



## Existence of solutions for a class of second-order impulsive Hamiltonian system with indefinite linear part



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### Abstract

We consider a class of second-order impulsive Hamiltonian system with indefinite linear part. By using saddle point theorem in critical point theory, an existence result is obtained, which extends and improves some existing results.

**Keywords:** Impulsive Hamiltonian system, saddle point theorem, solutions, existence.

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### 1. Introduction

Consider the following second-order impulsive Hamiltonian system with indefinite linear part

$$\begin{cases} \ddot{u}(t) - A(t)u(t) = -\nabla F(t, u(t)), & t \in [0, 1] \setminus \{t_1, t_2, \dots, t_m\}, \\ u(0) = u(1) = 0, u(t_i^+) = u(t_i^-), i = 1, \dots, m, \\ \dot{u}(t_i^+) - \dot{u}(t_i^-) = I_i(u(t_i)), i = 1, \dots, m, \end{cases} \quad (1.1)$$

where  $A \in C([0, 1], \mathbb{R}^{N \times N})$  is a continuous map with values in the space of  $N \times N$  symmetric matrices,  $u \in \mathbb{R}^N$ ,  $F : (0, 1) \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a Carathéodory function,  $I_i$  are continuous functions on  $\mathbb{R}^N$ , and

$$u(t_i^\mp) = \lim_{t \rightarrow t_i^\mp} u(t), \quad \dot{u}(t_i^\mp) = \lim_{t \rightarrow t_i^\mp} \dot{u}(t).$$

When  $A(t) = 0$  and  $u(t)$  is defined in  $(0, 1)$  with values in  $\mathbb{R}$ , (1.1) reduced to the following impulsive problems

$$\begin{cases} \ddot{u}(t) = -\nabla F(t, u(t)), & t \in [0, 1] \setminus \{t_1, t_2, \dots, t_m\}, \\ u(0) = u(1) = 0, u(t_i^+) = u(t_i^-), i = 1, \dots, m, \\ \dot{u}(t_i^+) - \dot{u}(t_i^-) = I_i(u(t_i)), i = 1, \dots, m. \end{cases} \quad (1.2)$$

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Impulsive problems arise naturally in real world. As we know, many mathematical models involve impulses. Since impulsive problems are important in real world, many researchers have extensively studied the theory and applications of impulsive differential equations, see [2, 4, 5, 8, 11] for more details. In recent years, variational methods have been widely used to study Hamiltonian system and impulsive problems, see [1, 3, 6, 7, 9, 12–17] and references therein. In [7], Kyritsi and Papageorgion investigated problem (1.2) and obtained an existence result by using Morse critical groups. In [1], Agarwal, Bhaskar and Perea studied problem (1.2) via Morse theory and got some results. Later, Wang and Wang [13] obtained an existence result for problem (1.2) under some conditions on  $\nabla F(t, u(t))$  by using saddle point theorem, which extends and improves the results in [1] and [7].

Motivated mainly by [1, 7, 13], we consider problem (1.1) with indefinite linear part by saddle point theorem and obtain an existence result, which seems not to be considered previously. For convenience,  $C_i$  denote different positive constants in the following.

Suppose the following hypotheses hold:

(H1)  $\nabla F(t, u)$  is asymptotically piecewise linear and

$$\nabla F(t, u) = \sum_{i=1}^{m+1} a_i \chi_i(t) u + g(t, u),$$

where  $a_1, \dots, a_{m+1} \in \mathbb{R}$ ,

$$\chi_i(t) = \begin{cases} 1, & t \in (t_{i-1}, t_i), \\ 0, & t \in (0, 1) \setminus (t_{i-1}, t_i) \end{cases}$$

is the characteristic function of the subinterval  $(t_{i-1}, t_i)$  and there are  $r \in (1, 2)$  and two positive constants  $C_1$  and  $C_2$  such that

$$|g(t, u)| \leq C_1 |u|^{r-1} + C_2 \quad \text{for a.e. } t \in [0, 1] \text{ and all } u \in \mathbb{R}^N. \quad (1.3)$$

(H2) There are positive constants  $C_3, C_4$  and  $\mu > 2$  such that

$$(u, I_i(u)) \geq C_3 |u|^\mu - C_4, \quad \forall u \in \mathbb{R}^N, \quad i = 1, 2, \dots, m. \quad (1.4)$$

We only consider the nonresonant case, where for all  $i \in \{1, \dots, m+1\}$ ,  $a_i$  is not in the set

$$\Sigma_i = \{\lambda_k^i : 0 \leq \lambda_1^i \leq \lambda_2^i \leq \dots \leq \lambda_k^i \rightarrow +\infty, k = 1, 2, \dots\}$$

of the eigenvalues of the problem

$$\begin{cases} -\ddot{u}(t) + A(t)u(t) = \lambda u(t), & t \in (t_{i-1}, t_i), \\ u(t_{i-1}) = u(t_i) = 0, & i = 1, \dots, m+1, \end{cases}$$

The main result is the following.

**Theorem 1.1.** Assume that (H1) and (H2) hold, and  $a_i \notin \Sigma_i$  for  $i = 1, 2, \dots, m+1$ , then problem (1.1) has one nontrivial homoclinic solution.

## 2. Preliminaries

Let  $W$  be the Sobolev space defined by

$$W = W_{\text{per}}^{1,2}((0, 1), \mathbb{R}^N) = \{u \in W^{1,2}((0, 1), \mathbb{R}^N) : u(0) = u(1)\}$$

with the inner product

$$(u, v) = \int_0^1 [(\dot{u}(t), \dot{v}(t)) + (A(t)u(t), v(t))] dt.$$

The (compact) embedding of  $W^{1,2}((0, 1), \mathbb{R}^N)$  into  $C([0, 1], \mathbb{R}^N)$  shows that this definition makes sense. It

is easy to see that weak solutions of problem (1.1) coincide with the critical points of the  $C^1$ -functional

$$\varphi(u) = \frac{1}{2} \int_0^1 [|\dot{u}(t)|^2 + (A(t)u(t), u(t))] dt - \int_0^1 F(t, u(t)) dt - \sum_{i=1}^m \hat{I}_i(u(t_i)), \quad u \in W,$$

where  $\hat{I}_i(u(t_i)) = \int_0^{u(t_i)} I(s) ds$ . Since every  $\hat{I}_i(0) = 0$  for  $i = 1, 2, \dots, m$ , the closed linear subspace

$$\mathcal{N} = \{u \in W : u(t_i) = 0, i = 1, 2, \dots, m\}$$

is important here. For  $i = 1, 2, \dots, m$ , the mapping  $W \mapsto \mathbb{R}^N$ ,  $u \mapsto u(t_i)$  is a bounded linear functional on  $W$  and hence there is a unique  $w_i$  such that  $u(t_i) = (u, w_i)$  by the Riesz-Frechet representation theorem. In fact,

$$w_i(t) = \begin{cases} (1 - t_i)t, & t \in [0, t_i], \\ t_i(1 - t), & t \in [t_i, 1]. \end{cases} \quad (2.1)$$

Since  $t_i$  are distinct,  $w_i$  are linearly independent, so  $\mathcal{N}$  is the orthogonal complement of the  $m$ -dimensional subspace  $\mathcal{M}$  that they span. Hence we have the orthogonal decomposition

$$W = \mathcal{N} \oplus \mathcal{M}, \quad u = v + w,$$

and

$$\begin{aligned} \varphi(u) = & \frac{1}{2} \int_0^1 [|\dot{v}(t)|^2 + (A(t)v(t), v(t))] dt + \frac{1}{2} \int_0^1 [|\dot{w}(t)|^2 + (A(t)w(t), w(t))] dt \\ & - \int_0^1 F(t, u(t)) dt - \sum_{i=1}^m \hat{I}_i(u(t_i)), \quad u \in W. \end{aligned} \quad (2.2)$$

By (2.1), each  $w \in \mathcal{M}$  is affine on the subintervals  $[t_{i-1}, t_i]$ . Since the space of continuous functions on  $[0, 1]$  that are affine on these subintervals and vanish at the endpoints is also  $m$ -dimensional, hence  $\mathcal{M}$  is precisely this space. Then we also have

$$\max_{t \in [0, 1]} |w(t)| = \max_{i=1, \dots, m} |w(t_i)|, \quad \forall w \in \mathcal{M},$$

and this is an equivalent norm on this finite dimensional space.

The subspace  $\mathcal{N}$  has the decomposition

$$\mathcal{N} = \bigoplus_{i=1}^{m+1} \mathcal{N}_i, \quad v = \sum_{i=1}^{m+1} v_i,$$

where  $\mathcal{N}_i = W^{1,2}((t_{i-1}, t_i), \mathbb{R}^N)$ ,  $v_i = \chi_i v$ . Combining this with (2.2) gives

$$\begin{aligned} \varphi(u) = & \frac{1}{2} \left[ \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (|\dot{v}(t)|^2 + (A(t)v(t), v(t))) dt \right] + \frac{1}{2} \int_0^1 [|\dot{w}(t)|^2 + (A(t)w(t), w(t))] dt \\ & - \int_0^1 F(t, u(t)) dt - \sum_{i=1}^m \hat{I}_i(u(t_i)) \\ = & \frac{1}{2} \|v\|^2 + \frac{1}{2} \|w\|^2 - \int_0^1 F(t, u(t)) dt - \sum_{i=1}^m \hat{I}_i(u(t_i)), \quad u \in W. \end{aligned} \quad (2.3)$$

The following lemma is saddle point theorem, which comes from [10] and is useful in the proof of our theorem.

**Lemma 2.1 ([10]).** Let  $E = V \oplus X$ , where  $E$  is a real Banach space and  $V \neq \{0\}$  is finite dimensional. Suppose  $I \in C^1(E, \mathbb{R})$ , satisfies (PS)-condition and

- (i) there is a constant  $\alpha > 0$  and a bounded neighborhood  $B$  of 0 in  $V$  such that  $I_{\partial B} \leq \alpha$ , and
- (ii) there is a constant  $\beta > \alpha$  such that  $I|_X \geq \beta$ .

Then  $I$  possesses a critical value  $c \geq \beta$ . Moreover  $c$  can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in \bar{B}} I(h(u)),$$

where  $\Gamma = \{h \in C(\bar{B}, E) \mid h = \text{id on } \partial B\}$ .

Recall that  $I$  satisfies (PS)-condition if every sequence  $\{u_n\}$  in  $W$  such that  $I(u_n)$  is bounded and  $I'(u_n) \rightarrow 0$  as  $n \rightarrow \infty$  has a convergent subsequence.

### 3. Proof of the main theorem

*Proof of Theorem 1.1.* Firstly, we prove that the functional  $\varphi$  satisfies (PS)-condition. Let  $\{u_n\} \subset W$  satisfying  $\varphi'(u_n) \rightarrow 0$  as  $n \rightarrow \infty$  and  $\varphi(u_n)$  is bounded. We will show that  $\{u_n\}$  has a convergent subsequence. From (H1), we have

$$F(t, u) = \sum_{i=1}^{m+1} \frac{1}{2} a_i \chi_i(t) |u|^2 + G(t, u), \quad (3.1)$$

where  $G(t, u) = \int_0^u g(t, s) ds$  satisfies

$$|G(t, u)| \leq C_5 |u|^r + C_6 \quad \text{for a.e. } t \in (0, 1) \text{ and all } u \in \mathbb{R}^N, \quad (3.2)$$

where  $C_5$  and  $C_6$  are positive constants. From (2.3) and (3.1), we have

$$\begin{aligned} \varphi(u) = & \frac{1}{2} \left[ \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (|\dot{v}_i(t)|^2 - a_i |v_i(t)|^2) dt + \int_0^1 |\dot{w}(t)|^2 dt - \sum_{i=1}^{m+1} a_i \int_{t_{i-1}}^{t_i} |w(t)|^2 dt \right] \\ & - \sum_{i=1}^{m+1} a_i \int_{t_{i-1}}^{t_i} (v_i(t), w(t)) dt + \frac{1}{2} \int_0^1 (A(t)w(t), w(t)) dt \\ & + \frac{1}{2} \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} (A(t)v_i(t), v_i(t)) dt - \int_0^1 G(t, u(t)) dt - \sum_{i=1}^m \hat{I}_i(w(t_i)), \quad u \in W. \end{aligned}$$

From (H2), there exist two positive constants  $C_7$  and  $C_8$  such that

$$\hat{I}_i(u) \geq C_7 |u|^\mu - C_8, \quad \forall u \in \mathbb{R}^N, \quad i = 1, 2, \dots, m. \quad (3.3)$$

Let  $J_0$  be the set of those  $i$  for which  $a_i < \lambda_1^i$  and let  $J_1 = \{1, \dots, m+1\} \setminus J_0$ . For each  $i \in J_1$ ,  $\lambda_{n_i}^i < a_i < \lambda_{n_i+1}^i$  for some  $n_i \geq 1$ , and we have the decomposition

$$\mathcal{N}_i = \mathcal{N}_i^+ \oplus \mathcal{N}_i^-, \quad v_i = v_i^+ + v_i^-,$$

where  $\mathcal{N}_i^-$  is the  $n_i$ -dimensional subspace spanned by the eigenfunctions of  $\lambda_1^i, \dots, \lambda_{n_i}^i$  and  $\mathcal{N}_i^+$  is its orthogonal complement. Then, we have

$$\begin{aligned} \varphi(u) = & \frac{1}{2} \left[ \sum_{i \in J_0} \int_{t_{i-1}}^{t_i} (|\dot{v}_i(t)|^2 - a_i |v_i(t)|^2) dt + \sum_{i \in J_1} \int_{t_{i-1}}^{t_i} (|\dot{v}_i^+(t)|^2 - a_i |v_i^+(t)|^2) dt \right. \\ & + \sum_{i \in J_1} \int_{t_{i-1}}^{t_i} (|\dot{v}_i^-(t)|^2 - a_i |v_i^-(t)|^2) dt + \int_0^1 |\dot{w}(t)|^2 dt - \sum_{i=1}^{m+1} a_i \int_{t_{i-1}}^{t_i} |w(t)|^2 dt \Big] \\ & - \sum_{i=1}^{m+1} a_i \int_{t_{i-1}}^{t_i} (v_i(t), w(t)) dt + \frac{1}{2} \int_0^1 (A(t)w(t), w(t)) dt \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{i \in J_0} \int_{t_{i-1}}^{t_i} (A(t)v_i(t), v_i(t)) dt + \frac{1}{2} \sum_{i \in J_1} \int_{t_{i-1}}^{t_i} (A(t)v_i^+(t), v_i^+(t)) dt \\
& + \frac{1}{2} \sum_{i \in J_1} \int_{t_{i-1}}^{t_i} (A(t)v_i^-(t), v_i^-(t)) dt - \int_0^1 G(t, u(t)) dt - \sum_{i=1}^m \hat{I}_i(w(t_i)),
\end{aligned}$$

for

$$u = \sum_{i \in J_0} v_i + \sum_{i \in J_1} (v_i^+ + v_i^-) + w \in \bigoplus_{i \in J_0} \mathcal{N}_i \oplus \bigoplus_{i \in J_1} (\mathcal{N}_i^+ \oplus \mathcal{N}_i^-) \oplus \mathcal{M}. \quad (3.4)$$

We have

$$\begin{aligned}
& \int_{t_{i-1}}^{t_i} (|\dot{v}_i(t)|^2 + (A(t)v_i(t), v_i(t))) dt \geq \lambda_1^i \int_{t_{i-1}}^{t_i} |v_i(t)|^2 dt, \quad i \in J_0, \\
& \int_{t_{i-1}}^{t_i} (|\dot{v}_i^+(t)|^2 + (A(t)v_i^+(t), v_i^+(t))) dt \geq \lambda_{n_i+1}^i \int_{t_{i-1}}^{t_i} |v_i^+(t)|^2 dt, \quad i \in J_1, \\
& \int_{t_{i-1}}^{t_i} (|\dot{v}_i^-(t)|^2 + (A(t)v_i^-(t), v_i^-(t))) dt \leq \lambda_{n_i}^i \int_{t_{i-1}}^{t_i} |v_i^-(t)|^2 dt, \quad i \in J_1,
\end{aligned}$$

hence, we have

$$\int_{t_{i-1}}^{t_i} (|\dot{v}_i(t)|^2 + (A(t)v_i(t), v_i(t)) - a_i |v_i(t)|^2) dt \geq c_i \|v_i\|^2, \quad i \in J_0, \quad (3.5)$$

$$\int_{t_{i-1}}^{t_i} (|\dot{v}_i^+(t)|^2 + (A(t)v_i^+(t), v_i^+(t)) - a_i |v_i^+(t)|^2) dt \geq c_i^+ \|v_i\|^2, \quad i \in J_1, \quad (3.6)$$

$$\int_{t_{i-1}}^{t_i} (|\dot{v}_i^-(t)|^2 + (A(t)v_i^-(t), v_i^-(t)) - a_i |v_i^-(t)|^2) dt \leq -c_i^- \|v_i\|^2, \quad i \in J_1, \quad (3.7)$$

where the constants

$$\begin{aligned}
c_i &= 1 - \frac{\max\{a_i, 0\}}{\lambda_1^i}, \quad i \in J_0, \\
c_i^+ &= 1 - \frac{a_i}{\lambda_{n_i+1}^i}, \quad c_i^- = \frac{a_i}{\lambda_{n_i}^i} - 1, \quad i \in J_1,
\end{aligned}$$

are all positive. Referring to the decomposition (3.4), write

$$u_n = \sum_{i \in J_0} v_{ni} + \sum_{i \in J_1} (v_{ni}^+ + v_{ni}^-) + w_n, \quad \bar{u}_n = \sum_{i \in J_0} v_{ni} + \sum_{i \in J_1} (v_{ni}^+ - v_{ni}^-) - w_n.$$

Then, we have

$$\begin{aligned}
(\varphi'(u_n), \bar{u}_n) &= \sum_{i \in J_0} \int_{t_{i-1}}^{t_i} (|\dot{v}_{ni}(t)|^2 - a_i |v_{ni}(t)|^2) dt + \sum_{i \in J_1} \int_{t_{i-1}}^{t_i} (|\dot{v}_{ni}^+(t)|^2 - a_i |v_{ni}^+(t)|^2) dt \\
&\quad - \sum_{i \in J_1} \int_{t_{i-1}}^{t_i} (|\dot{v}_{ni}^-(t)|^2 - a_i |v_{ni}^-(t)|^2) dt - \int_0^1 |\dot{w}_n(t)|^2 dt + \sum_{i=1}^{m+1} a_i \int_{t_{i-1}}^{t_i} |w_n(t)|^2 dt \\
&\quad + 2 \sum_{i \in J_1} a_i \int_{t_{i-1}}^{t_i} (v_{ni}^-(t), w_n(t)) dt + \int_0^1 (A(t)w_n(t), w_n(t)) dt \\
&\quad + \sum_{i \in J_0} \int_{t_{i-1}}^{t_i} (A(t)v_{ni}(t), v_{ni}(t)) dt + \sum_{i \in J_1} \int_{t_{i-1}}^{t_i} (A(t)v_{ni}^+(t), v_{ni}^+(t)) dt
\end{aligned} \quad (3.8)$$

$$\begin{aligned}
& - \sum_{i \in J_1} \int_{t_{i-1}}^{t_i} (A(t)v_{ni}^-(t), v_{ni}^-(t)) dt - \int_0^1 (g(t, u_n(t)), \bar{u}_n) dt \\
& - \sum_{i=1}^m (I_i(w_n(t_i)), w_n(t_i)),
\end{aligned}$$

From  $\varphi'(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ , (1.3), (1.4), (3.5), (3.6), (3.7) and (3.8), we have

$$\begin{aligned}
& \sum_{i \in J_0} c_i \|v_{ni}\|^2 + \sum_{i \in J_1} (c_i^+ \|v_{ni}^+\|^2 + c_i^- \|v_{ni}^-\|^2) + C_3 \sum_{i=1}^m |w_n(t_i)|^\mu \\
& \leq C_8 (\|w_n\|^2 + \sum_{i \in J_1} \|v_{ni}^-\| \|w_n\| + \|u_n\|^{r-1} \|\bar{u}_n\| + \|\bar{u}_n\| + 1),
\end{aligned}$$

where  $C_8$  is a positive constant. Since  $\max_i |w(t_i)|$  defines an equivalent norm on  $\mathcal{M}$ ,  $\mu > 2$ ,  $\|\bar{u}_n\| = \|u_n\|$ , and  $r < 2$ , the boundedness of

$$\|u_n\| = \sum_{i \in J_0} \|v_{ni}\|^2 + \sum_{i \in J_1} (\|v_{ni}^+\|^2 + \|v_{ni}^-\|^2) + \|w_n\|^2$$

follows. That is  $\varphi$  satisfies (PS)-condition.

Next, we will apply saddle point theorem to the splitting

$$W = \left( \bigoplus_{i \in J_1} \mathcal{M}_i^- \oplus \mathcal{M} \right) \oplus \left( \bigoplus_{i \in J_0} \mathcal{N}_i \oplus \bigoplus_{i \in J_1} \mathcal{N}_i^+ \right) = W_1 \oplus W_2.$$

For  $u = \sum_{i \in J_1} \mathcal{N}_i^- + w \in W_1$ , from (3.2), (3.3), (3.5), (3.6) and (3.7), we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2} \left[ \sum_{i \in J_1} \int_{t_{i-1}}^{t_i} (|\dot{v}_i^-(t)|^2 - a_i |v_i^-(t)|^2 + (A(t)v_i^-(t), v_i^-(t))) dt \right. \\
& \quad \left. + \int_0^1 (A(t)w(t), w(t)) dt + \int_0^1 |\dot{w}(t)|^2 dt - \sum_{i=1}^{m+1} a_i \int_{t_{i-1}}^{t_i} |w(t)|^2 dt \right] \\
& \quad - \sum_{i \in J_1} a_i \int_{t_{i-1}}^{t_i} (v_i^-(t), w(t)) dt - \int_0^1 G(t, u(t)) dt - \sum_{i=1}^m \hat{I}_i(w(t_i)) \\
& \leq -\frac{1}{2} \sum_{i \in J_1} c_i^- \|v_i^-\|^2 - C_6 \sum_{i=1}^m |w(t_i)|^\mu + C_9 (\|w\|^2 + \sum_{i \in J_1} \|v_i^-\| \|w\| + \|u\|^r + 1).
\end{aligned}$$

Since  $\max_i |w(t_i)|$  defines an equivalent norm on  $\mathcal{M}$ ,  $\mu > 2$ , and  $r < 2$ , it follows that  $\varphi(u) \rightarrow -\infty$  as  $\|u\|^2 = \sum_{i \in J_1} \|v_i^-\|^2 + \|w\|^2 \rightarrow \infty$ . On the other hand, for  $u = \sum_{i \in J_0} v_i + \sum_{i \in J_1} v_i^+ \in W_2$ , from (3.2), (3.5), (3.6) and (3.7), we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2} \left[ \sum_{i \in J_0} \int_{t_{i-1}}^{t_i} (|\dot{v}_i(t)|^2 - a_i |v_i(t)|^2 + (A(t)v_i(t), v_i(t))) dt \right. \\
& \quad \left. + \sum_{i \in J_1} \int_{t_{i-1}}^{t_i} (|\dot{v}_i^+(t)|^2 - a_i |v_i^+(t)|^2 + (A(t)v_i^+(t), v_i^+(t))) dt \right] - \int_0^1 G(t, u(t)) dt \\
& \geq \frac{1}{2} \left[ \sum_{i \in J_0} c_i \|v_i\|^2 + \sum_{i \in J_1} c_i^+ \|v_i^+\|^2 \right] - C_{10} (\|u\|^r + 1).
\end{aligned} \tag{3.9}$$

Since  $r < 2$ , it follows from (3.9) that  $\varphi(u)$  is bounded below on  $W_2$ . Thus,  $\varphi(u)$  has a critical point by Lemma 2.1. Then problem (1.1) has at least one solution.  $\square$

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