + i\delta(y)x^*.$$

Putting $x = ix$ in (2.9), we find that

$$ix\mathcal{L}(y^*) - i\mathcal{L}(y)x^* = ix\delta(y^*) - i\delta(y)x^*.$$

Comparing the two above equation, we get the identity (2.5), which completes the proof. □

Theorem 2.2. *Let $\mathcal{A}$ be a semiprime unital complex Banach $*$-algebra. Assume that mappings $\Phi : \mathcal{A}\times\mathcal{A} \to [0,\infty)$ and $\varphi : \mathcal{A}\times\mathcal{A} \to [0,\infty)$ satisfy the assumptions of Theorem 2.1. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is mapping subjected to inequalities (2.1) and (2.2). Then δ is a linear derivation.*

Proof. Since $\mathcal{A}$ has a unit element, by setting $x = e$ in (2.5), we see that $\delta = \mathcal{L}$. In particular, we obtain from (2.3) that

$$\delta(xy^*+yx^*) = \delta(x)y^* + x\delta(y^*) + \delta(y)x^* + y\delta(x^*), \quad \forall\, x, y \in \mathcal{A}. \tag{2.10}$$

Considering $y = x^*$ in (2.10), we get

$$J(x) + J(x^*) = 0, \quad \forall\, x \in \mathcal{A}, \tag{2.11}$$

where $J(x)$ stands for

$$J(x) = \delta(x^2) - \delta(x)x - x\delta(x).$$

Letting $y = xy^* + yx^*$ in (2.10), we have

$$\delta(x(y+y^*)x^*) = -J(x)y^* - yJ(x^*) + \delta(x)(y+y^*)x^* + x(y+y^*)\delta(x^*) + x\delta(y+y^*)x^*.$$

Replacing y by $y - y^*$ in the above equation, we get

$$J(x)(y-y^*) - (y-y^*)J(x^*) = 0. \tag{2.12}$$

Multiplying by i on both sides in (2.12), we obtain that

$$iJ(x)(y-y^*) - i(y-y^*)J(x^*) = 0.$$

Putting $y = iy$ in (2.12), we find that

$$iJ(x)(y+y^*) - i(y+y^*)J(x^*) = 0.$$

Combining the above relation, we see that

$$J(x)y = yJ(x^*), \quad \forall\, x, y \in \mathcal{A}. \tag{2.13}$$

Since $\mathcal{A}$ contains a unit element, by letting $y = e$ in (2.13), we have $J(x) = J(x^*)$. By virtue of (2.11), we know that a mapping δ satisfies the equation

$$\delta(x^2) = \delta(x)x + x\delta(x), \quad \forall\, x \in \mathcal{A}.$$

So δ is a ring Jordan derivation. The semiprimeness of $\mathcal{A}$ guarantees that δ is a ring derivation, that is,

$$\delta(xy) = \delta(x)y + x\delta(y), \quad \forall\, x, y \in \mathcal{A}. \tag{2.14}$$

From (2.1), we see that

$$\|\delta(2te)\| = \lim_{n\to\infty} \frac{1}{2^n}\|\delta(2^n \cdot 2te)\| \leqslant \lim_{n\to\infty} \frac{1}{2^n}\Phi(2^n te, 2^n te) = 0,$$

for $t \in \mathbb{C}$. This implies that $\delta(te) = 0$. Let $y = te$ in (2.14). Then $\delta(tx) = t\delta(x)$ for all $x \in \mathcal{A}$ and for $t \in \mathbb{C}$. Therefore, δ is linear, which concludes the proof. □

Remark 2.3. Note that any linear derivation on semi-simple Banach algebra is continuous [15]. It is well-known that semisimple algebras are semiprime [6].

We get the following result.

Corollary 2.4. *Let $\mathcal{A}$ be a semisimple unital complex Banach $*$-algebra. Assume that mappings $\Phi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ and $\varphi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ satisfy the assumptions of Theorem 2.1. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is mapping subjected to inequalities (2.1) and (2.2). Then δ is continuous.*

Theorem 2.5. *Let $\mathcal{A}$ be either a semiprime complex Banach $*$-algebra or a unital complex Banach $*$-algebra. Assume that mappings $\Phi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ and $\varphi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ satisfy the assumptions of Theorem 2.1. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is mapping such that*

$$\|\delta(tx+ty) - t\delta(x) - t\delta(y)\| \leqslant \Phi(x,y) \tag{2.15}$$

for all $x, y \in \mathcal{A}$ and all $t \in \mathbb{T}_\varepsilon$ and the inequality (2.2). Then δ is a linear derivation.

Proof. We consider $t = 1$ in (2.15). According to Theorem 2.1, we see that there exists a unique additive mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ satisfying (2.3), (2.4) and (2.5).

It is suffices to show that $\mathcal{L}$ is linear. The inequality (2.15) yields that for all $x \in \mathcal{A}$ and all $t \in \mathbb{T}_\varepsilon$,

$$\|\mathcal{L}(tx) - t\mathcal{L}(x)\| = \lim_{n\to\infty} \frac{1}{2^n}\|\delta(2^n tx) - 2t\delta(2^{n-1}x)\| \leqslant \lim_{n\to\infty} \frac{1}{2^n}\Phi(2^{n-1}x, 2^{n-1}x) = 0.$$

Hence $\mathcal{L}(tx) = t\mathcal{L}(x)$. Then the mapping $\mathcal{L}$ is linear (refer to [13]).

If $\mathcal{A}$ is unital, set $y = e$ in (2.5). Then $\delta = \mathcal{L}$. If $\mathcal{A}$ is non-unital, then, by (2.5), we see that $\mathcal{L}(y) - \delta(y)$ lies in the right annihilator $\mathrm{ran}(\mathcal{A})$ of $\mathcal{A}$. If $\mathcal{A}$ is semiprime, $\mathrm{ran}(\mathcal{A}) = 0$, so that $\delta = \mathcal{L}$.

From (2.3), we get (2.10). Considering $y = iy$ in (2.10), we have

$$-i\delta(xy^*) + i\delta(yx^*) = -i\delta(x)y^* - ix\delta(y^*) + i\delta(y)x^* + iy\delta(x^*).$$

Multiplying i on both sides in the above relation, we see that

$$\delta(xy^*) - \delta(yx^*) = \delta(x)y^* + x\delta(y^*) - \delta(y)x^* - y\delta(x^*). \tag{2.16}$$

Combining (2.10) and (2.16), we obtain that

$$\delta(xy^*) = \delta(x)y^* + x\delta(y^*).$$

Letting $y = y^*$ in the above equation, we find that

$$\delta(xy) = \delta(x)y + x\delta(y), \quad \forall\, x, y \in \mathcal{A}.$$

Thereby, δ is a linear derivation. This completes the proof. □

Corollary 2.6. *Let $\mathcal{A}$ be a semisimple complex Banach $*$-algebra. Assume that mappings $\Phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ and $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfy the assumptions of Theorem 2.1. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is mapping subjected to the inequalities (2.15) and (2.2). Then δ is continuous.*

We now demonstrate the following proposition quoted in this work.

Proposition 2.7 ([9, Proposition 1.6.]). *Let $\mathcal{R}$ be a ring, $\mathcal{X}$ be a left $\mathcal{R}$-module and $\delta : \mathcal{R} \to \mathcal{X}$ be a left derivation.*

(i) *Suppose that $a\mathcal{R}x = 0$ with $a \in \mathcal{R}$, $x \in \mathcal{X}$ implies $a = 0$ or $x = 0$. If $\delta \neq 0$, then $\mathcal{R}$ is commutative.*

(ii) *Suppose that $\mathcal{X} = \mathcal{R}$ is a semiprime ring. Then δ is a derivation which maps $\mathcal{R}$ into its center.*

Theorem 2.8. *Let $\mathcal{A}$ be a semiprime complex Banach $*$-algebra. Assume that mappings $\Phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ and $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfy the assumptions of Theorem 2.1. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is mapping subjected to the inequality (2.15) and*

$$\|\delta(xy^* + yx^*) - y^*\delta(x) - x\delta(y^*) - x^*\delta(y) - y\delta(x^*)\| \leqslant \varphi(x, y), \quad (x, y \in \mathcal{A}). \tag{2.17}$$

Then δ is a linear derivation which maps $\mathcal{A}$ into the intersection of its center $Z(\mathcal{A})$ and its radical $\mathrm{rad}(\mathcal{A})$.

Proof. We let $t = 1$ in (2.15). As in the proof of Theorem 2.1, we see that there exists a unique additive mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ satisfying (2.4) and (2.5) with

$$\mathcal{L}(xy^* + yx^*) = y^*\mathcal{L}(x) + x\mathcal{L}(y^*) + x^*\mathcal{L}(y) + y\mathcal{L}(x^*), \quad \forall\, x, y \in \mathcal{A}. \tag{2.18}$$

Employing the same method as the proof of Theorem 2.5, we find that $\mathcal{L}$ is linear.

By (2.5), $\mathcal{L}(y) - \delta(y)$ lies in the right annihilator $\mathrm{ran}(\mathcal{A})$ of $\mathcal{A}$. Since $\mathcal{A}$ is semiprime, $\mathrm{ran}(\mathcal{A}) = 0$, so that $\delta = \mathcal{L}$. It follows from (2.18) that

$$\delta(xy^* + yx^*) = y^*\delta(x) + x\delta(y^*) + x^*\delta(y) + y\delta(x^*), \quad \forall\, x, y \in \mathcal{A}. \tag{2.19}$$

Letting $y = iy^*$ in (2.19), we have

$$-i\delta(xy^*) + i\delta(yx^*) = -iy^*\delta(x) - ix\delta(y^*) + ix^*\delta(y) + iy\delta(x^*).$$

Multiplying i on both sides in the above relation, we see that

$$\delta(xy^*) - \delta(yx^*) = y^*\delta(x) + x\delta(y^*) - x^*\delta(y) - y\delta(x^*). \tag{2.20}$$

Combining (2.18) and (2.20), we obtain that

$$\delta(xy^*) = y^*\delta(x) + x\delta(y^*).$$

Letting $y = y^*$ in the above equation, we find that

$$\delta(xy) = y\delta(x) + x\delta(y), \quad \forall\, x, y \in \mathcal{A}.$$

Thereby, δ is a linear left derivation.

On the other hand, from Proposition 2.7, we see that δ is a linear derivation with $\delta(\mathcal{A}) \subseteq Z(\mathcal{A})$. Since $Z(\mathcal{A})$ is a commutative Banach algebra, the Singer-Wermer theorem tells us that $\delta|_{Z(\mathcal{A})}$ maps $Z(\mathcal{A})$ into $\mathrm{rad}(Z(\mathcal{A})) = Z(\mathcal{A}) \cap \mathrm{rad}(\mathcal{A})$ and thus $\delta^2(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$. Using the semiprimeness of $\mathrm{rad}(\mathcal{A})$ as well as the identity

$$2\delta(x)y\delta(x) = \delta^2(xyx) - x\delta^2(yx) - \delta^2(xy)x + x\delta^2(y)x, \quad (x, y \in \mathcal{A}),$$

we have $\delta(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$. Therefore, $\delta(\mathcal{A}) \subseteq Z(\mathcal{A}) \cap \mathrm{rad}(\mathcal{A})$, which concludes the proof. □

Theorem 2.9. *Let $\mathcal{A}$ be a noncommutative prime unital complex Banach $*$-algebra. Assume that mappings $\Phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ and $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfy the assumptions of Theorem 2.1. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is mapping subjected to the inequalities (2.15) and (2.17). Then δ is identically zero.*

Proof. As we did in the proof of Theorem 2.8, there exists a unique linear mapping $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ satisfying (2.4) and (2.5) with the inequality (2.18). Since $\mathcal{A}$ contains the unit element, we have by (2.5) that $\delta = \mathcal{L}$. So (2.18) implies (2.19). Using the same method as the proof of Theorem 2.8, we see that δ is a linear left derivation.

Therefore, by Proposition 2.7, δ is identically zero, which ends the proof. □

Acknowledgment

The authors would like to thank the referees for giving useful suggestions and for the improvement of this manuscript. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (grant number 2017028238).

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