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Weakly (s,r)-contractive multi-valued operators on b-metric space

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Abstract

In this paper we introduce the notion of weakly (s,r)-contractive multi-valued operator on b-metric space and establish some fixed point theorems for this operator. In addition, an application to the differential equation is given to illustrate usability of obtained results.

Keywords: b-metric space, weakly (s,r)-contractive multi-valued operator, fixed point theorem.

2010 MSC: 47H04, 47H09, 47H10.

1. Introduction

Banach fixed point theorem [1] says that every contractive mapping on a complete metric space has a unique fixed point. As it is well known, the Banach fixed point theorem is a very useful, simple and classical tool in modern analysis. There are a large number of generalizations for this interesting theorem, for example see [5, 9–11, 16, 20]. On the one hand, to get an analog result for multi-valued mappings, one has to equip the powerset of a set with some suitable metric. One such a metric is a Hausdorff metric. Markin [13] for the first time used the Hausdorff metric to study the fixed point theory of the multi-valued contractive mapping; Nadler [15] and Reich [18, 19] respectively introduced the fixed point theorem of the multi-valued contractive operator and generalized the compression conditions given by Nadler; Rus [23] introduced multi-valued weakly Picard operator; Popescu [17] introduced the definition of the (s,r)-contractive multi-valued operator and showed that this operator is a weakly Picard operator. On the other hand, Czerwik [3] introduced the notions of the contractive mapping and the set-valued contractive mapping on b-metric space. Recently Kamran and Hussain [12] generalized the (s,r)-contractive multi-valued operator and introduced the notion of the weakly (s,r)-contractive multi-valued operator. They also obtained fixed points and strict fixed point theorems for the weakly (s,r)-contractive multi-valued

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operator. Thus it is worth for us to research fixed point theorems of the multi-valued operator in b-metric space.

Next, we present some elementary definitions and results which will be used throughout this paper. Details can be seen in [2–4, 6–8, 21, 24, 25].

Definition 1.1 ([3]). Let X be a nonempty set and $K \geqslant 1$ be a given constant. A function $d : X \times X \to \mathbb{R}^+$ is called a b-metric if the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x,\ y \in X$;
(3) $d(x, y) \leqslant K[d(x, z) + d(z, y)]$ for all $x,\ y,\ z \in X$.

The pair (X, d) is called a b-metric space (with constant K).

It is easy to see that any metric space is a b-metric space with $K = 1$. The following example shows that a b-metric on X need not be a metric on X.

Example 1.2. The set $\mathbb{R}$ of real numbers together with the function

$$d(x, y) := |x - y|^2$$

for all $x,\ y \in \mathbb{R}$ is a b-metric space with constant $K = 2$ but not a metric space.

Definition 1.3 ([2]). Let (X, d) be a b-metric space and $\{x_n\}$ be a sequence of X such that

(1) $\{x_n\}$ is convergent if there exists an x in X such that for any $\varepsilon > 0$, there exists an $n(\varepsilon) \in \mathbb{N}$, such that $n \geqslant n(\varepsilon)$, $d(x_n, x) < \varepsilon$.
(2) $\{x_n\}$ is a Cauchy sequence if for any $\varepsilon > 0$, there exists an $n(\varepsilon) \in \mathbb{N}$, such that for all $m, n \geqslant n(\varepsilon)$, $d(x_n, x_m) < \varepsilon$.
(3) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

In contrast to the general metric, b-metric is not continuous. However we introduce the following lemma.

Lemma 1.4 ([21]). *Let (X, d) be a b-metric space with the constant $K \geqslant 1$, and suppose that the sequences $\{x_n\}$ and $\{y_n\}$ converge to x, y, respectively. Then*

$$\frac{1}{K^2} d(x, y) \leqslant \underline{\lim}_{n\to\infty} d(x_n, y_n) \leqslant \overline{\lim}_{n\to\infty} d(x_n, y_n) \leqslant K^2 d(x, y).$$

In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$,

$$\frac{1}{K} d(x, z) \leqslant \underline{\lim}_{n\to\infty} d(x_n, z) \leqslant \overline{\lim}_{n\to\infty} d(x_n, z) \leqslant K d(x, z).$$

In order to study fixed point theorems of the multi-valued mapping, we introduce the concept of Hausdorff metric.

Definition 1.5. Let (X, d) be a metric space and $CB(X)$ be the class of all nonempty closed and bounded subsets of X. For any $A, B \in CB(X)$, set

$$H(A, B) = \max\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\},$$

where $d(x, B) = \inf_{y\in B} d(x, y)$, then $(CB(X), H)$ is a metric space and $H(A, B)$ is called a Hausdorff metric.

Similarly, if (X, d) is a b-metric space, then $(CB(X), H)$ is a b-metric space. $H(A, B)$ is called a b-Hausdorff metric on $CB(X)$. In the following, unless stated in particular, $H(A, B)$ will always denote a b-Hausdorff metric.

Remark 1.6. Suppose that (X, d) is a metric space, then $H(A, B){=}0$ iff $A = B$.

Definition 1.7 ([4])**.** Let X be a b-metric space and $T : X \to CB(X)$ be a multi-valued operator. If there exists $k \in [0, 1]$ such that $H(Tx, Ty) \leqslant kd(x, y)$ for all $x, y \in X$, then T is called a contractive multi-valued operator.

Definition 1.8. Let (X, d) be a b-metric space and $T : X \to CB(X)$ be a multi-valued operator. If there exist constants s, r with $r \in [0, 1], s \geqslant r$ such that for all $x, y \in X$,

$$d(y, Tx) \leqslant Ksd(x, y) \Rightarrow H(Tx, Ty) \leqslant rM_T(x, y),$$

where

$$M_T(x, y) = \max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{d(x, Ty) + d(y, Tx)}{2K}\right\},$$

then T is called a (s, r)-contractive multi-valued operator.

The purpose of this paper is to generalize the results of Kamran [12] and introduce the notion of weakly (s, r)-contractive multi-valued operator and establish some fixed point theorems for this operator on b-metric space.

2. Main results

In this section we introduce the notion of weakly (s, r)-contractive multi-valued operator and present our results. We start this section with the following definition.

Definition 2.1. Let (X, d) be a b-metric space and $T : X \to CB(X)$ be a multi-valued operator. If there exist $r \in [0, 1]$ and $s \geqslant r, L \geqslant 0$ such that for any $x, y \in X$,

$$d(x, Ty) \leqslant Ksd(x, y) \Rightarrow H(Tx, Ty) \leqslant rM_T(x, y) + L\min\{d(x, y),\ d(y, Tx)\},$$

where

$$M_T(x, y) = \max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{d(x, Ty) + d(y, Tx)}{2K}\right\},$$

then T is a weakly (s, r)-contractive multi-valued operator on X.

Remark 2.2. When $L = 0$, the above definition reduces to Definition 1.8.

The following example shows that the notion of weakly (s, r)-contractive operator properly generalizes the notion of (s, r)-contractive operator.

Example 2.3. Let $X = \{1, 2, 3\}$ endowed with the b-metric $d(x, y) = |x - y|^2$. Then (X, d) is a complete b-metric space with the constant $K = 2$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{1, 2\}, & x \in \{1, 2\}, \\ \{3\}, & x = 3. \end{cases}$$

Then $H(T1, T1) = H(T2, T2) = H(T3, T3) = H(T1, T2) = H(T2, T1) = 0$. By choosing $s = 0.4$,

$$d(1, T3) = 4 > 3.2 = K \cdot s \cdot d(1, 3), \quad d(2, T3) = 1 > 0.8 = K \cdot s \cdot d(2, 3), \quad d(3, T2) = 1 < 0.8 = K \cdot s \cdot d(3, 2).$$

Further, $d(3, T1) = 1 < 3.2 = K \cdot s \cdot d(3, 1)$. Now if we choose $L = 1$ and $r = 0.2$, then

$$H(T3, T1) = 1 < 4.8 = r\max\left\{d(3, 1),\ d(3, T3),\ d(1, T1),\ \frac{d(3, T1) + d(1, T3)}{2K}\right\} + L\min\{d(3, 1),\ d(1, T3)\}.$$

This shows that T is weakly $(0.4, 0.2)$-contractive map with $L = 1$, but not $(0.4, 0.2)$-contractive. Since

$d(3,T1) = 1 < 3.2 = Ksd(3,1)$ but

$$H(T3,T1) = 1 > 0.8 = r\max\left\{d(3,1),\ d(3,T3),\ d(1,T1),\ \frac{d(3,T1)+d(1,T3)}{2K}\right\}.$$

Lemma 2.4 ([14, 21]). *Let (X,d) be a complete b-metric space with the constant $K \geqslant 1$ and $\{x_n\}$ be a sequence in X such that $d(x_{n+1},x_{n+2}) \leqslant \alpha d(x_n,x_{n+1})$ for all $n = 0,1,2,\ldots$, where $0 \leqslant \alpha < 1$. If $K\alpha < 1$, then $\{x_n\}$ is a Cauchy sequence in X.*

The following theorem generalizes the result of Kamran and Hussain [12] to the setting of b-metric space.

Theorem 2.5. *Let (X,d) be a complete b-metric space and $T : X \to CB(X)$ be a weakly (s,r)-contractive operator with $r < \min\{\frac{1}{K}, s\}$. Then T has fixed points.*

Proof. Take a real number $r_1 > 1$ such that $0 \leqslant r < r_1 < \min\{\frac{1}{K}, s\}$. Let $x_1 \in X$ and $x_2 \in Tx_1$. Then $d(x_2,Tx_1) = 0 \leqslant Ksd(x_2,x_1)$ and using hypothesis,

$$\begin{aligned} d(x_2,Tx_2) \leqslant H(Tx_1,Tx_2) &\leqslant rM_T(x_1,x_2) + L\min\{d(x_1,x_2),\ d(x_2,Tx_1)\} \\ &= r\max\left\{d(x_1,x_2),\ d(x_2,Tx_2),\ d(x_1,Tx_1),\ \frac{d(x_1,Tx_2)+d(x_2,Tx_1)}{2K}\right\} \\ &\leqslant r\max\left\{d(x_1,x_2),\ d(x_2,Tx_2),\ \frac{d(x_1,x_2)+d(x_2,Tx_2)}{2}\right\}. \end{aligned}$$

(1) If $d(x_1,x_2) \leqslant d(x_2,Tx_2)$, then $d(x_2,Tx_2) \leqslant rd(x_2,Tx_2)$. Since $r < 1$, we have $d(x_2,Tx_2) = 0$, and x_2 is a fixed point of T.
(2) If $d(x_1,x_2) > d(x_2,Tx_2)$, then $d(x_2,Tx_2) \leqslant rd(x_1,x_2)$. Since $r < 1$, it follows that there exists $x_3 \in Tx_2$ such that $d(x_2,x_3) \leqslant r_1 d(x_1,x_2)$. Continuing in this manner a sequence $\{x_n\}$ can be constructed in X such that $x_{n+1} \in Tx_n$ and $d(x_{n+1},x_{n+2}) \leqslant r_1 d(x_n,x_{n+1})$ for all $n \in \mathbb{N}$.

Since $Kr_1 < 1$, it implies $\{x_n\}$ is a Cauchy sequence by using Lemma 2.4. Since X is a complete, there is $z \in X$ such that $\{x_n\}$ converges to z. Now, we claim that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(z,Tx_{n_k}) \leqslant Ksd(z,x_{n_k}),\ \forall k \in \mathbb{N}.$$

If not, there exists a positive integer $N \in \mathbb{N}$ such that

$$d(z,Tx_n) > Ksd(z,x_n),\ \forall n \geqslant N.$$

This implies

$$d(z,x_{n+1}) > Ksd(z,x_n,),\ \forall n \geqslant N.$$

By induction, we obtain

$$d(z,x_{n+p}) > (Ks)^p d(z,x_n),\ \forall n \geqslant N, p \geqslant 1. \tag{2.1}$$

Since

$$\begin{aligned} d(x_n,x_{n+p}) &\leqslant K(d(x_n,x_{n+1}) + d(x_{n+1},x_{n+p})) \\ &\leqslant Kd(x_n,x_{n+1})(1 + Kr_1 + \cdots + K^{p-1}r_1^{p-1} \\ &= \frac{K[1-(Kr_1)^p]}{1-Kr_1} d(x_n,x_{n+1}),\quad \forall n \geqslant N, p \geqslant 1. \end{aligned}$$

Let $p \to \infty$, using Lemma 1.4,

$$\frac{1}{K} d(z,x_n) \leqslant \underline{\lim}_{p\to\infty} d(x_n,x_{n+p}) \leqslant \frac{K}{1-Kr_1} d(x_n,x_{n+1}),\ \forall n \geqslant N.$$

Thus

$$d(z, x_{n+p}) \leqslant \frac{K^2}{1-Kr_1} d(x_{n+p}, x_{n+p+1}) \leqslant \frac{K^2 r_1^p}{1-Kr_1} d(x_n, x_{n+1}), \forall n \geqslant N, p \geqslant 1. \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$d(z, x_n) < \frac{K^2 r_1^p}{(Ks)^p(1-Kr_1)} d(x_n, x_{n+1}).$$

Set $p \to \infty$, $d(z, x_n) = 0$, $\forall n \geqslant N$, which contradicts to (1). Therefore there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(z, Tx_{n_k}) \leqslant Ksd(z, x_{n_k}), \ \forall k \in \mathbb{N}.$$

Thus

$$d(x_{n_k+1}, Tz) \leqslant H(Tx_{n_k}, Tz) \leqslant r \max\left\{d(z, x_{n_k}),\ d(z, Tz),\ d(x_{n_k}, Tx_{n_k}),\ \frac{d(z, Tx_{n_k}) + d(z, Tz)}{2K}\right\} + L\min\{d(x_{n_k}, z),\ d(x_{n_k}, Tz)\}.$$

Letting $k \to \infty$,

$$\overline{\lim_{k\to\infty}}\, d(x_{n_k+1}, Tz) \leqslant r\max\left\{d(z, Tz), \frac{d(z, Tz)}{2K}\right\} = rd(z, Tz).$$

By the triangle inequality,

$$d(z, Tz) \leqslant K[d(z, x_{n_k+1}) + d(x_{n_k+1}, Tz)].$$

Thus

$$\lim_{k\to\infty} \frac{1}{K} d(z, Tz) \leqslant \overline{\lim_{k\to\infty}}\, [d(z, x_{n_k+1}) + d(x_{n_k+1}, Tz)], \quad \frac{1}{K} d(z, Tz) \leqslant \overline{\lim_{k\to\infty}}\, d(x_{n_k+1}, Tz) \leqslant rd(z, Tz).$$

As $Kr < 1$, $d(z, Tz) = 0$. Since $Tz \in CB(X), z \in Tz$, T has fixed point. □

From the following example, one can see that under the condition of Theorem 2.5, the fixed point may not be unique.

Example 2.6. Let $X = [1, \infty)$ and $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then d is a complete b-metric but not a metric on X with the constant $K = 2$. Define $T : X \to CB(X)$ by

$$Tx = [2, 2 + \frac{x}{3}]$$

for all $x \in X$. Consider $H(Tx, Ty) = \frac{1}{9}(x-y)^2 = \frac{1}{9}d(x, y)$, where we choose $r = \frac{1}{9} \in [0, 1)$, $s = \frac{1}{5} > r$, $L = 1 \geqslant 0$. Then the conditions of Theorem 2.5 are satisfied. Moreover, 2 and 3 are the two fixed points of T.

It is necessary for us to consider the uniqueness of the fixed point of the weakly (s, r)-contractive multi-valued operator.

Corollary 2.7. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a weakly (s, r)-contractive single-valued operator with $r < \min\{\frac{1}{K}, s\}$. Then T has a fixed point. Moreover, if $Ks \geqslant 1$ and $r + L < 1$, then T has a unique fixed point.*

Proof. From Theorem 2.5, T has a fixed point. Let $Ks \geqslant 1$ and $(r + L) < 1$. Suppose that there exist two different fixed points x and y of T. Then

$$d(y, Tx) = d(y, x) \leqslant Ksd(y, x).$$

Thus

$$\begin{aligned}
d(Tx,Ty)) &\leqslant rM_T(x,y)+L\min\{d(x,y),\ d(y,Tx)\},\\
d(x,y) &\leqslant rM_T(x,y)+L\min\{d(x,y),\ d(y,Tx)\}\\
&= r\max\left\{d(x,y),\ d(x,Tx),\ d(y,Ty),\ \frac{d(x,Ty)+d(y,Tx)}{2K}\right\}+L\min\{d(x,y),d(y,Tx)\}\\
&= rd(x,y)+Ld(x,y)=(r+L)d(x,y).
\end{aligned}$$

It is a contradiction, since $(r+L)<1$. □

Next, we introduce the other theorem about the weakly (s,r)-contractive multi-valued operator.

Theorem 2.8. *Let* (X,d) *be a complete* b*-metric space and* $T: X\to CB(X)$ *be a multi-valued operator. Assume that there exist constants* $r,\ s\in[0,1)$ *and* $r<s<\frac{1}{K}$ *such that*

$$\frac{1}{K(1+Kr)}d(x,Tx)\leqslant d(x,y)\leqslant \frac{K^2}{1-Ks}d(Tx,x)$$

implies

$$H(Tx,Ty)\leqslant rM_T(x,y)+L\min\{d(x,y),d(y,Tx)\},$$

where

$$M_T(x,y)=\max\left\{d(x,y),\ d(x,Tx),\ d(y,Ty),\ \frac{d(x,Ty)+d(y,Tx)}{2K}\right\}.$$

Then T *has a fixed point.*

Proof. Take a real number r_1 such that $0\leqslant r<r_1<s<\frac{1}{K}$. Since $\frac{1-Kr_1}{1-Ks}>1$, it follows that for $x_1\in X$ there exists $x_2\in Tx_1$ such that

$$d(x_1,x_2)\leqslant \frac{1-Kr_1}{1-Ks}d(x_1,Tx_1).$$

Then

$$\frac{1}{K(1+Kr)}d(x_1,Tx_1)\leqslant d(x_1,Tx_1)\leqslant d(x_1,x_2)\leqslant \frac{1}{1-Ks}d(x_1,Tx_1)\leqslant \frac{K^2}{1-Ks}d(x_1,Tx_1),$$

and by hypothesis

$$\begin{aligned}
d(x_1,Tx_2)\leqslant H(Tx_1,Tx_2)&\leqslant rM_T(x_1,x_2)+L\min\{d(x_1,x_2),\ d(x_2,Tx_1)\}\\
&\leqslant r\max\left\{d(x_1,x_2),\ d(x_2,Tx_2),\ \frac{d(x_1,x_2)-d(x_2,Tx_2)}{2K}\right\}\\
&\leqslant r\max\left\{d(x_1,Tx_1),\ d(x_2,Tx_2),\ \frac{d(x_1+x_2)+d(x_2,Tx_2)}{2}\right\}.
\end{aligned}$$

(1) If $d(x_1,x_2)\leqslant d(x_2,Tx_2)$, then $d(x_2,Tx_2)\leqslant rd(x_2,Tx_2)$. Since $r<1$, we have $d(x_2,Tx_2)=0$. Then x_2 is the fixed point of T.

(2) If $d(x_1,x_2)>d(x_2,Tx_2)$, then $d(x_2,Tx_2)\leqslant rd(x_1,x_2)$. Since $r<1$, it follows that there exists $x_3\in Tx_2$ such that

$$d(x_2,x_3)\leqslant r_1d(x_1,x_2),\quad d(x_2,x_3)\leqslant \frac{1-Kr_1}{1-Ks}d(x_2,Tx_2).$$

Therefore a sequence $\{x_n\}$ can be constructed in X such that $x_{n+1}\in Tx_n$ and

$$\begin{aligned}
d(x_{n+1},x_{n+2})&\leqslant r_1d(x_n,x_{n+1}),\ \forall n\in\mathbb{N},\\
d(x_n,x_{n+1})&\leqslant \frac{1-Kr_1}{1-Ks}d(x_n,Tx_n),\ \forall n\in\mathbb{N}.
\end{aligned}\tag{2.3}$$

Since $Kr_1 < 1$, it implies $\{x_n\}$ is a Cauchy sequence by using Lemma 2.4. Since X is complete, there is $z \in X$ such that x_n converges to z, that is

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

Since

$$d(x_{n+p}, x_n) \leqslant Kd(x_n, x_{n+1}) + K^2 d(x_{n+1}, x_{n+2}) + \cdots + K^p d(x_{n+p-1}, x_{n+p}),$$
$$d(x_{n+p}, x_n) \leqslant Kd(x_n, x_{n+1})(1 + Kr_1 + K^2 r_1^2 + \cdots + K^{p-1} r_1^{p-1}) = \frac{K[1-(Kr_1)^p]}{1-Kr_1} d(x_n, x_{n+1}), \ \forall n \geqslant N, p \geqslant 1.$$

Set $p \to \infty$,

$$\frac{1}{K} d(z, x_n) \leqslant \overline{\lim_{p\to\infty}}\, d(x_{n+p}, x_n) \leqslant \frac{K}{1-Kr_1} d(x_n, x_{n+1}).$$

Thus

$$d(z, x_n) \leqslant \frac{K^2}{1-Kr_1} d(x_n, x_{n+1}), \quad \forall n \geqslant 1.$$

From (2.3),

$$d(z, x_n) \leqslant \frac{K^2}{1-Ks} d(x_n, Tx_n), \ \forall n \in \mathbb{N}. \tag{2.4}$$

Now suppose that there exists $N > 0$ such that

$$d(z, x_n) \leqslant \frac{1}{K(1+Kr)} d(x_n, Tx_n), \quad \forall n \geqslant N.$$

Therefore

$$\begin{aligned} d(x_n, x_{n+1}) \leqslant K(d(x_n, z) + d(z, x_{n+1})) &< \frac{1}{1+Kr}[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})] \\ &\leqslant \frac{1}{1+Kr}[d(x_n, Tx_n) + rd(x_n, x_{n+1})]. \end{aligned}$$

This implies

$$d(x_n, x_{n+1}) < d(x_n, Tx_n),$$

which is impossible. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(z, x_{n_k}) > \frac{1}{K(1+Kr)} d(x_{n_k}, Tx_{n_k}), \quad \forall k \geqslant \ N. \tag{2.5}$$

From (2.4) and (2.5) and using the hypothesis,

$$\begin{aligned} d(x_{n_k+1}, Tz) \leqslant H(Tx_{n_k}, Tz) &\leqslant rM_T(x_{n_k}, z) + L\min\{d(x_{n_k}, z),\ d(x_{n_k}, Tz)\} \\ &= r\max\left\{d(x_{n_k}, z),\ d(x_{n_k}, Tx_{n_k}),\ d(z, Tz),\ \frac{d(x_{n_k}, Tz) + d(z, Tx_{n_k}}{2K}\right\} \\ &\quad + L\min\{d(x_{n_k}, z),\ d(x_{n_k}, Tz)\}. \end{aligned}$$

Therefore

$$\frac{1}{K} d(z, Tz) \leqslant \overline{\lim_{k\to\infty}}\, d(x_{n_k+1}, Tz) \leqslant r\max\left\{d(z, Tz), \frac{d(z, Tz)}{2K}\right\} = rd(z, Tz).$$

As $Kr < 1$, we get $d(z, Tz) = 0$. Since $Tz \in CB(X), z \in Tz$, T has the fixed point. □

Corollary 2.9. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a weakly (s, r)-contractive single-valued operator. Assume there exists $r \in [0, 1)$ and $r < \frac{1}{K}$ such that $\forall x, y \in X$*

$$\frac{1}{K(1+Kr)} d(x, Tx) \leqslant d(x, y) \leqslant \frac{K^2}{1-Kr} d(x, Tx)$$
$$\Rightarrow H(Tx, Ty) \leqslant r M_T(x, y) + L \min\{d(x, y),\ d(y, Tx)\},\ \ \forall x, y \in X,$$

where

$$M_T(x, y) = \max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{d(x, Ty) + d(y, Tx)}{2K}\right\}.$$

Then there exists $z \in X$ such that $Tz = z$.

Proof. For every $x_1 \in X$ the sequence $\{x_n\}$ is defined by $x_{n+1} = Tx_n$. One can easily prove that $d(x_{n+1}, x_{n+2}) \leqslant r d(x_n, x_{n+1})$ and $\{x_n\}$ is a Cauchy sequence. Then there is a point $z \in X$ such that $\lim_{n\to\infty} x_n = z$. From above theorem we have $d(x_n, z) \leqslant \frac{K^2}{1-Kr} d(x_n, x_{n+1})$ for all $n \geqslant 1$ and there exists a subsequence $\{x_{n_k}\}$ such that

$$d(z, x_{n_k}) \geqslant \frac{1}{K(1-Kr)} d(x_{n_k}, x_{n_k+1}),\quad \forall k \geqslant N.$$

Therefore

$$d(x_{n_k+1}, Tz) \leqslant H(Tx_{n_k}, Tz) \leqslant r \max\left\{d(x_{n_k}, z),\ d(x_{n_k}, Tx_{n_k}),\ d(z, Tz),\ \frac{d(x_{n_k}, Tz) + d(z, Tx_{n_k})}{2K}\right\}$$
$$+ L \min\{d(x_{n_k}, z),\ d(x_{n_k}, Tz)\}.$$

Letting $k \to \infty$, using the triangle inequality,

$$\frac{1}{K} d(z, Tz) \leqslant \overline{\lim_{k\to\infty}}\, d(x_{n_k+1}, Tz) \leqslant r \max\left\{d(z, Tz),\ \frac{d(z, Tz)}{2K}\right\}.$$

Then we get $d(z, Tz) = 0$ as $Kr < 1$. Since $Tz \in CB(X)$, $z \in Tz$, T has a fixed point. □

3. Application

For fixed point theorems, there are a number of applications in differential equations and integral equations.

Let X be a set of the continuous functions on the closed interval $[a, b]$ and we define the b-metric by

$$d(x, y) = \max_{t\in[a,b]} |x(t) - y(t)|^2,\ \ \forall x, y \in X.$$

Then (X, d) is a complete b-metric space with the constant $K = 2$.

Consider the differential equation

$$\begin{cases} \frac{dx}{dy} = f(x, y), \\ y(x_0) = y_0. \end{cases} \tag{3.1}$$

The equation (3.1) is equivalent to the following integral equation,

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(t))dt. \tag{3.2}$$

We choose a constant $0 < \delta < 1$, and define a map T on the continuous functional space $C[x_0 - \delta, x_0 + \delta]$ by

$$Ty(x) = y_0 + \int_{x_0}^{x} f(x, y(t))dt.$$

Then the integral equation (3.2) has a solution which is equivalent to that the map T has a fixed point. Now we suppose that

(1) there exist constants $r \in [0,1]$, $s > 0$ and $r < \min\{\frac{1}{2}, s\}$, such that for all $y_1, y_2 \in X$,

$$|y_2 - [y_0 + \int_{x_0}^{x} f(x, y(t))dt]|^2 \leqslant 2s|y_1 - y_2|^2 \Rightarrow |f(z, y_1) - f(z, y_2)|^2 \leqslant r|y_1 - y_2|^2.$$

We have

$$\begin{aligned} d(Ty_1, Ty_2) &= \max_{|x-x_0|<\delta} |\int_{x_0}^{x} [f(t, y_1(t)) - f(t, y_2(t))]dt|^2 \\ &\leqslant \max_{|x-x_0|<\delta} \int_{x_0}^{x} |[f(t, y_1(t)) - f(t, y_2(t))]|^2 dt \\ &\leqslant \max_{|x-x_0|<\delta} \int_{x_0}^{x} r|y_1(t) - y_2(t)|^2 dt \\ &\leqslant r\delta \max_{|t-x_0|<\delta} |y_1(t) - y_2(t)|^2 \\ &= r\delta d(y_1(t), y_2(t)) \\ &\leqslant rM_T(y_1(t), y_2(t)) + L\min\{d(y_1(t), y_2(t)), d(y_2(t), Ty_1(t))\}, \end{aligned}$$

where

$$M_T(y_1, y_2) = \max\left\{d(y_1, y_2), d(y_1, Ty_1), d(y_2, Ty_2), \frac{d(y_1, Ty_2) + d(y_2, Ty_1)}{4}\right\}.$$

Then T satisfies the conditions of Theorem 2.5 and T has a fixed point. So there exists a continuous function $y_0(t)$ such that

$$y_0(t) = \int_{x_0}^{x} f(x, y_0(t))dt, \ \forall x \in [x_0 - \delta, x_0 + \delta].$$

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References

[1] S. Banach, *Sur les operations dans les ensembles abstraits et leures applications aux equations integrales*, Fundam. Math., **3** (1922), 133–181. 1

[2] M. Boriceanu, M. Bota, A. Petruşel, *Multivalued fractals in b-metric spaces*, Cent. Eur. J. Math., **8** (2010), 367–377. 1, 1.3

[3] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1** (1993), 5–11. 1, 1.1

[4] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Semin. Mat. Fis. Univ. Modena, **46** (1998), 263–276. 1, 1.7

[5] P. N. Dutta, B. S. Choudhury, *A generalization of contraction principle in metric spaces*, Fixed Point Theory Appl., **2008** (2008), 8 pages. 1

[6] N. Hussain, D. Dorić, Z. Kadelburg, S. Radenović, *Suzuki-type fixed point results in metric type spaces*, Fixed Point Theory Appl., **2012** (2012), 12 pages. 1

[7] N. Hussain, V. Parvaneh, J. R. Roshan, Z. Kadelburg, *Fixed points of cyclic weakly (ψ, ϕ, L, A, B)-contractive mappings in ordered b-metric spaces with applications*, Fixed Point Theory Appl., **2013** (2013), 18 pages.

[8] N. Hussain, N. Yasmin, N. Shafqat, *Multi-valued Ćirić contractions on metric spaces with applications*, Filomat, **28** (2014), 1953–1964. 1

[9] H. Işık, D. Türkoğlu, *Fixed point theorems for weakly contractive mappings in partially ordered metric-like spaces*, Fixed Point Theory Appl., **2013** (2013), 12 pages. 1

[10] H. Işık, D. Türkoğlu, *Coupled fixed point theorems for new contractive mixed monotone mappings and applications to integral equations*, Filomat, **28** (2014), 1253–1264.

[11] H. Işık, D. Türkoğlu, *Generalized weakly alfa-contractive mappings and applications to ordinary differential equations*, Miskolc Math. Notes, **17** (2016), 365–379. 1

[12] T. Kamran, S. Hussain, *Weakly (s, r)-contractive multi-valued operators*, Rend. Circ. Mat. Palermo., **64** (2015), 475–482. 1, 1, 2

[13] J. T. Markin, *Continuous dependence of fixed point sets*, Proc. Amer. Math. Soc., **38** (1973), 545–547. 1

[14] R. Miculescu, A. Mihail, *New fixed point theorems for set-valued contractions in b-metric spaces*, J. Fixed Point Theory Appl., **19** (2017), 2153–2163. 2.4

[15] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488. 1

[16] J. J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation*, Order, **22** (2005), 223–239. 1

[17] O. Popescu, *A new type of contractive multivalued operators*, Bull. Sci. Math., **137** (2013), 30–44. 1

[18] S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital., **5** (1972), 26–42. 1

[19] S. Reich, A. J. Zaslavski, *Genericity in Nonlinear Analysis*, Springer, New York, (2014). 1

[20] B. H. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., **47** (2001), 2683–2693. 1

[21] J. R. Roshan, V. Parvaneh, Z. Kadelburg, *Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces*, J. Nonlinear Sci. Appl., **7** (2014), 229–245. 1, 1.4, 2.4

[22] J. R. Roshan, V. Parvaneh, Z. Kadelburg, N. Hussain, *New fixed point results in b-rectangular metric spaces*, Nonlinear Anal. Model. Control, **21** (2016), 614–634.

[23] I. A. Rus, *Basic problems of the metric fixed point theory revisited (II)*, Studia Univ. Babeş-Bolyai Math., **36** (1991), 81–89. 1

[24] S. L. Singh, S. Czerwik, K. Król, A. Singh, *Coincidences and fixed points of hybrid contractions*, Tamsui Oxf. J. Math. Sci., **24** (2008), 401–416. 1

[25] L. Wang, *The fixed point method for intuitionistic fuzzy stability of a quadratic functional equation*, Fixed Point Theory Appl., **2010** (2010), 7 pages. 1