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Some identities of degenerate Fubini polynomials arising from differential equations

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Abstract

Recently, Kim et al. have studied degenerate Fubini polynomials in [T. Kim, D. V. Dolgy, D. S. Kim, J. J. Seo, J. Nonlinear Sci. Appl., **9** (2016), 2857–2864]. Jang and Kim presented some identities of Fubini polynomials arising from differential equations in [G.-W. Jang, T. Kim, Adv. Studies Contem. Math., **28** (2018), to appear]. In this paper, we drive differential equations from the generating function of the degenerate Fubini polynomials. In addition, we obtain some identities from those differential equations.

Keywords: Differential equations, Fubini polynomials, degenerate Fubini polynomials.

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1. Introduction

As is well known that *the Fubini polynomials* $F_n(y)$ are defined by the generating function to be

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}. \tag{1.1}$$

In the spacial case $y = 1$,

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} F_n(1)\frac{t^n}{n!} = \sum_{n=0}^{\infty} F_n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} k!S_2(n,k)\frac{t^n}{n!},$$

where $S_2(n,k)$ denotes the Stirling numbers of the second kind and $F_n(1) = F_n$ are called the *Fubini numbers* or the *ordered Bell numbers*.

From the definition of the generating function of Fubini polynomials, the equation (1.1), we obtain

$$\frac{1}{1-y(e^t-1)} = \sum_{k=0}^{\infty} y^k(e^t-1)^k = \sum_{k=0}^{\infty} y^k k!\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} k!S_2(n,k)y^k\frac{t^k}{k!}. \tag{1.2}$$

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The equations (1.1) and (1.2) indicate that

$$F_n(y) = \sum_{k=0}^{n} k!S_2(n,k)y^k.$$

For a nonnegative integer n and any real x, the symbol $(x)_n$ denotes the falling factorials, that is,

$$(x)_0 = 1,\ (x)_n = x(x-1)(x-2)\cdots(x-n+1).$$

In [7], Kim introduced λ-*analogue falling factorials* and presented several results regarding it. The λ-analogue of falling factorials are defined as follows:

$$(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda).$$

The degenerate Stirling numbers of the second kind are given by the generating function

$$\frac{1}{k!}((1+\lambda t)^{\frac{1}{\lambda}}-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}.$$

It is well known that the Fubini numbers count the number of weak orderings on a set of n elements [2]. Velleman and Call consider combination locks with a numeric keypad [23]. Pippenger traces the problem of counting weak orderings to the work of Whitworth [19]. Recently, several authors have studied Fubini numbers and polynomials (see [3, 5, 18]).

In [1], Carlitz presented degenerate Stirling, Bernoulli, and Eulerian numbers. After Carlitz, a group of mathematician have studied the degenerate special numbers. For example, degenerate Fubini polynomials are studied in [11], degenerated Bell polynomials in [14], degenerate Cauchy numbers in [20, 21], and degenerate Daehee numbers in [16, 22]. Lim studied degenerate Gennochi polynomials [17].

In [11], Kim et al. introduced the degenerate Fubini polynomials as follows:

$$\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)} = \sum_{n=0}^{\infty} F_{n,\lambda}(y)\frac{t^n}{n!}. \tag{1.3}$$

Note that $\lim_{\lambda\to 0} F_{n,\lambda} = F_n(y)$.

Several recurrence relations and some properties of the degenerate Fubini polynomials are presented, for example,

$$F_{n,\lambda}(y) = \sum_{k=0}^{n} y^k k!S_{2,\lambda}(n,k), \tag{1.4}$$

where $S_{2,\lambda}(n,k)$ denotes the degenerate Stirling numbers of the second kind [11].

Recently, many mathematicians have studied special numbers using differential equations. Bernoulli numbers of the second kind are presented in [9], Frobenius-Euler polynomials are presented in [6], Mittag-Leffer polynomials are presented in [12], Changhee numbers and polynomials are presented in [8, 10], and Daehee and degenerate Daehee numbers are presented in [16, 20].

In this paper, we drive differential equations from the generating function of the degenerate Fubini polynomials. In addition, we obtain some identities from those differential equations.

2. Preliminaries

Throughout this article, for a positive integer N and a function Y, we denote Y^N for the N-th power of Y, and $Y^{(N)}$ for the N-th partial derivative of Y with respect to y,

$$Y^N = \underbrace{Y\times\cdots\times Y}_{\text{N-times}},\quad Y^{(0)} = Y,\ Y^{(N)} = \frac{\partial}{\partial y}Y^{(N-1)}.$$

In [4], Jang and Kim presented some identities of Fubini polynomials arising from differential equations. They showed that the function $Y(t,y)=\frac{1}{1-y(e^t-1)}$ satisfies the following differential equation.

$$\frac{\partial^N Y}{\partial t^N}=\sum_{k=0}^{N}(-1)^{N-k}a_k(N)(1+y)^k Y^{k+1},$$

where $a_0(N)=1, a_N(N)=N!$, and

$$a_k(N)=\sum_{i_1=0}^{N-k}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}}(k+1)^{N-k-i_1}\left(\prod_{j=1}^{k-1}(k-j+1)^{i_j-i_{j+1}+1}\right)$$

for $1\leqslant k\leqslant N-1$.

In this paper, the generating function of degenerate Fubini polynomials is used to derive some differential equations through the differentiation with respect to y.

For any positive integer r, the *higher order degenerate Fubini polynomials*, denoted by $F^{od(r)}_{n,\lambda}$, are defined by the generating function to be:

$$\left(\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right)^r=\sum_{n=0}^{\infty}F^{od(r)}_{n,\lambda}(y)\frac{t^n}{n!}.$$

Let us consider the case when $r=2$.

$$\begin{aligned}\left(\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right)^2&=\left(\sum_{n=0}^{\infty}F_{n,\lambda}(y)\frac{t^n}{n!}\right)^2=\left(\sum_{l_2=0}^{\infty}F_{l_2,\lambda}(y)\frac{t^{l_2}}{l_2!}\right)\left(\sum_{l_1=0}^{\infty}F_{l_1,\lambda}(y)\frac{t^{l_1}}{l_1!}\right)\\&=\sum_{l_2=0}^{\infty}\sum_{l_1=0}^{l_2}\binom{l_2}{l_1}F_{l_2-l_1,\lambda}(y)F_{l_1,\lambda}(y)\frac{t^{l_2}}{l_2!}.\end{aligned}\tag{2.1}$$

From the previous equation (2.1) the following is obtained naturally.

$$\sum_{n=0}^{\infty}F^{od(3)}_{n,\lambda}(y)\frac{t^n}{n!}=\sum_{l_3=0}^{\infty}\sum_{l_2=0}^{l_3}\sum_{l_1=0}^{l_2}\binom{l_3}{l_2}\binom{l_2}{l_1}F_{l_3-l_2,\lambda}(y)F_{l_2-l_1,\lambda}(y)F_{l_1,\lambda}(y)\frac{t^{l_3}}{l_3!}.$$

For brevity set $l_0=0$ and $l_k=n$. It is not difficult to obtain the following identity:

$$\sum_{n=0}^{\infty}F^{od(k)}_{n,\lambda}(y)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{l_{k-1}=0}^{n}\cdots\sum_{l_1=0}^{l_2}\left(\prod_{i=0}^{k-1}\binom{l_{i+1}}{l_i}F_{l_{i+1}-l_i,\lambda}(y)\right)\frac{t^n}{n!}.\tag{2.2}$$

The equation (2.2) yields an explicit formula for higher order Fubini polynomials.

Theorem 2.1. *For any positive integer k, nonnegative integer n and real λ,*

$$F^{od(k)}_{n,\lambda}(y)=\sum_{l_{k-1}=0}^{n}\cdots\sum_{l_1=0}^{l_2}\left(\prod_{i=0}^{k-1}\binom{l_{i+1}}{l_i}F_{l_{i+1}-l_i,\lambda}(y)\right),$$

where $l_0=0$ and $l_k=n$.

In [5], some product of Fubini polynomials are considered. Kargin showed that following identity holds.

$$(y+1)\sum_{k=0}^{n}\binom{n}{k}F_k(y)F_{n-k}(y)=F_{n+1}(y)+F_n(y)\ (n\geqslant 0).$$

Here we obtain the k-th power of $F_{n,\lambda}(y)$.

Theorem 2.2. *For any positive integer* k*, nonnegative integer* n*, and real* λ*,*

$$F^k_{n,\lambda}(y) = \sum_{l_k=0}^{kn} \sum_{l_{k-1}=0}^{l_k} \cdots \sum_{l_1=0}^{l_2} l_k! \left(\prod_{i=0}^{k-1} (l_{i+1}-l_i)! S_{2,\lambda}(n, l_{i+1}-l_i) \right) y^{l_k},$$

where $l_0 = 0$.

Proof. From the expression (1.4), we get

$$\begin{aligned} F^2_{n,\lambda}(y) &= \left(\sum_{l_2=0}^{n} l_2! S_{2,\lambda}(n, l_2) y^{l_2} \right) \left(\sum_{l_1=0}^{n} l_1! S_{2,\lambda}(n, l_1) y^{l_1} \right) \\ &= \sum_{l_2=0}^{2n} \sum_{l_1=0}^{l_2} (l_2-l_1)! l_1! S_{2,\lambda}(n, l_2-l_1) S_{2,\lambda}(n, l_1) y^{l_2}. \end{aligned}$$

Let us assume that

$$F^{k-1}_{n,\lambda}(y) = \sum_{l_{k-1}=0}^{(k-1)n} \sum_{l_{k-1}=0}^{l_k} \cdots \sum_{l_1=0}^{l_2} \left(\prod_{i=0}^{k-2} (l_{i+1}-l_i)! S_{2,\lambda}(n, l_{i+1}-l_i) \right) y^{l_k},$$

where $l_0 = 0$. Then we have the conclusion as follows:

$$\begin{aligned} F^k_{n,\lambda}(y) &= \left(\sum_{l_k=0}^{n} l_k! S_{2,\lambda}(n, l_k) y^{l_k} \right) F^{k-1}_{n,\lambda}(y) \\ &= \left(\sum_{l_k=0}^{n} S_{2,\lambda}(n, l_k) y^{l_3} \right) \left(\sum_{l_{k-1}=0}^{(k-1)n} \cdots \sum_{l_1=0}^{l_2} \left(\prod_{i=0}^{k-2} (l_{i+1}-l_i)! S_{2,\lambda}(n, l_{i+1}-l_i) \right) y^{l_{k-1}} \right) \\ &= \sum_{l_k=0}^{kn} \sum_{l_{k-1}=0}^{l_k} \cdots \sum_{l_1=0}^{l_2} l_k! \left(\prod_{i=0}^{k-1} (l_{i+1}-l_i)! S_{2,\lambda}(n, l_{i+1}-l_i) \right) y^{l_k}. \end{aligned}$$

□

3. Differential equations associated with the generating function of degenerate Fubini polynomials

From now on, we use Y to denote the generating function of the degenerate Fubini polynomials,

$$Y = Y(y,t) = \frac{1}{1 - y((1+\lambda t)^{\frac{1}{\lambda}} - 1)}. \tag{3.1}$$

By taking derivative with respect to y of (3.1), we get

$$Y^{(1)} = \frac{\partial Y}{\partial y} = \frac{((1+\lambda t)^{\frac{1}{\lambda}} - 1)}{(1 - y((1+\lambda t)^{\frac{1}{\lambda}} - 1))^2} = \frac{1 - 1 + y((1+\lambda t)^{\frac{1}{\lambda}} - 1)}{y(1 - y((1+\lambda t)^{\frac{1}{\lambda}} - 1))^2}. \tag{3.2}$$

From (3.2), we have

$$yY^{(1)} = Y^2 - Y. \tag{3.3}$$

Let us take derivative both sides of (3.3), and then multiply both sides by y, then we have

$$yY^{(1)} + y^2 Y^{(2)} = 2yYY^{(1)} - yY^{(1)}. \tag{3.4}$$

By (3.3) and (3.4), we get

$$y^2 Y^{(2)} = Y - Y^2 + 2Y(Y^2 - Y) - (Y^2 - Y) = 2Y - 3Y^2 + 2Y^3. \tag{3.5}$$

One more time, differentiation of both sides of (3.5) and multiplying both sides by y, then we have:

$$2y^2Y^{(2)} + y^3Y^{(3)} = 2yY^{(1)} - 6yYY^{(1)} + 6yY^2Y^{(1)}.$$

From this, we get

$$\begin{aligned} y^3Y^{(3)} &= -2y^2Y^{(2)} + 2yY^{(1)} - 6yYY^{(1)} + 6yY^2Y^{(1)} \\ &= -2(2Y - 3Y^2 + 2Y^3) + 2(Y^2 - Y) - 6Y(Y^2 - Y) + 6Y^2(Y^2 - Y) \\ &= -6Y + 14Y^2 - 16Y^3 + 6Y^4. \end{aligned}$$

To obtain $Y^{(N)}$, assume that

$$y^NY^{(N)} = \sum_{k=0}^{N} a_k(N)Y^{k+1}. \tag{3.6}$$

Let us take derivative of both sides of (3.6), and then multiply by y,

$$Ny^NY^{(N)} + y^{N+1}Y^{(N+1)} = \sum_{k=0}^{N} a_k(N)(k+1)Y^k yY^{(1)}. \tag{3.7}$$

By (3.3) and (3.7), we obtain

$$\begin{aligned} y^{N+1}Y^{(N+1)} &= -N\sum_{k=0}^{N} a_k(N)Y^{k+1} + \sum_{k=0}^{N} a_k(N)(k+1)Y^k(Y^2 - Y) \\ &= -N\sum_{k=0}^{N} a_k(N)Y^k + \sum_{k=1}^{N+1} a_{k-1}(N)kY^{k+1} - \sum_{k=0}^{N} a_k(N)(k+1)Y^{k+1} \\ &= \sum_{k=1}^{N} (-(N+k+1)a_k + ka_{k-1}(N))Y^{k+1} - (N+1)a_0(N)Y + a_{N+1}(N)(N+1)Y^{N+2}. \end{aligned} \tag{3.8}$$

Substituting $N+1$ instead of N in (3.6), then the equation (3.6) becomes

$$y^{N+1}Y^{(N+1)} = \sum_{k=0}^{N+1} a_k(N+1)Y^k. \tag{3.9}$$

Comparing coefficients on the both sides of (3.8) and (3.9), we get the following recurrence relations.

$$\begin{aligned} &a_0(N+1) = -(N+1)a_0(N), \quad a_{N+1}(N+1) = (N+1)a_{N+1}(N), \\ &a_k(N+1) = -(N+k+1)a_k(N) + ka_{k-1}(N) \quad \text{for } 1 \leqslant k \leqslant N. \end{aligned} \tag{3.10}$$

From (3.3), we know that

$$a_0(1) = -1, \quad a_1(1) = 1. \tag{3.11}$$

The recurrence relations (3.10) and initial conditions (3.11) yield the following:

$$a_0(N) = (-1)^{N+1}N!, \quad a_N(N) = N!. \tag{3.12}$$

Let us observe the recurrence relation $a_k(N)(1 \leqslant k \leqslant N)$ with respect to N in (3.10).

$$\begin{aligned}
a_k(N) &= -(N+k)a_k(N-1) + ka_{k-1}(N-1)\\
&= -(N+k)\Big(-(N+k-1)a_k(N-2) + ka_{k-1}(N-2)\Big) + ka_{k-1}(N-1)\\
&= (N+k)(N+k-1)a_k(N-2) - k(N+k)a_{k-1}(N-2) + ka_{k-1}(N-1)\\
&= (N+k)(N+k-1)\Big(-(N+k-2)a_k(N-3) + ka_{k-1}(N-3)\Big)\\
&\quad - k(N+k)a_{k-1}(N-2) + ka_{k-1}(N-1)\\
&= (-1)^3(N+k)_3 a_k(N-3) + k(-1)^2(N+k)_2 a_{k-1}(N-3)\\
&\quad - k(N+k)a_{k-1}(N-2) + ka_{k-1}(N-1)\\
&\;\vdots\\
&= (-1)^{N-k}(N+k)_{N-k}a_k(k)\\
&\quad + \sum_{l_1=1}^{N-k-1}(-1)^{l_1}k(N+k)_{l_1}a_{k-1}(N-l_1-1) + ka_{k-1}(N-1)\\
&= \sum_{l_1=0}^{N-k}(-1)^{l_1}k(N+k)_{l_1}a_{k-1}(N-l_1-1).
\end{aligned} \tag{3.13}$$

Applying the processing in the equation (3.13) to $a_{k-1}(N-l_1-1)$,

$$a_{k-1}(N-l_1-1) = \sum_{l_2=0}^{N-l_1-k}(-1)^{l_2}(k-1)(N-l_1+k-2)_{l_2}a_{k-2}(N-l_1-l_2-2).$$

For brevity set $N_0 = N, N_i = N - \sum_{m=0}^{i} l_i$ with $l_0 = 0$, then we have

$$\begin{aligned}
a_k(N_0-0) &= \sum_{l_1=0}^{N_0-k}(-1)^{l_1}k(N_0+k)_{l_1}a_{k-1}(N_1-1),\\
a_{k-1}(N_1-1) &= \sum_{l_2=0}^{N_1-k}(-1)^{l_2}(k-1)(N_1+k-2)_{l_2}a_{k-2}(N_2-2),\\
a_{k-2}(N_2-2) &= \sum_{l_3=0}^{N_2-k}(-1)^{l_3}(k-2)(N_2+k-4)_{l_3}a_{k-3}(N_3-3).
\end{aligned}$$

From (3.12) and $a_0(N_k-k) = (-1)^{N_k-k+1}(N_k-k)!$, we get

$$a_k(N) = \sum_{l_1=0}^{N_0-k}\sum_{l_2=0}^{N_1-k}\cdots\sum_{l_k=0}^{N_{k-1}-k}(-1)^{N-k+1}k!(N_k-k)!\prod_{i=0}^{k}(N_i+k-2i)_{l_i}, \tag{3.14}$$

where $N_0 = N, N_i = N - \sum_{m=0}^{i} l_m$, and $l_0 = 0$. We note that the coefficients $a_k(N)$ are similar to those in [10].

By (3.9) and (3.14), we obtain the following theorem.

Theorem 3.1. *For $N \in \mathbb{N}$, let us consider the following differential equation with respect to y:*

$$y^N Y^{(N)} = \sum_{k=0}^{N} a_k(N)Y^{k+1}, \tag{3.15}$$

where

$$a_k(N)=\sum_{l_1=0}^{N_0-k}\sum_{l_2=0}^{N_1-k}\cdots\sum_{l_k=0}^{N_{k-1}-k}(-1)^{N-N_k}k!(N_k-k)!\prod_{i=0}^{k}(N_i+k-2i)_{l_i}$$

with $N_0=N, N_i=N-\sum_{m=0}^{i}l_m,\ l_0=0$, *and* $a_0(N)=(-1)^{N+1}N!$. *Then* $Y=\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}$ *is a solution of* (3.15).

From (1.3), we have

$$\begin{aligned}Y^{(N)}=\frac{\partial^N}{\partial y^N}\left(\frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}\right)&=\frac{\partial^N}{\partial y^N}\sum_{n=0}^{\infty}F_{n,\lambda}(y)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\frac{\partial^N}{\partial y^N}\left(\sum_{k=0}^{n}y^k k!S_{2,\lambda}(n,k)\right)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=N}^{n}(k)_N y^{k-N}k!S_{2,\lambda}(n,k)\frac{t^n}{n!}.\end{aligned}\tag{3.16}$$

The equation (3.16) yields

$$y^NY^{(N)}=\sum_{n=0}^{\infty}\sum_{k=N}^{n}(k)_N y^k k!S_{2,\lambda}(n,k)\frac{t^n}{n!}.\tag{3.17}$$

From the definition of Y, Y^{k+1} is the generating function of higher-order Fubini polynomials. The higher-order Fubini polynomials expressed in (2.2).

$$\begin{aligned}\sum_{k=0}^{N}a_k(N)Y^{k+1}&=\sum_{k=0}^{N}a_k(N)\sum_{n=0}^{\infty}F_{n,\lambda}^{od(k+1)}(y)\frac{t^n}{n!}\\&=\sum_{k=0}^{N}a_k(N)\sum_{n=0}^{\infty}\left(\sum_{l_k=0}^{n}\cdots\sum_{l_1=0}^{l_2}\left(\prod_{i=0}^{k}\binom{l_{i+1}}{l_i}F_{l_{i+1}-l_i,\lambda}(y)\right)\right)\frac{t^n}{n!}\\&=\sum_{k=0}^{N}\sum_{n=0}^{\infty}\sum_{l_k=0}^{n}\cdots\sum_{l_1=0}^{l_2}a_k(N)\left(\prod_{i=0}^{k}\binom{l_{i+1}}{l_i}F_{l_{i+1}-l_i,\lambda}(y)\right)\frac{t^n}{n!}.\end{aligned}\tag{3.18}$$

By comparing the coefficients on the both sides in (3.17) and (3.18), we have the following identity.

Theorem 3.2. *For any nonnegative integer* n, *positive integer* N, *and real* λ,

$$\sum_{k=0}^{n}(k)_N y^k k!S_{2,\lambda}(n,k)=\sum_{k=0}^{N}\sum_{l_k=0}^{n}\cdots\sum_{l_1=0}^{l_2}a_k(N)\left(\prod_{i=1}^{k+1}\binom{l_{i+1}}{l_i}F_{l_i-l_{i-1},\lambda}(y)\right),$$

where

$$a_k(N)=\sum_{l_1=0}^{N_0-k}\sum_{l_2=0}^{N_1-k}\cdots\sum_{l_k=0}^{N_{k-1}-k}(-1)^{N-N_k}k!(N_k-k)!\prod_{i=0}^{k}(N_i+k-2i)_{l_i}$$

with $N_0=N, N_i=N-\sum_{m=0}^{i}l_m,\ l_0=0$, *and* $a_0(N)=(-1)^{N+1}N!$.

Now we consider the inversion formula of Theorem 3.1. From (3.3), we have

$$Y^2=Y+yY^{(1)}\tag{3.19}$$

Differentiating both sides of (3.19), then multiplying both sides by y, and then applying (3.3), we have

$$2Y(Y^2-Y)=2yY^{(1)}+y^2Y^{(2)}. \tag{3.20}$$

Substituting $Y+yY^{(1)}$ instead of y^2, the equation (3.20) becomes

$$2Y^3=2Y^2+2yY^{(1)}+y^2Y^{(2)}=2(Y+yY^{(1)})+2yY^{(1)}+y^2Y^{(2)}=2Y+4yY^{(1)}+y^2Y^{(2)}.$$

Continuing this process, we can set

$$N!Y^{N+1}=\sum_{k=0}^{N} b_k(N)y^kY^{(k)}. \tag{3.21}$$

Let us take differentiate of both sides of (3.20), we have

$$N!(N+1)Y^NY^{(1)}=\sum_{k=0}^{N} b_k(N)\left(ky^{k-1}Y^{(k)}+y^kY^{(k+1)}\right).$$

Multiply both sides by y, then we have

$$\begin{aligned}(N+1)!Y^NyY^{(1)}&=\sum_{k=0}^{N} b_k(N)\left(ky^{k-1}yY^{(k)}+y^{k+1}Y^{(k+1)}\right)\\&=\sum_{k=1}^{N} b_k(N)ky^kY^{(k)}+\sum_{k=0}^{N} b_k(N)y^{k+1}Y^{(k+1)}\\&=\sum_{k=1}^{N} b_k(N)ky^kY^{(k)}+\sum_{k=1}^{N+1} b_{k-1}(N)y^kY^{(k)}.\end{aligned} \tag{3.22}$$

Since $yY^{(1)}=Y^2-Y$, the equation (3.22) becomes

$$\begin{aligned}(N+1)!Y^{N+2}&=(N+1)\sum_{k=0}^{N} b_k(N)y^kY^{(k)}+\sum_{k=1}^{N} b_k(N)ky^kY^{(k)}+\sum_{k=1}^{N+1} b_{k-1}(N)y^kY^{(k)}\\&=\sum_{k=1}^{N}\left((N+k+1)b_k(N)+b_{k-1}(N)y^k\right)Y^{(k)}\\&\quad+(N+1)b_0(N)Y+b_N(N)y^{N+1}Y^{(N+1)}.\end{aligned} \tag{3.23}$$

Substituting $N+1$ instead of N in (3.21),

$$(N+1)!Y^{N+2}=\sum_{k=0}^{N+1} b_k(N+1)y^kY^{(k)}. \tag{3.24}$$

Comparing the coefficients between (3.23) and (3.24), we get

$$\begin{aligned}&b_0(N+1)=(N+1)b_0(N),\quad b_{N+1}(N+1)=b_N(N),\\&b_k(N+1)=(N+k+1)b_k(N)+b_{k-1}(N)\ \text{ for } 1\geqslant k\geqslant N.\end{aligned} \tag{3.25}$$

From (3.25) and $b_0(0)=b_0(1)=1$, we note that

$$b_0(N)=N!,\quad b_N(N)=b_{N-1}(N-1)=\cdots=b_0(0)=1.$$

Let us observe the recurrence relation $b_k(N+1)$ with respect to N

$$\begin{aligned}b_k(N+1) &= (N+k+1)b_k(N) + b_{k-1}(N)\\ &= (N+k+1)\Big((N+k)b_k(N-1) + b_{k-1}(N-1)\Big) + b_{k-1}(N)\\ &= (N+k+1)_2\Big((N+k-1)b_k(N-2) + b_{k-1}(N-2)\Big)\\ &\quad + (N+k+1)b_{k-1}(N-1) + b_{k-1}(N)\\ &= (N+k+1)_3 b_k(N-2) + (N+k+1)_2 b_{k-1}(N-2)\\ &\quad + (N+k+1)b_{k-1}(N-1) + b_{k-1}(N).\end{aligned}$$

Continuing this process, we have

$$\begin{aligned}b_k(N+1) &= (N+k+1)_{N-k+1} b_k(k) + \sum_{l_1=1}^{N-k} (N+k+1)_{l_1} b_{k-1}(N-l_1) + b_{k-1}(N)\\ &= \sum_{l_1=0}^{N-k+1} (N+k+1)_{l_1} b_{k-1}(N-l_1).\end{aligned} \tag{3.26}$$

The recurrence relation (3.26) yields, for any positive integer N and nonnegative integer k,

$$b_k(N) = \sum_{l_1=0}^{N-k} (N+k)_{l_1} b_{k-1}(N-l_1-1). \tag{3.27}$$

From (3.27), we get

$$\begin{aligned}b_{k-1}(N-l_1-1) &= \sum_{l_2=0}^{N-l_1-k} (N-l_1+k-2)_{l_2} b_{k-2}(N-l_1-l_2-2),\\ b_{k-2}(N-l_1-l_2-2) &= \sum_{l_3=0}^{N-l_1-l_2-k} (N-l_1-l_2+k-4)_{l_3} b_{k-3}(N-l_1-l_2-l_3-3),\\ b_0\Big(N-\sum_{i=1}^{k} l_i - k\Big) &= \Big(N-\sum_{i=1}^{k} l_i - k\Big)!.\end{aligned}$$

By substituting sequentially b_k, we obtain

$$b_k(N) = \sum_{l_1=0}^{N_1-k}\sum_{l_2=0}^{N_2-k}\cdots\sum_{l_{k-1}=0}^{N_{k-1}-k} (N_k-k)!\Big(\prod_{i=0}^{k-1}(N_i+k-2i)_{l_i}\Big), \tag{3.28}$$

where

$$N_i = N - \sum_{m=0}^{i} l_m, \quad N_0 = N, \quad \text{and } l_0 = 0.$$

By (3.21) and (3.28), we get the following theorem.

Theorem 3.3. *For $N \in \mathbb{N}$, let us consider the following differential equation with respect to y:*

$$N!Y^{N+1} = \sum_{k=0}^{N} b_k(N) y^k Y^{(k)}, \tag{3.29}$$

where

$$b_k(N) = \sum_{l_1=0}^{N_1-k} \sum_{l_2=0}^{N_2-k} \cdots \sum_{l_{k-1}=0}^{N_{k-1}-k} (N_k-k)!\left(\prod_{i=0}^{k-1}(N_i+k-2i)_{l_i}\right)$$

with

$$N_i = N - \sum_{m=0}^{i} l_m, \; N_0 = N, \; \textit{and } l_0 = 0.$$

Then $Y = \frac{1}{1-y((1+\lambda t)^{\frac{1}{\lambda}}-1)}$ *is a solution of* (3.29).

From (3.17), we know that

$$y^k Y^{(k)} = \sum_{n=0}^{\infty} \sum_{m=k}^{n} (m)_N y^m m! S_{2,\lambda}(n,m) \frac{t^n}{n!},$$

and

$$\sum_{k=0}^{N} b_k(N) y^k Y^{(k)} = \sum_{k=0}^{N} \sum_{n=0}^{\infty} \sum_{m=k}^{n} b_k(N)(m)_k y^m m! S_{2,\lambda}(n,m) \frac{t^n}{n!}. \tag{3.30}$$

From Theorem 2.1, we have

$$Y^N = \sum_{n=0}^{\infty} F_{n,\lambda}^{od(N)}(y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{l_{N-1}=0}^{n} \cdots \sum_{l_1=0}^{l_2} \left(\prod_{i=1}^{N} \binom{l_{i+1}}{l_i} F_{l_i - l_{i-1},\lambda}(y) \right) \frac{t^n}{n!}. \tag{3.31}$$

By (3.30) and (3.31), we have the following theorem.

Theorem 3.4. *For any nonnegative integer* n *and positive integer* N,

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} b_k(N)(m)_k y^m m! S_{2,\lambda}(n,m) = \sum_{n=0}^{\infty} \sum_{l_{N-1}=0}^{n} \cdots \sum_{l_1=0}^{l_2} \left(\prod_{i=1}^{N} \binom{l_{i+1}}{l_i} F_{l_i - l_{i-1},\lambda}(y) \right),$$

where

$$b_k(N) = \sum_{l_1=0}^{N_1-k} \sum_{l_2=0}^{N_2-k} \cdots \sum_{l_{k-1}=0}^{N_{k-1}-k} (N_k-k)!\left(\prod_{i=0}^{k-1}(N_i+k-2i)_{l_i}\right)$$

with

$$N_i = N - \sum_{m=0}^{i} l_m, \; N_0 = N, \; \textit{and } l_0 = 0.$$

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