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Simultaneous iteration for variational inequalities over common solutions for finite families of nonlinear problems

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Abstract

In this paper, we apply Theorem 3.2 of [G. M. Lee, L.-J. Lin, J. Nonlinear Convex Anal., **18** (2017), 1781–1800] to study the variational inequality over split equality fixed point problems for three finite families of strongly quasi-nonexpansive mappings. Then we use this result to study variational inequalities over split equality for three various finite families of nonlinear mappings. We give a unified method to study split equality for three various finite families of nonlinear problems. Our results contain many results on split equality fixed point problems and multiple sets split feasibility problems as special cases. Our results can treat large scale of nonlinear problems by group these problems into finite families of nonlinear problems, then we use simultaneous iteration to find the solutions of these problems. Our results will give a simple and quick method to study large scale of nonlinear problems and will have many applications to study large scale of nonlinear problems.

Keywords: Split equality fixed point problem, split fixed point problem, quasi-pseudocontractive mapping, demicontractive mapping, pseudo-contractive mapping.

2010 MSC: 47H06, 47H09, 47H10, 47J25, 65K15.

1. Introduction

Let $T : H_1 \to H_1$, and let $\text{Fix}(T) = \{x \in H_1 : x = Tx\}$ denote the fixed point set of T. For each $i \in \{1, 2, 3\}$, let H_i be a real Hilbert space. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively and $A : H_1 \to H_2$ be a bounded linear operator.

The split feasibility problem (**SFP**) in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [6] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction.

The split feasibility problem (**SFP**) is the problem:

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in C \text{ and } A\bar{x} \in Q.$$

Let $F : C \to H_1$ be an operator. The variational inequality problem **VIP(F, C)** is the following problem:

$$\text{Find } \bar{x} \in C \text{ such that } \langle F\bar{x}, u - \bar{x} \rangle \geqslant 0 \text{ for all } u \in C.$$

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The solution set of the variational inequality problem is denoted by $VI(F, C)$. The variational inequality problem **VIP(F, C)** has many applications in engineering, optimization, and signal recovery problem, see for example, Chuang et al. [11] and references therein.

Let $A : H_1 \to H_3, B : H_2 \to H_3$ be bounded linear operators, the split equality problem (**SEFP)** which was first introduced by Moudafi [18] is the problem:

$$\text{Find } \bar{x} \in C, \bar{y} \in Q \text{ such that } A\bar{x} = B\bar{y}.$$

The split equality problem has many applications such as decomposition method for PDE, application in image science, game theory, and intensity-modulated radiation [18]. It is easy to see that when $B = I$, and $H_2 = H_3$, then (**SEFP)** is reduced to (**SFP)**. Moudafi [18] introduced an iteration process to establish a weak convergence theorem for split equality problem under suitable assumptions.

Let $T : H_1 \to H_1, S : H_2 \to H_2$ be firmly quasi-nonexpansive mappings such that $Fix(T) \neq \emptyset$, $Fix(S) \neq \emptyset$, and let $A : H_1 \to H_3, B : H_2 \to H_3$ be bounded linear operators. Moudafi and AI-Shemas [19] introduced an iteration process and established a weak convergence theorem for split equality fixed point problem (**SEFPP)**:

$$\text{Find } \bar{x} \in Fix(T), \bar{y} \in Fix(S) \text{ such that } A\bar{x} = B\bar{y}.$$

When $B = I$, and $H_2 = H_3$, then (**SEFPP**) is reduced to the split common fixed point problem (**SCFPP**) [7, 17]:

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in Fix(U) \text{ and } A\bar{x} \in Fix(W).$$

Recently, many results on split equality fixed point problem have been found and one is referred to [8, 10, 23, 24, 26, 27] and references therein.

Recently, Lee and Lin [14], studied variational inequality problem over split equality fixed point sets of strongly quasi-nonexpansive mappings with applications to variational inequality problem over split equality fixed point for the same type of m nonlinear operators.

In this paper, we apply Lee and Lin [14, Theorem 3.2] to study the variational inequality over split equality fixed point problems for three finite families of strongly quasi-nonexpansive mappings. Then we use this result to study variational inequalities over split equality for three various finite families of nonlinear mappings. We give a unified method to study split equality for three various finite families of nonlinear problems. Our results contain many results on split equality fixed point problems and multiple sets split feasibility problems as special cases. Our results can treat large scale of nonlinear problems by group these problems into finite families of nonlinear problems, then we use simultaneous iteration to find the solutions of these problems. Our results will give a simple and quick method to study large scale of nonlinear problems and will have many applications to study large scale of nonlinear problems.

2. Preliminaries

For each $i \in \{1, 2, 3, 4\}$, let H_i be a (real) Hilbert space with inner products $\langle \cdot, \cdot \rangle$ and norms $\| \cdot \|$, and let $I_i : H_i \to H_i$ be the identity mapping on H_i. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in H_i and $x \in H_i$, we denote the strongly convergence and the weak convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in H_i$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Throughout this paper, we use these notations unless specified otherwise. Let C be a nonempty subset of a real Hilbert space H_1, and let $T : C \to H_1$. Then T is

(1) nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$;
(2) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and $\|Tx - y\| \leqslant \|x - y\|$ for all $x \in C$ and for all $y \in Fix(T)$;
(3) ρ-strongly quasi-nonexpansive (in short ρ- SQNE), where $\rho \geqslant 0$, if $Fix(T) \neq \emptyset$ and

$$\|Tx - y\|^2 \leqslant \|x - y\|^2 - \rho\|Tx - x\|^2$$

for all $x \in C, y \in Fix(T)$;

(4) monotone if $\langle x-y, Tx-Ty\rangle \geqslant 0$ for all $x, y \in C$;
(5) γ-strongly monotone if there exists $\gamma > 0$ such that $\langle x-y, Tx-Ty\rangle \geqslant \gamma\|x-y\|^2$ for all $x, y \in C$;
(6) pseudocontractive if $\|Tx-Ty\|^2 \leqslant \|x-y\|^2 + \|x-Tx-(y-Ty)\|^2$ for all $x, y \in C$;
(7) k-demicontractive if $Fix(T) \neq \emptyset$ and there exists $-\infty < k < 1$ such that $\|Tx-y\|^2 \leqslant \|x-y\|^2 + k\|Tx-x\|^2$ for all $x \in C$ and for all $y \in Fix(T)$;
(8) k-strictly pseudononspreading [20] if there exists $k \in (0,1)$ such that $\|Tx-Ty\|^2 \leqslant \|x-y\|^2 + k\|x-Tx-(y-Ty)\|^2 + \langle x-Tx, y-Ty\rangle$ for all $x, y \in C$;
(9) firmly nonexpansive if $\|Tx-Ty\|^2 + \|(I_1-T)x-(I_1-T)y\|^2 \leqslant \|x-y\|^2$ for all $x, y \in C$;
(10) directed if $Fix(T) \neq \emptyset$, and $\langle Tx-y, Tx-x\rangle \leqslant 0$ for all $x \in C$ and for all $y \in Fix(T)$;
(11) demiclosed if for each sequence $\{x_n\}$ and x in C with $x_n \rightharpoonup x$ and $(I-T)x_n \to 0$ implies that $(I-T)x = 0$;
(12) α-averaged if there exist $\alpha \in (0,1)$ and a nonexpansive mapping $S : C \to H_1$ such that $T = (1-\alpha)I + \alpha S$;
(13) hemicontinuous if, for all $x, y \in C$, the mapping $g : [0,1] \to H_1$, defined by $g(t) = T(tx+(1-t)y)$ is continuous with respect to weak topology on H_1;
(14) quasi-pseudocontractive if $Fix(T) \neq \emptyset$ and $\|Tx-y\|^2 \leqslant \|x-y\|^2 + \|Tx-x\|^2$ for all $x \in C$ and for all $y \in Fix(T)$;
(15) α-inverse-strongly monotone (in short α-ism) if $\langle x-y, Tx-Ty\rangle \geqslant \alpha\|Tx-Ty\|^2$ for all $x, y \in C$ and $\alpha > 0$.

Lemma 2.1 ([3]). *Let C be a nonempty closed convex subset of a real Hilbert space H_1. Let $T : C \to H_1$ be a nonexpansive mapping, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in C. If $x_n \rightharpoonup w$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tw = w$.*

Let $f : H_1 \to (-\infty, \infty]$ be a proper, lower-semicontinuous, and convex function. Then the subdifferential ∂f of f is defined by

$$\partial f(x) = \{u \in H_1 : f(y) \geqslant f(x) + \langle y-x, u\rangle \text{ for all } y \in H_1\}.$$

Let C be a nonempty closed convex subset of a real Hilbert space H_1. For each $x \in H_1$, there is a unique element $u \in C$ such that $u = \arg\min_{y\in C} \|x-y\|$. The mapping $P_C : H_1 \to C$ which is defined by $P_C x = \arg\min_{y\in C} \|x-y\|$ for $x \in H_1$ is called the metric projection from H_1 onto C.

Proposition 2.2 ([1]). *Let C be a nonempty subset of a Hilbert space H_1, and let $T : C \to H_1$ be nonexpansive, and $\alpha \in (0,1)$. Then the following are equivalent:*

(i) *T is α-averaged;*
(ii) $(\forall x \in C)(\forall y \in C)$, $\|Tx-Ty\|^2 \leqslant \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(I_1-T)x-(I_1-T)y\|^2$.

Lemma 2.3 ([15]). *Let $T : H_1 \to H_1$ be a k-demicontractive operator with $k < 1$. Denote $T_\lambda = (1-\lambda)I_1 + \lambda T$ for $\lambda \in (0, 1-k)$. Then for any $x \in H_1, z \in Fix(T)$,*

$$\|T_\lambda x - z\|^2 \leqslant \|x-z\|^2 - (1-k-\lambda)\|T_\lambda x - x\|^2.$$

Lemma 2.4 ([20]). *Let C be a nonempty closed convex subset of H_1 and $T : C \to C$ be a k-strictly pseudononspreading mapping with $Fix(T) \neq \emptyset$. Set $T_\lambda = \lambda I_1 + (1-\lambda)T, \lambda \in [k,1)$. Then the following hold:*

(i) $Fix(T_\lambda) = Fix(T)$;
(ii) *T_λ is demiclosed;*
(iii) $\|T_\lambda x - T_\lambda y\|^2 \leqslant \|x-y\|^2 + \frac{2}{1-\lambda}\langle x-T_\lambda x, y-T_\lambda y\rangle - (\lambda-k)\|x-T_\lambda x-(y-T_\lambda y)\|^2$.

The equilibrium problem (**EP**) [2] is the problem:

$$\text{Find } z \in C \text{ such that } g(z,y) \geqslant 0 \text{ for each } y \in C,$$

where $g : C \times C \to \mathbb{R}$ is a bifunction. The solution set of equilibrium problem (**EP**) is denoted by $EP(C, g)$. We say that $g : C \times C \to \mathbb{R}$ satisfies the following conditions (A1)-(A4) if the following conditions hold:

(A1) $g(x, x) = 0$ for each $x \in C$;
(A2) g is monotone, i.e., $g(x, y) + g(y, x) \leqslant 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$, $\limsup_{t \downarrow 0} g(tz + (1 - t)x, y) \leqslant g(x, y)$;
(A4) for each $x \in C$, the scalar function $y \to g(x, y)$ is convex and lower semicontinuous.

Theorem 2.5 ([12]). *Let $g : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4). For $r > 0$, define $T_r^g : H_1 \to C$ by*

$$T_r^g x = \left\{ z \in C : g(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geqslant 0, \ \forall \, y \in C \right\}$$

for all $x \in H$. Then the following hold:

(i) *T_r^g is single-valued;*
(ii) *T_r^g is firmly nonexpansive, that is, $\|T_r^g x - T_r^g y\|^2 \leqslant \langle x - y, T_r^g x - T_r^g y\rangle$ for all $x, y \in H$;*
(iii) *$\{x \in H : \ T_r^g x = x\} = \{x \in C : \ g(x, y) \geqslant 0, \ \forall y \in C\}$;*
(iv) *$\{x \in C : \ g(x, y) \geqslant 0, \ \forall y \in C\}$ is a closed and convex subset of C.*

Here, T_r^g is called the resolvent of g for $r > 0$.

Theorem 2.6 ([14]). *Let $M : C \to H_1$ be a hemicontinuous and monotone mapping. Suppose that M is locally bounded on C. Then, for $r > 0$ and $x \in H_1$, define $T_r : H_1 \to C$ by*

$$T_r x = \left\{ z \in C : \langle y - z, Mz \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geqslant 0, \ \forall \, y \in C \right\}$$

for all $x \in H$. Then the following hold:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive, that is, $\|T_r x - T_r y\|^2 \leqslant \langle x - y, T_r x - T_r y\rangle$ for all $x, y \in H$;*
(iii) *$\{x \in H : \ T_r x = x\} = VI(M, C)$;*
(iv) *$VI(M, C)$ is a closed and convex subset of C.*

Theorem 2.7 ([14]). *Let $T : C \to H_1$ be a hemi-continuous and pseudocontractive mapping. Suppose that T is locally bounded on C. Then, for each $r > 0$ and each $x \in H_1$, define $F_r : H_1 \to C$ by*

$$F_r x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leqslant 0, \ \ \forall \, y \in C \right\}$$

for all $x \in H_1$. Then the following hold:

(i) *F_r is single-valued;*
(ii) *F_r is firmly nonexpansive;*
(iii) *$Fix(F_r) = Fix(T)$;*
(iv) *$Fix(T)$ is a closed and convex subset of C.*

Proposition 2.8 ([5]). *Let $A : H_1 \to H_2$ be a bounded linear operator with $\|A\| > 0$ and $T : H_2 \to H_2$ be an operator satisfying $TAw = Aw$ for some $w \in H_1$. Further let $V = I_1 - \frac{1}{\|A\|^2} A^*(I_2 - T)A$. If T is an α-SQNE operator for some $\alpha \geqslant 0$, then*

(i) *$Fix(V) = A^{-1} Fix(T)$;*
(ii) *V is α-SQNE.*

If T is demiclosed, then V is demiclosed .

Define $L = \{1, 2, \ldots, m\}$ and $\triangle_m = \{\omega = (\omega_1.\omega_2, \ldots, \omega_m) \in \mathbb{R}^m, \omega_i \geqslant 0, i \in L$, and $\sum_{i=1}^m \omega_i = 1\}$.

Proposition 2.9 ([5]). *For each $i \in L$, let $S_i : H_1 \to H_1$ be demiclosed and ρ_i-SQNE Suppose that $\bigcap_{i\in L} \mathrm{Fix}(S_i) \neq \emptyset$. Let $S = \sum_{i=1}^m \omega_i S_i$, where $\omega \in \triangle_m$. Then S is a ρ-SQNE operator with $\rho = \sum_{i=1}^m (\frac{\omega_i}{\rho_i+1})^{-1} - 1$ and S is demiclosed .*

Proposition 2.10 ([1]). *Let C be a nonempty subset of H_1, let $\{T_i\}_{i\in I}$ be a finite family of quasi-nonexpansive operators from C to H_1 such that $\bigcap_{i\in I} \mathrm{Fix}(T_i) \neq \emptyset$, let $\{\omega_i : i \in I\}$ be strict positive numbers such that $\sum_{i\in I} \omega_i = 1$. Then $\mathrm{Fix}(\sum_{i\in I} \omega_i T_i) = \bigcap_{i\in I} \mathrm{Fix}(T_i)$.*

Lemma 2.11 ([9]). *Let $T : H_1 \to H_1$ be a L_1-Lipschitz continuous mapping with $L_1 > 0$. Denote $K = (1-\xi)I_1 + \xi T(1-\eta)I_1 + \eta T)$. If $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L_1^2}}$, then*

(i) $\mathrm{Fix}(T) = \mathrm{Fix}(K)$;

(ii) *if T is demiclosed, then K is also demiclosed;*

(iii) *in addition, if $T : H_1 \to H_1$ is quasi-pseudocontractive, then K is quasi-nonexpansive.*

Let C be a nonempty closed convex subset of H_1, and let the indicate function $\iota_C : H_1 \to [0, \infty]$ be defined by

$$\iota_C x = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then ι_C is a proper, lower semicontinuous, and convex function and $J_\lambda^{\partial \iota_C} = P_C$.

Let $g \in \Gamma_0(H_1)$ and $\lambda \in (0, \infty)$. The proximal operator of $g \in \Gamma_0(H_1)$ of order $\lambda \in (0, \infty)$ is

$$\mathrm{prox}_{\lambda g} x = \arg\min_{v\in H_1} \{g(v) + \frac{1}{2\lambda}\|v - x\|^2\},\ x \in H_1.$$

Lemma 2.12 ([1]). *Let $g \in \Gamma_0(H_1)$ and $\lambda \in (0, \infty)$. Then*

(i) $\mathrm{prox}_{\lambda g} = (I_1 + \lambda \partial g)^{-1} = J_\lambda^{\partial g}$;

(ii) $\mathrm{prox}_{\lambda g}$ *is firmly nonexpansive;*

(iii) *if C is a nonempty closed convex subset of H_1 and $g = i_C$, then $\mathrm{prox}_{\lambda g} = P_C$ for all $\lambda \in (0, \infty)$.*

Lemma 2.13 ([16]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a k-strictly pseudocontractive mapping. Then T is demiclosed.*

3. Variational inequalities over split equality fixed point for finite families of nonlinear mappings

For each $i \in \{1, 2, 3, 4\}$, let H_i be a Hilbert space, I_i be the identity mapping on H_i, $V_i : H_i \to H_i$ be L_i-Lipschitz continuous, $F_i : H_i \to H_i$ be κ_i-Lipschitz continuous and η_i-strongly monotone with $\kappa_i > 0$, and $\eta_i > 0$. Let

$$L = \max_{i\leqslant i\leqslant 3} L_i,\ \kappa = \max_{1\leqslant i\leqslant 3} \kappa_i,\ \eta = \min_{1\leqslant i\leqslant 3} \eta_i,\ \mu \in (0, \tfrac{2\eta}{\kappa^2}) \text{ and } \gamma \in (0, \tfrac{\tau}{L}),$$

where $\tau = \mu(\eta - \frac{1}{2}\mu\kappa^2)$. For each $i \in \{1, 2, 3\}$, let $A_i : H_i \to H_4$ be a bounded linear operator with adjoint A_i^*. Suppose that $\|A_i\| > 0$, $0 < \xi < \frac{1}{\sum_{i=1}^3 \|A_i\|^2}$. Let $B_1 : H_1 \to H_2$ be a bounded linear operator with adjoint B_1^* and let $B_2 : H_1 \to H_3$ be a bounded linear operator with adjoint B_2^*. Suppose that $\|B_1\| > 0$ and $\|B_2\| > 0$. The product $\bigotimes_{1\leqslant i\leqslant 3} H_i = H_1 \times H_2 \times H_3$ is a Hilbert space with inner product and norm given by

$$\langle x, y\rangle = \sum_{i=1}^3 \langle x_i, y_i\rangle \text{ and } \|x\|^2 = \sum_{i=1}^3 \|x_i\|^2$$

for any $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \bigotimes_{1\leqslant i\leqslant 3} H_i$. For any $x = (x_1, x_2, x_3) \in \bigotimes_{1\leqslant i\leqslant 3} H_i$, let $V, F, I = I_1 \times I_2 \times I_3 : \bigotimes_{1\leqslant i\leqslant 3} H_i \to \bigotimes_{1\leqslant i\leqslant 3} H_i$ be defined by

$$\begin{cases} I(x) = (x_1, x_2, x_3), \\ V(x) = (V_1(x_1), V_2(x_2), V_3(x_3)), \\ F(x) = (F_1(x_1), F_2(x_2), F_3(x_3)). \end{cases}$$

Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $(0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. In the following, we use these notations and assumptions unless specified otherwise.

Theorem 3.1 ([14]). *For each $i \in \{1, 2, 3\}$, let $\rho_i > 0$ and let $T_i : H_i \to H_i$ be a demiclosed ρ_i-strongly quasi-nonexpansoive mapping. Suppose that*

$$\Lambda = \{(x, y, z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \mathrm{Fix}(T_1), y \in \mathrm{Fix}(T_2), z \in \mathrm{Fix}(T_3)), A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, and let the sequences $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)T_1(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_2 - \mu\alpha_n F_2)T_2(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_3 - \mu\alpha_n F_3)T_3(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Remark 3.2. The assumption " $\rho_i > 0$ " is needed in [14, Theorem 3.2].

The following theorem and corollary are essential tools in this paper.

Theorem 3.3. *For each $i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, \ell\}, k \in \{1, 2, \dots, s\}$, let $\sigma_i > 0, r_j > 0$ and $\delta_k > 0$, and let*

(i) $M_i : H_1 \to H_1$ *be a demiclosed σ_i-strongly quasi-nonexpansive mapping;*

(ii) $Q_j : H_2 \to H_2$ *be a demiclosed r_j-strongly quasi-nonexpansive mapping;*

(iii) $G_k : H_3 \to H_3$ *be a demiclosed δ_k-strongly quasi-nonexpansive mapping.*

Let $(\zeta_1, \zeta_2, \dots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \dots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \dots, \omega_s) \in \Delta_s$. Suppose that

$$\Lambda = \{(x, y, z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} \mathrm{Fix}(M_i), y \in \bigcap_{j=1}^{\ell} \mathrm{Fix}(Q_j),\ z \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_k), A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, and let the sequences $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m} \zeta_i M_i(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_2 - \mu\alpha_n F_2)\sum_{j=1}^{\ell} \theta_j Q_j(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_3 - \mu\alpha_n F_3)\sum_{k=1}^{s} \omega_k G_k(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Let

(i) $T_1 = \sum_{i=1}^{m} \zeta_i M_i$;

(ii) $T_2 = \sum_{j=1}^{\ell} \theta_j Q_j$;

(iii) $T_3 = \sum_{k=1}^{s} \omega_k G_k$.

By Proposition 2.9,

(i) T_1 is a demiclosed ρ_1-strongly quasi-nonexpansive mapping for some $\rho_1 > 0$;

(ii) T_2 is a demiclosed ρ_2-strongly quasi-nonexpansive mapping for some $\rho_2 > 0$;
(iii) T_3 is a demiclosed ρ_3-strongly quasi-nonexpansive mapping for some $\rho_3 > 0$.

By Proposition 2.10,

(i) $\mathrm{Fix}(T_1) = \bigcap_{i=1}^{m} \mathrm{Fix}(M_i)$;
(ii) $\mathrm{Fix}(T_2) = \bigcap_{j=1}^{\ell} \mathrm{Fix}(Q_j)$;
(iii) $\mathrm{Fix}(T_3) = \bigcap_{k=1}^{s} \mathrm{Fix}(G_k)$.

Let $\Omega = \{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \mathrm{Fix}(T_1), y \in \mathrm{Fix}(T_2), z \in \mathrm{Fix}(T_3)),\ A_1(x) = A_2(y) = A_3(z)\}$.
It is easy to see that $\Omega = \Lambda \neq \emptyset$. Then Theorem 3.3 follows from Theorem 3.1. □

Corollary 3.4. *For each $i \in \{1,2,3,\}$, let $V_i : H_1 \to H_1$ be L_i-Lipschitz continuous, $F_i : H_1 \to H_1$ be κ_i-Lipschitz continuous and η_i-strongly monotone with $\kappa_i > 0$, and $\eta_i > 0$.*
For each $i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,\ell\}, k \in \{1,2,\dots,s\}$, let $\sigma_i > 0, r_j > 0$ and $\delta_k > 0$, and let

(i) *$M_i : H_1 \to H_1$ be a demiclosed σ_i-strongly quasi-nonexpansive mapping;*
(ii) *$Q_j : H_1 \to H_1$ be a demiclosed r_j-strongly quasi-nonexpansive mapping;*
(iii) *$G_k : H_1 \to H_1$ be a demiclosed δ_k-strongly quasi-nonexpansive mapping.*

Let $(\zeta_1, \zeta_2, \dots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \dots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \dots, \omega_s) \in \Delta_s$. Suppose that

$$\Lambda = \{(x,x,x) : x \in H_1, x \in \bigcap_{i=1}^{m} \mathrm{Fix}(M_i) \bigcap \bigcap_{j=1}^{\ell} \mathrm{Fix}(Q_j) \bigcap \bigcap_{k=1}^{s} \mathrm{Fix}(G_k)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, and let the sequences $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m} \zeta_i M_i(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ *for all* $n \in \mathbb{N}$;
(ii) $y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_1 - \mu\alpha_n F_2)\sum_{j=1}^{\ell} \theta_j Q_j(y_n - \frac{\xi}{3}(2y_n - x_n - z_n))$ *for all* $n \in \mathbb{N}$;
(iii) $z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_1 - \mu\alpha_n F_3)\sum_{k=1}^{s} \omega_k G_k(z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Let $H_1 = H_2 = H_3 = H_4, A_1 = I_1 = A_2 = A_3$ in Theorem 3.3, then Corollary 3.4 follows from Theorem 3.3. □

Theorem 3.5. *For each $i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,\ell\}, k \in \{1,2,\dots,s\}$, let*

(i) *$M_i : H_1 \to H_1$ be a demiclosed σ_i-demicontractive mapping;*
(ii) *$Q_j : H_2 \to H_2$ be a demiclosed r_j-demicontractive mapping;*
(iii) *$G_k : H_3 \to H_3$ be a demiclosed δ_k-demicontractive mapping.*

Let

(i) $M_{i\lambda_i} = (1-\lambda_i)I_1 + \lambda_i M_i$ *for* $\lambda_i \in (0, 1-\sigma_1)$;
(ii) $Q_{j\beta_j} = (1-\beta_j)I_2 + \beta_j Q_j$ *for* $\beta_j \in (0, 1-r_j)$;
(iii) $G_{k\eta_k} = (1-\eta_k)I_3 + \eta_k G_k$ *for* $\eta_k \in (0, 1-\delta_k)$.

Let $(\zeta_1, \zeta_2, \dots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \dots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \dots, \omega_s) \in \Delta_s$. Suppose that $\Lambda = \{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} \mathrm{Fix}(M_i), y \in \bigcap_{j=1}^{\ell} \mathrm{Fix}(Q_j),\ z \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_k), A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset$.
Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, and let the sequences $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m} \zeta_i M_{i\lambda_i}(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;
(ii) $y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_2 - \mu\alpha_n F_2)\sum_{j=1}^{\ell} \theta_j Q_{j\beta_j}(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_3 - \mu\alpha_n F_3)\sum_{k=1}^{s} \omega_k G_{k\eta_k}(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Since for each $i \in \{1, 2, \ldots, m, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}$,

(i) $M_i : H_1 \to H_1$ be a demiclosed σ_i-demicontractive mapping;
(ii) $Q_j : H_2 \to H_2$ be a demiclosed r_j-demicontractive mapping;
(iii) $G_k : H_3 \to H_3$ be a demiclosed δ_k-demicontractive mapping.

It follows from Lemma 2.3 for each $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}$, $\lambda_i \in (0, 1 - \sigma_i)$, $\beta_j \in (0, 1 - r_j)$, and $\eta_k \in (0, 1 - \delta_k)$ that

(i) $M_{i\lambda_i}$ is a demiclosed $(1 - \sigma_i - \lambda_i)$-strongly quasi-nonexpansive mapping;
(ii) $Q_{j\beta_j}$ is a demiclosed $(1 - r_j - \beta_j)$-strongly quasi-nonexpansive mapping;
(iii) $G_{k\eta_k}$ is a demiclosed $(1 - \delta_k - \eta_k)$-strongly quasi-nonexpansive mapping.

It is easy to see that

(i) $Fix(M_{i\lambda_i}) = Fix(M_i)$;
(ii) $Fix(Q_{j\beta_j}) = Fix(Q_j)$;
(iii) $Fix(G_{k\eta_k}) = Fix(G_k)$.

Let $\Omega = \{(x, y, z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} Fix(M_{i\lambda_i}), y \in \bigcap_{j=1}^{\ell} Fix(Q_{j\beta_j}),\ z \in \bigcap_{k=1}^{s} Fix(G_{k\eta_k})), A_1(x) = A_2(y) = A_3(z)\}$.

It is easy to see that $\Omega = \Lambda \neq \emptyset$. Then Theorem 3.5 follows from Theorem 3.3. □

Remark 3.6.

(i) Theorem 3.5 improves and generalizes [24, Theorem 3.2] and [22, Theorem 3.3]. In [22, Theorem 3.3], $V = (f_1, f_2)$ is a contraction mapping and $F = (I_1, I_2)$.
(ii) Theorem 3.5 extends [14, Theorem 4.1] from variational inequality over split equality fixed points of m demicontractive mappings to variational inequality over split equality of three families of demicontractive mappings.

Corollary 3.7. *For each* $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}$, *let*

(i) $M_i : H_1 \to H_1$ *be a demiclosed quasi-nonexpansive mapping;*
(ii) $Q_j : H_2 \to H_2$ *be a demiclosed quasi-nonexpansive m mapping;*
(iii) $G_k : H_3 \to H_3$ *be a demiclosed quasi-nonexpansive mapping.*

Let

(i) $M_{i\lambda_i} = (1 - \lambda_i)I_1 + \lambda_i M_i$ *for* $\lambda_i \in (0, 1)$;
(ii) $Q_{j\beta_j} = (1 - \beta_j)I_2 + \beta_j Q_j$ *for* $\beta_j \in (0, 1)$;
(iii) $G_{k\eta_k} = (1 - \eta_k)I_3 + \eta_k G_k$ *for* $\eta_k \in (0, 1)$.

Let $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$. *Suppose that*

$$\Lambda = \{(x, y, z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} Fix(M_i), y \in \bigcap_{j=1}^{\ell} Fix(Q_j),\ z \in \bigcap_{k=1}^{s} Fix(G_k), A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m} \zeta_i M_{i\lambda_i}(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;
(ii) $y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_2 - \mu\alpha_n F_2)\sum_{j=1}^{\ell} \theta_j Q_{j\beta_j}(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_3 - \mu\alpha_n F_3)\sum_{k=1}^{s} \omega_k G_{k\eta_k}(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Since a quasi-nonexpansive mapping is a 0-demicontractive mapping, Corollary 3.7 follows from Theorem 3.5. □

Remark 3.8.

(i) Corollary 3.7 improves and generalizes [13, Theorem 3.1]. [13, Theorem 3.1] established a strong convergence theorem for $VI(L - \gamma f, \Lambda)$, where L is a strongly positive bounded self-adjoint linear operator, f is a contraction mapping, and Λ is a multiple sets split fixed point of quasi-nonexpansive mappings. Since a strongly positive bounded self-adjoint operator is a Lipschitz continuous and strongly monotone operator.

(ii) Corollary 3.7 generalizes [5, Corollary 5.1] which established a weak convergence of multiple sets split fixed point theorem of quasi-nonexpansive mappings.

(iii) Corollary 3.7 also extends [27, Theorems 3.2 and 3.4]. In [27, Theorems 3.2 and 3.4], $V = (f_1, f_2)$ is a contraction mapping and $F = (I_1, I_2)$.

Theorem 3.9. *For each* $i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, \ell\}, k \in \{1, 2, \dots, s\}$, *let*

(i) $M_i : H_1 \to H_1$ *be a* σ_i*-strictly pseudo-nonspreading mapping;*

(ii) $Q_j : H_2 \to H_2$ *be a* r_j*-strictly pseudo-nonspreading mapping;*

(iii) $G_k : H_3 \to H_3$ *be a demiclosed* δ_k*-strictly pseudo-nonspreading mapping.*

Let

(i) $M_{i\lambda_i} = (1 - \lambda_i)I_1 + \lambda_i M_i$ *for* $\lambda_i \in (\sigma_i, 1)$;

(ii) $Q_{j\beta_j} = (1 - \beta_j)I_2 + \beta_j Q_j$ *for* $\beta_j \in (r_j, 1)$;

(iii) $G_{k\eta_k} = (1 - \eta_k)I_3 + \eta_k G_k$ *for* $\eta_k \in (\delta_k, 1)$.

Let $(\zeta_1, \zeta_2, \dots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \dots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \dots, \omega_s) \in \Delta_s$. *Suppose that*

$$\Lambda = \{(x, y, z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} Fix(M_i), y \in \bigcap_{j=1}^{\ell} Fix(Q_j),\ z \in \bigcap_{k=1}^{s} Fix(G_k), A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m} \zeta_i M_{i\lambda_i}(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_2 - \mu\alpha_n F_2)\sum_{j=1}^{\ell} \theta_j Q_{j\beta_j}(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_3 - \mu\alpha_n F_3)\sum_{k=1}^{s} \omega_k G_{k\eta_k}(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. By assumptions and Lemma 2.4, for each $i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, \ell\}, k \in \{1, 2, \dots, s\}$, and for $\lambda_i \in (\sigma_i, 1), \beta_j \in (r_j, 1)$ and $\eta_k \in (\delta_k, 1)$ we have that

(i) $Fix(M_{i\lambda_i}) = Fix(M_i)$, $Fix(Q_{j\beta_j}) = Fix(Q_j)$, $Fix(G_{k\eta_k}) = Fix(G_k)$;

(ii) $M_{i\lambda_i}, Q_{j\beta_j}$, and $G_{k\eta_k}$ are demiclosed;

(iii) $\|M_{i\lambda_i}x - M_{i\lambda_i}y\|^2 \leqslant \|x - y\|^2 + \frac{2}{1-\lambda_i}\langle x - M_{i\lambda_i}x, y - M_{i\lambda_i}y\rangle - (\lambda_i - \sigma_i)\|x - M_{i\lambda_i}x - (y - M_{i\lambda_i}y)\|^2$;

(iv) $\|Q_{j\beta_j}x - Q_{j\beta_j}y\|^2 \leqslant \|x - y\|^2 + \frac{2}{1-\beta_j}\langle x - Q_{j\beta_j}x, y - Q_{j\beta_j}y\rangle - (\beta_j - r_j)\|x - Q_{j\beta_j}x - (y - Q_{j\beta_j}y)\|^2$;

(v) $\|G_{k\beta_j}x - Q_{j\beta_j}y\|^2 \leqslant \|x - y\|^2 + \frac{2}{1-\eta_k}\langle x - G_{k\eta_k}x, y - G_{k\eta_k}y\rangle - (\eta_k - \delta_k)\|x - G_{k\eta_j}x - (y - G_{j\eta_k}y)\|^2$.

It is easy to see that for each $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}$, and for $\lambda_i \in (\sigma_i, 1), \beta_j \in (r_j, 1)$ and $\eta_k \in (\delta_k, 1)$ that

(i) $M_{i\lambda_i}$ is a demiclosed $(\lambda_i - \sigma_i)$-strongly quasi-nonexpansive mapping;
(ii) $Q_{j\beta_j}$ is a demiclosed $(\beta_j - r_j)$-strongly quasi-nonexpansive mapping;
(iii) $G_{k\eta_k}$ is a demiclosed $(\eta_k - \delta_k)$-strongly quasi-nonexpansive mapping.

Let

$$\Omega=\{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} \mathrm{Fix}(M_{i\lambda_i}), y \in \bigcap_{j=1}^{\ell} \mathrm{Fix}(Q_{j\beta_j}),\ z \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_{k\delta_k}), A_1(x) = A_2(y) = A_3(z)\}.$$

It is easy to see that $\Omega = \Lambda \neq \emptyset$. Then Theorem 3.9 follows from Theorem 3.3. □

Remark 3.10.

(i) Theorem 3.9 improves and generalizes [10, Theorem 3.1]. [10, Theorem 3.1] established a weak convergence theorem for split equality multiple sets fixed point of strictly pseudo-nonspreading mappings. The simultaneous iteration in Theorem 3.9 is different from the simultaneous iterations in [10, Theorem 3.1].
(ii) In [14, Theorem 3.7], the authors studied variational inequality problem over split equality fixed point for m strictly pseudo-nonspreading mappings, but Theorem 3.9 studies variational inequality problem over split equality fixed point for three finite families of strictly pseudo-nonspreading mappings.

Theorem 3.11. *For each $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}$, let*

(i) $M_i : H_1 \to H_1$ *be a σ_i-strictly pseudo-nonspreading mapping;*
(ii) $Q_j : H_2 \to H_2$ *be a r_j-strictly pseudo-contractive mapping;*
(iii) $G_k : H_3 \to H_3$ *be a demiclosed δ_k-demicontractive mapping.*

Let

(i) $M_{i\lambda_i} = (1 - \lambda_i)I_1 + \lambda_i M_i$ *for* $\lambda_i \in (\sigma_i, 1)$;
(ii) $Q_{j\beta_j} = (1 - \beta_j)I_2 + \beta_j Q_j$ *for* $\beta_j \in (0, 1 - r_j)$;
(iii) $G_{k\eta_k} = (1 - \eta_k)I_3 + \eta_k G_k$ *for* $\eta_k \in (0, 1 - \delta_k)$.

Let $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$. *Suppose that*

$$\Lambda = \{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} \mathrm{Fix}(M_i), y \in \bigcap_{j=1}^{\ell} \mathrm{Fix}(Q_j),\ z \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_k), A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m} \zeta_i M_{i\lambda_i}(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;
(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_2 - \mu\alpha_n F_2)\sum_{j=1}^{\ell} \theta_j Q_{j\beta_j}(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;
(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_3 - \mu\alpha_n F_3)\sum_{k=1}^{s} \omega_k G_{k\eta_k}(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. We see in the proof of Theorem 3.9 that $M_{i\lambda_i}$ is a demiclosed $(\lambda_i - \sigma_i)$-strongly quasi-nonexpansive mapping and $\mathrm{Fix}(M_{i\lambda_i}) = \mathrm{Fix}(M_i)$ for each $i \in \{1, 2, \ldots, m\}$.

Since for each $j \in \{1, 2, \ldots, \ell\}$, $Q_j : H_2 \to H_2$ is a r_j-strictly pseudo-contractive mapping. It is easy to see that Q_j is a r_j-demicontractive mapping for each $j \in \{1, 2, \ldots, \ell\}$. For each $j \in \{1, 2, \ldots, \ell\}$, by Lemma 2.13, Q_j is demiclosed.

It follows from Lemma 2.3 for each $j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}$, $\beta_j \in (0, 1 - r_j)$ and $\eta_k \in (0, 1 - \delta_k)$ that

(i) $Q_{j\beta_j}$ is a demiclosed $(1-r_j-\beta_j)$-strongly quasi-nonexpansive mapping;
(ii) $G_{k\eta_k}$ is a demiclosed $(1-\delta_k-\eta_k)$-strongly quasi-nonexpansive mapping.

Let

$$\Omega=\{(x,y,z)\in\bigotimes_{1\leqslant i\leqslant 3}H_i : x\in\bigcap_{i=1}^{m}\mathrm{Fix}(M_{i\lambda_i}), y\in\bigcap_{j=1}^{\ell}\mathrm{Fix}(Q_{j\beta_j}),\ z\in\bigcap_{k=1}^{s}\mathrm{Fix}(G_{k\delta_k}), A_1(x)=A_2(y)=A_3(z)\}.$$

It is easy to see that $\Omega=\Lambda\neq\emptyset$. Then Theorem 3.11 follows from Theorem 3.3. □

Corollary 3.12. *Let $V_i : H_1\to H_1$ be L_i-Lipschitz continuous, $F_i : H_1\to H_1$ be κ_i-Lipschitz continuous and η_i-strongly monotone with $\kappa_i>0$, and $\eta_i>0$. For each $i\in\{1,2,\dots,m\}, j\in\{1,2,\dots,\ell\}, k\in\{1,2,\dots,s\}$, let*

(i) *$M_i : H_1\to H_1$ be a σ_i-strictly pseudo-nonspreading mapping;*
(ii) *$Q_j : H_1\to H_1$ be a r_j-strictly pseudo-contractive mapping;*
(iii) *$G_k : H_1\to H_1$ be a demiclosed δ_k-demicontractive mapping.*

Let

(i) $M_{i\lambda_i}=(1-\lambda_i)I_1+\lambda_iM_i$ *for* $\lambda_i\in(\sigma_i,1)$;
(ii) $Q_{j\beta_j}=(1-\beta_j)I_1+\beta_jQ_j$ *for* $\beta_j\in(0,1-r_j)$;
(iii) $G_{k\eta_k}=(1-\eta_k)I_1+\eta_kG_k$ *for* $\eta_k\in(0,1-\delta_k)$.

Let $(\zeta_1,\zeta_2,\dots,\zeta_m)\in\Delta_m,(\theta_1,\theta_2,\dots,\theta_\ell)\in\Delta_\ell,(\omega_1,\omega_2,\dots,\omega_s)\in\Delta_s$. *Suppose that*

$$\Lambda=\{(x,x,x) : x\in H_1, x\in\bigcap_{i=1}^{m}\mathrm{Fix}(M_i)\bigcap\bigcap_{j=1}^{\ell}\mathrm{Fix}(Q_j)\bigcap\bigcap_{k=1}^{s}\mathrm{Fix}(G_k)\}\neq\emptyset.$$

Let $x_1\in H_1$, $y_1\in H_1$, $z_1\in H_1$, *and let the sequences* $\{(x_n,y_n,z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1}=\alpha_n\gamma V_1(x_n)+(I_1-\mu\alpha_nF_1)\sum_{i=1}^{m}\zeta_iM_{i\lambda_i}(x_n-\frac{\xi}{3}(2x_n-y_n-z_n))$ *for all* $n\in\mathbb{N}$;
(ii) $y_{n+1}=\alpha_n\gamma V_2(y_n)+(I_1-\mu\alpha_nF_2)\sum_{j=1}^{\ell}\theta_jQ_{j\beta_j}(y_n-\frac{\xi}{3}(2y_n-x_n-z_n))$ *for all* $n\in\mathbb{N}$;
(iii) $z_{n+1}=\alpha_n\gamma V_3(z_n)+(I_1-\mu\alpha_nF_3)\sum_{k=1}^{s}\omega_kG_{k\eta_k}(z_n-\frac{\xi}{3}(2z_n-x_n-y_n))$ *for all* $n\in\mathbb{N}$.

Then $\lim_{n\to\infty}(x_n,y_n,z_n)\in VI(\mu F-\gamma V,\Lambda)$.

Proof. Let $H_1=H_2=H_3=H_4, A_1=I_1=A_2=A_3$ in Theorem 3.11, then Corollary 3.12 follows from Theorem 3.11. □

Theorem 3.13. Let $V_i : H_1\to H_1$ be L_i-Lipschitz continuous, $F_i : H_1\to H_1$ be κ_i-Lipschitz continuous and η_i-strongly monotone with $\kappa_i>0$, and $\eta_i>0$. For each $i\in\{1,2,\dots,m\}, j\in\{1,2,\dots,\ell\}, k\in\{1,2,\dots,s\}$, let

(i) $M_i : H_1\to H_1$ be a σ_i-strictly pseudo-nonspreading mapping;
(ii) $Q_j : H_2\to H_2$ be a r_j-strictly pseudo-contractive mapping;
(iii) $G_k : H_3\to H_3$ be a demiclosed δ_k-demicontractive mapping.

Let

(i) $M_{i\lambda_i}=(1-\lambda_i)I_1+\lambda_iM_i$ for $\lambda_i\in(\sigma_i,1)$;
(ii) $Q_{j\beta_j}=(1-\beta_j)I_2+\beta_jQ_j$ for $\beta_j\in(0,1-r_j)$;
(iii) $G_{k\eta_k}=(1-\eta_k)I_3+\eta_kG_k$ for $\eta_k\in(0,1-\delta_k)$.

Let $(\zeta_1,\zeta_2,\dots,\zeta_m)\in\Delta_m,(\theta_1,\theta_2,\dots,\theta_\ell)\in\Delta_\ell,(\omega_1,\omega_2,\dots,\omega_s)\in\Delta_s$. Suppose that

$$\Lambda=\{(x,x,x) : x\in H_1, x\in\bigcap_{i=1}^{m}\mathrm{Fix}(M_i), B_1(x)\in\bigcap_{j=1}^{\ell}\mathrm{Fix}(Q_j), B_2(x)\in\bigcap_{k=1}^{s}\mathrm{Fix}(G_k)\}\neq\emptyset.$$

Let $x_1\in H_1$, $y_1\in H_1$, $z_1\in H_1$, and let the sequences $\{(x_n,y_n,z_n)\}_{n\in\mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m} \zeta_i M_{i\lambda_i}(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ for all $n \in \mathbb{N}$;
(ii) $y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_1 - \mu\alpha_n F_2)\sum_{j=1}^{\ell} \theta_j (I_1 - \frac{1}{\|B_1\|^2} B_1^*(I_2 - Q_{j\beta_j})B_1)(y_n - \frac{\xi}{3}(2y_n - x_n - z_n))$ for all $n \in \mathbb{N}$;
(iii) $z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_1 - \mu\alpha_n F_3)\sum_{k=1}^{s} \omega_k (I_1 - \frac{1}{\|B_2\|^2} B_2^*(I_3 - G_{k\eta_k})B_2)(z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ for all $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Theorem 3.11 shows that for each $j \in \{1,2,\dots,\ell\}$, $k \in \{1,2,\dots,s\}$, $\beta_j \in (0, 1-r_j)$, and $\eta_k \in (0, 1-\delta_k)$,

(i) $Q_{j\beta_j}$ is a demiclosed $(1 - r_j - \beta_j)$-strongly quasi-nonexpansive mapping;
(ii) $G_{k\eta_k}$ is a demiclosed $(1 - \delta_k - \eta_k)$-strongly quasi-nonexpansive mapping.

Let

(i) $V_{j\beta_j} = (I_1 - \frac{1}{\|B_1\|^2} B_1^*(I_2 - Q_{j\beta_j})B_1)$;
(ii) $W_{k\eta_k} = (I_1 - \frac{1}{\|B_2\|^2} B_2^*(I_3 - G_{k\eta_k})B_2)$.

By Proposition 2.8, for $\beta_j \in (0, 1-r_j)$, and for $\eta_k \in (0, 1-\delta_k)$,

(i) $\mathrm{Fix}(V_{j\beta_j}) = B_1^{-1}\mathrm{Fix}(Q_j)$ and $\mathrm{Fix}(W_{k\eta_k}) = B_2^{-1}\mathrm{Fix}(G_k)$.
(ii) $V_{j\beta_j} : H_1 \to H_1$ is a demiclosed $(1 - r_j - \beta_j)$-strongly quasi-nonexpansive mapping, and $W_{k\eta_k} : H_1 \to H_1$ is a demiclosed $(1 - \delta_k - \eta_k)$-strongly quasi-nonexpansive mapping.

Theorem 3.11 shows that $M_{i\lambda_i}$ is a demiclosed $(\lambda_i - \sigma_i)$-strongly quasi-nonexpansive mapping for $\lambda_i \in (\sigma_i, 1)$, and $\mathrm{Fix}(M_{i\lambda_i}) = \mathrm{Fix}(M_i)$.

Let

$$\Omega = \{(x,x,x) : x \in H_1, x \in \bigcap_{i=1}^{m} \mathrm{Fix}(M_{i\lambda_i}),\ x \in \bigcap_{j=1}^{\ell} \mathrm{Fix}(V_{j\beta_j}), x \in \bigcap_{k=1}^{s} \mathrm{Fix}(W_{k\eta_k})\}.$$

It is easy to see that $\Omega = \Lambda \neq \emptyset$. Then Theorem 3.13 follows from Corollary 3.12. □

Theorem 3.14. *For each* $i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,\ell\}, k \in \{1,2,\dots,s\}$, *let*

(i) $P_i : H_1 \to H_1$ *be a* σ_i*-Lipschitz continuous demiclosed quasi-pseudocontractive mapping;*
(ii) $R_j : H_2 \to H_2$ *be a* ρ_j*-Lipschitz continuous demiclosed quasi-pseudocontractive mapping;*
(iii) $W_k : H_3 \to H_3$ *be a* δ_k*-Lipschitz continuous demiclosed quasi-pseudocontractive mapping.*

For each $i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,\ell\}$, *and* $k \in \{1,2,\dots,s\}$, *let*

(i) $M_i = (1-\xi_i)I_1 + \xi_i P_i(1-\gamma_i)I_1 + \gamma_1 P_i$;
(ii) $Q_j = (1-\sigma_j)I_2 + \sigma_j R_j(1-\nu_j)I_2 + \nu_j R_j$;
(iii) $G_k = (1-\rho_k)I_3 + \rho_k W_k(1-\delta_k)I_3 + \delta_k W_k$,

where $0 < \xi_i < \gamma_i < \frac{1}{1+\sqrt{1+\sigma_i^2}}, i \in \{1,2,\dots,m\}, 0 < \omega_j < \nu_j < \frac{1}{1+\sqrt{1+\rho_j^2}}, j \in \{1,2,\dots,\ell\}$, *and* $0 < \rho_k < \xi_k < \frac{1}{1+\sqrt{1+\delta_k^2}}, k \in \{1,2,\dots,s\}$, *and let*

(i) $M_{i\lambda_i} = (1-\lambda_i)I_1 + \lambda_i M_i$ *for* $\lambda_i \in (0,1)$;
(ii) $Q_{j\beta_j} = (1-\beta_j)I_2 + \beta_j Q_j$ *for* $\beta_j \in (0,1)$;
(iii) $G_{k\eta_k} = (1-\eta_k)I_3 + \eta_k G_k$ *for* $\eta_k \in (0,1)$.

Let $(\zeta_1, \zeta_2, \dots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \dots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \dots, \omega_s) \in \Delta_s$. *Suppose that*

$$\Lambda = \{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} \mathrm{Fix}(P_i), y \in \bigcap_{j=1}^{\ell} \mathrm{Fix}(R_j),\ z \in \bigcap_{k=1}^{s} \mathrm{Fix}(W_k)), A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m} \zeta_i M_{i\lambda_i}(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;
(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_2 - \mu\alpha_n F_2)\sum_{j=1}^{\ell} \theta_j Q_{j\beta_j}(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;
(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_3 - \mu\alpha_n F_3)\sum_{k=1}^{s} \omega_k G_{k\eta_k}(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. By Lemma 2.11, for each $i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,\ell\}, k \in \{1,2,\dots,s\}$,

(i) $Fix(M_i) = Fix(P_i), Fix(R_j) = Fix(Q_j), Fix(W_k) = Fix(G_k)$;
(ii) M_i, Q_j, and G_k are demiclosed at 0;
(iii) M_i, Q_j, and G_k are quasi-nonexpansive mappings.

Let

$$\Omega = \{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} Fix(M_i), y \in \bigcap_{j=1}^{\ell} Fix(Q_j),\ z \in \bigcap_{k=1}^{s} Fix(G_k), A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.$$

It is easy to see that $\Omega = \Lambda \neq \emptyset$ and Theorem 3.14 follows from Corollary 3.7. □

Remark 3.15. Since firmly quasinonexpansive mapping, pseudo-contractive mappings, k-strict pseudo-contractive mapping, k-strict pseudo-nonspreading mapping, demi-contractive mappings, and directed operators are special cases of quasi-pseudo-contractive mappings, we see that Theorem 3.14 extends many results on fixed point problems, multiple sets split fixed point problems and split equality fixed point problems existing in the literature.

Corollary 3.16. *For each* $i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,\ell\}, k \in \{1,2,\dots,s\}$, *let*

(i) $P_i : H_1 \to H_1$ *be a* σ_i*-Lipschitz continuous demiclosed quasi-pseudocontractive mapping;*
(ii) $R_j : H_2 \to H_2$ *be a* ρ_j*-Lipschitz continuous demiclosed quasi-pseudocontractive mapping;*
(iii) $W_k : H_3 \to H_3$ *be a* δ_k*-Lipschitz continuous demiclosed quasi-pseudocontractive mapping.*

For each $i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,\ell\}$, *and* $k \in \{1,2,\dots,s\}$, *let*

(i) $M_i = (1-\xi_i)I_1 + \xi_i P_i(1-\gamma_i)I_1 + \gamma_1 P_i$;
(ii) $Q_j = (1-\sigma_j)I_2 + \sigma_j R_j(1-\nu_j)I_2 + \nu_j R_j$;
(iii) $G_k = (1-\rho_k)I_3 + \rho_k W_k(1-\delta_k)I_3 + \delta_k W_k$.

where $0 < \xi_i < \gamma_i < \frac{1}{1+\sqrt{1+\sigma_i^2}}, i \in \{1,2,\dots,m\}$, $0 < \omega_j < \nu_j < \frac{1}{1+\sqrt{1+\rho_j^2}}, j \in \{1,2,\dots,\ell\}$, *and* $0 < \rho_k < \xi_k < \frac{1}{1+\sqrt{1+\delta_k^2}}, k \in \{1,2,\dots,s\}$, *and let*

(i) $M_{i\lambda_i} = (1-\lambda_i)I_1 + \lambda_i M_i$ *for* $\lambda_i \in (0,1)$;
(ii) $Q_{j\beta_j} = (1-\beta_j)I_2 + \beta_j Q_j$ *for* $\beta_j \in (0,1)$;
(iii) $G_{k\eta_k} = (1-\eta_k)I_3 + \eta_k G_k$ *for* $\eta_k \in (0,1)$.

Let $(\zeta_1,\zeta_2,\dots,\zeta_m) \in \Delta_m, (\theta_1,\theta_2,\dots,\theta_\ell) \in \Delta_\ell, (\omega_1,\omega_2,\dots,\omega_s) \in \Delta_s$.
Suppose that

$$\Lambda = \{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} Fix(P_i), y \in \bigcap_{j=1}^{\ell} Fix(R_j),\ z \in \bigcap_{k=1}^{s} Fix(W_k)), A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, *and let the sequences* $\{(x_n,y_n,z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n x_1 + (1-\alpha_n)\sum_{i=1}^{m} \zeta_i M_{i\lambda_i}(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n y_1 + (1-\alpha_n)\sum_{j=1}^{\ell}\theta_j Q_{j\beta_j}(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n z_1 + (1-\alpha_n)\sum_{k=1}^{s}\omega_k G_{k\eta_k}(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) = P_\Lambda(x_1, y_1, z_1)$.

Proof. Let $V(x,y,z) = (V_1(x), V_2(y), V_3(z)) = (x_1, y_1, z_1)$ for any $(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i$, and let $F = I_1 \times I_2 \times I_3$, then V is $\frac{1}{6}$-Lipschitz continuous. We choose $\mu = 1$, $\gamma = 1$, and $\tau = \frac{1}{2}$. Then Corollary 3.16 follows from Theorem 3.14. □

Remark 3.17. Chang et al. [9] introduced an iteration process to study the split equality fixed point of quasi-pseudocontractive mappings and established a weak convergence theorem, they also established a strong convergence theorem under the assumption that both the quasi-pseudocontractive mappings which are considered by them are semicompactness, but we don't have the assumption of semi-compact on any one of operators in Theorem 3.14 and Corollary 3.16. We give a different proof to establish strong convergence theorem for the split equality fixed point of quasi-pseudocontractive mappings.

Theorem 3.18. For each $i \in \{1,2,\dots,m\}$, $j \in \{1,2,\dots,\ell\}$, $k \in \{1,2,\dots,s\}$, let

(i) $M_i : H_1 \to H_1$ be a hemicontinuous, locally bounded pseudocontractive mapping;

(ii) $Q_j : H_2 \to H_2$ be a hemicontinuous, locally bounded monotone mapping;

(iii) $G_k : H_3 \to H_3$ be a demiclosed δ_k- demicontractive mapping.

Let $r > 0$, for $x \in H_1$, and $u \in H_2$, and set

(i) $S_i(x) = \{z \in H_1 : \langle y - z, M_i z\rangle - \frac{1}{r}\langle y - z, (1+r)z - x\rangle \leqslant 0,\ \forall\, y \in H_i\}$;

(ii) $P_j(u) = \{z \in D_j : \langle y - z, Q_j(z)\rangle + \frac{1}{r}\langle y - z, z - u\rangle \geqslant 0,\ \forall\, y \in D_j\}$;

(iii) $G_{k\eta_k} = (1-\eta_k)I_3 + \eta_k G_k$ for $\eta_k \in (0, 1-\delta_k)$.

Let $(\zeta_1, \zeta_2, \dots, \zeta_m) \in \Delta_m$, $(\theta_1, \theta_2, \dots, \theta_\ell) \in \Delta_\ell$, $(\omega_1, \omega_2, \dots, \omega_s) \in \Delta_s$. Suppose that

$$\Lambda = \{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} \mathrm{Fix}(M_i), y \in \bigcap_{j=1}^{\ell} VI(Q_j, D_j),\ z \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_k), A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, and let the sequences $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i S_i(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ for all $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_2 - \mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_j P_j(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ for all $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_3 - \mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k G_{k\eta_k}(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ for all $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. We see in Theorem 3.5, for each $\eta \in (1-\lambda_k)$, $G_{k\eta_k}$ is a demiclosed $(1-\delta_k-\eta_k)$-strongly quasi-nonexpansive mapping. By theorem 2.6, and Theorem 2.7, we show that for each $i \in \{1,2,\dots,m\}$, and $j \in \{1,2,\dots,s\}$,

(i) S_i and P_j are single-valued;

(ii) S_i and P_j are firmly nonexpansive;

(iii) $\mathrm{Fix}(P_j) = VI(Q_j, D_j)$, and $\mathrm{Fix}(S_i) = \mathrm{Fix}(M_i)$.

Then for each $i \in \{1,2,\dots,m\}$, and $j \in \{1,2,\dots,s\}$, S_i and P_j are averaged. Therefore S_i and P_j are nonexpansive mappings. Then by Lemma 2.1, S_i and P_j are demiclosed. By Proposition 2.2, we show that

(i) S_i is a ρ_i-strongly quasi-nonexpansive mapping for some $\rho_i > 0$;

(ii) P_j is a demiclosed γ_j-strongly quasi-nonexpansive mapping for some $\gamma_j > 0$.

Let

$$\Omega = \{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m} \mathrm{Fix}(S_i), y \in \bigcap_{j=1}^{\ell} \mathrm{Fix}(P_j),\ z \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_{k\eta_k}), A_1(x) = A_2(y) = A_3(z)\}.$$

It is easy to see that $\Omega = \Lambda \neq \emptyset$. Then Theorem 3.18 follows from Theorem 3.3. □

We apply Theorem 3.3 and argue as Theorems 3.5 and 3.13, we can study the variational inequality problem over split fixed point of three finite families of demicontractive mappings.

Theorem 3.19. *Let $V_i : H_1 \to H_1$ be L_i-Lipschitz continuous, $F_i : H_1 \to H_1$ be κ_i-Lipschitz continuous and η_i-strongly monotone with $\kappa_i > 0$, and $\eta_i > 0$. For each $i \in \{1,2,\dots,m\}, j \in \{1,2,\dots,\ell\}, k \in \{1,2,\dots,s\}$, let*

(i) $M_i : H_1 \to H_1$ *be a demiclosed σ_i-demicontractive;*
(ii) $Q_j : H_2 \to H_2$ *be a demiclosed r_j-demicontractive mapping;*
(iii) $G_k : H_3 \to H_3$ *be a demiclosed δ_k-demicontractive mapping.*

Let

(i) $M_{i\lambda_i} = (1-\lambda_i)I_1 + \lambda_i M_i$ *for* $\lambda_i \in (0, 1-\sigma_i)$;
(ii) $Q_{j\beta_j} = (1-\beta_j)I_2 + \beta_j Q_j$ *for* $\beta_j \in (0, 1-r_j)$;
(iii) $G_{k\eta_k} = (1-\eta_k)I_3 + \eta_k G_k$ *for* $\eta_k \in (0, 1-\delta_k)$.

Let $(\zeta_1, \zeta_2, \dots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \dots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \dots, \omega_s) \in \Delta_s$. *Suppose that*

$$\Lambda = \{(x,x,x) : x \in H_1, x \in \bigcap_{i=1}^{m} \mathrm{Fix}(M_i),\ B_1(x) \in \bigcap_{j=1}^{\ell} \mathrm{Fix}(Q_j), B_2(x) \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_k)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, and let the sequences $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m} \zeta_i M_{i\lambda_i}(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ *for all* $n \in \mathbb{N}$;
(ii) $y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_1 - \mu\alpha_n F_2)\sum_{j=1}^{\ell} \theta_j (I_1 - \frac{1}{\|B_1\|^2} B_1^*(I_2 - Q_{j\beta_j})B_1)(y_n - \frac{\xi}{3}(2y_n - x_n - z_n))$ *for all* $n \in \mathbb{N}$;
(iii) $z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_1 - \mu\alpha_n F_3)\sum_{k=1}^{s} \omega_k (I_1 - \frac{1}{\|B_2\|^2} B_2^*(I_3 - G_{k\eta_k})B_2)(z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Remark 3.20. Theorem 3.19 improves and generalizes [24, Theorem 3.2]. In [24, Theorem 3.2], the authors studied multiple sets split fixed point of demicontractive mappings and quasi-nonexpansive mappings.

4. Variational inequality over split equality solutions for finite families of nonlinear mappings

Let $B : H_1 \multimap H_1$ be a multivalued mapping. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H_1 : Bx \neq \emptyset\}$. We say $B : H_1 \multimap H_1$ is monotone on H_1 if $\langle x - y, u - v\rangle \geqslant 0$ for all $x, y \in D(B), u \in Bx$, and $v \in By$. B is maximal monotone on H_1 if B is a monotone operator on H_1 and its graph is not properly contained in the graph of any other monotone operator on H_1. For a maximal monotone operator $B : H_1 \multimap H_1$ and $r > 0$, we may define a single-valued mapping $J_r^B : H_1 \to D(B)$ by $J_r^B = (I + rB)^{-1}$, and it is called the resolvent mapping of B for r.

For each $i \in \{1,2,\dots,m\}$, let C_i be a closed convex subset of H_1, let $M_{1i} : H_1 \multimap H_1$ be a maximum monotone multivalued mapping such that $D(M_{1i}) \subset C_i$, $L_{1i} : C_i \to H_1$ be a γ_{1i}-inverse strongly monotone operator, $h_{1i} \in \Gamma_0(H_1), g_{1i} \in \Gamma_0(H_1)$, and let g_{1i} be Fréchet differentiable with σ_{1i}-Lipschitz continuous Fréchet derivative ∇g_{1i}.

For each $j \in \{1,2,\dots,\ell\}$, let D_j be a closed convex subset of H_2, $M_{2j} : H_2 \multimap H_2$ be a maximum monotone operator such that $D(M_{2j}) \subset D_j$, $L_{2j} : D_j \to H_2$ be a γ_{2j}-inverse strongly monotone operator, $h_{2j} \in \Gamma_0(H_2), g_{2j} \in \Gamma_0(H_2)$, and let g_{2j} be Fréchet differentiable with σ_{2j}-Lipschitz continuous Fréchet derivative ∇g_{2j}. For each $k \in \{1,2,\dots,s\}$, let E_k be a closed convex subset of H_3, $M_{3k} : H_3 \multimap H_3$ be a maximum monotone operator such that $D(M_{3k}) \subset E_k$, $L_{3k} : E_k \to H_3$ be a γ_{3k}-inverse strongly monotone operator, $h_{3k} \in \Gamma_0(H_3), g_{3k} \in \Gamma_0(H_3)$, and let g_{3k} be Fréchet differentiable with σ_{3k}-Lipschitz continuous ∇g_{3k}. For each $k \in \{1,2,\dots,s\}$, let $G_k : H_3 \to H_3$ be a demiclosed δ_k-demicontractive mapping. Throughout this section we use these notations and assumptions unless specified otherwise.

Theorem 4.1. *Let* $(\zeta_1,\zeta_2,\dots,\zeta_m) \in \Delta_m, (\theta_1,\theta_2,\dots,\theta_\ell) \in \Delta_\ell, (\omega_1,\omega_2,\dots,\omega_s) \in \Delta_s$ *and let* $\zeta > 0$. *Suppose that*

$$\begin{aligned}\Lambda = \{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in \bigcap_{i=1}^{m}(M_{1i}+L_{1i})^{-1}0, z \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_k),\ y \in \bigcap_{j=1}^{\ell}(M_{2j}+L_{2j})^{-1}0,\\ A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.\end{aligned}$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, *and let the sequences* $\{(x_n,y_n,z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i J_\zeta^{M_{1i}}(I_1 - \zeta L_{1i})(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_2 - \mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_j J_\zeta^{M_{2j}}(I_2 - \zeta L_{2j})(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_3 - \mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k G_{k\eta_k}(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n,y_n,z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Argue as [25, Theorem 4.1], we see that $J_\zeta^{M_{1i}}(I_1 - \zeta L_{1i})$ and $J_\zeta^{M_{2j}}(I_2 - \zeta L_{2j})$ are averaged for each $i \in \{1,2,,\dots,m\}$ and each $j \in \{1,2,\dots,\ell\}$.

It is easy to see that

(i) $\mathrm{Fix}(J_\zeta^{M_{1i}}(I_1 - \zeta L_{1i})) = (M_{1i}+L_{1i})^{-1}0$;

(ii) $\mathrm{Fix}(J_\zeta^{M_{2j}}(I_2 - \zeta L_{2j})) = (M_{2j}+L_{2j})^{-1}0$.

By Proposition 2.2,

(i) $J_\zeta^{M_{1i}}(I_1 - \zeta L_{1i})$ is a λ_i-strongly quasi-nonexpansive mapping for some $\lambda_i > 0$;

(ii) $J_\zeta^{M_{2j}}(I_2 - \zeta L_{2j})$ is a β_j-strongly quasi-nonexpansive mapping for some $\beta_j > 0$.

For each $i \in \{1,2,,\dots,m\}$, and each $j \in \{1,2,\dots,\ell\}$, since $J_\zeta^{M_{1i}}(I_1 - \zeta L_{1i})$ and $J_\zeta^{M_{2j}}(I_2 - \zeta L_{2j})$ are averaged, $J_\zeta^{M_{1i}}(I_1 - \zeta L_{1i})$ and $J_\zeta^{M_{2j}}(I_2 - \zeta L_{2j})$ are nonexpansive. By Lemma 2.1, $J_\zeta^{M_{1i}}(I_1 - \zeta L_{1i})$ and $J_\zeta^{M_{2j}}(I_2 - \zeta L_{2j})$ are demiclosed.

It follows from Lemma 2.3 for each $k \in \{1,2,\dots,s\}$, $\eta_k \in (0, 1-\delta_k)$ that $G_{k\eta_k}$ is a demiclosed $(1-\delta_k-\eta_k)$-strongly quasi-nonexpansive mapping. Let

$$\Omega = \{x \in \bigcap_{i=1}^{m}\mathrm{Fix}(J_\zeta^{M_{1i}}(I_1 - \zeta L_{1i})), z \in \bigcap_{k=1}^{s}\mathrm{Fix}(G_{k\eta_k}), y \in \bigcap_{j=1}^{\ell}\mathrm{Fix}(J_\zeta^{M_{2j}}(I_2 - \zeta L_{2j})),\ A_1(x) = A_2(y) = A_3(z)\}.$$

It is easy to see that $\Lambda = \Omega \neq \emptyset$ and Theorem 4.1 follows from Theorem 3.3. □

Theorem 4.2. *Let* $(\zeta_1,\zeta_2,\dots,\zeta_m) \in \Delta_m, (\theta_1,\theta_2,\dots,\theta_\ell) \in \Delta_\ell, (\omega_1,\omega_2,\dots,\omega_s) \in \Delta_s$*, let* $V_i : H_1 \to H_1$ *be* L_i*-Lipschitz continuous,* $F_i : H_1 \to H_1$ *be* κ_i*-Lipschitz continuous and* η_i*-strongly monotone with* $\kappa_i > 0$*, and* $\eta_i > 0$*, and let* $\kappa > 0$*. Suppose that*

$$\Lambda = \{(x,x,x) : x \in H_1, x \in \bigcap_{i=1}^{m} (M_{1i}+L_{1i})^{-1}0, B_1(x) \in \bigcap_{j=1}^{\ell} (M_{2j}+L_{2j})^{-1}0,\ B_2(x) \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_k)\} \neq \emptyset.$$

Let $x_1 \in H_1$*,* $y_1 \in H_1$*,* $z_1 \in H_1$*, and let the sequences* $\{(x_n,y_n,z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1-\mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i J_\kappa^{M_{1i}}(I_1-\kappa L_{1i})(x_n - \frac{\xi}{3}(2x_n-y_n-z_n))$ *for all* $n\in\mathbb{N}$*;*

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_1-\mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_j(I_1 - \frac{1}{\|B_1\|^2}B_1^*(I_2-(J_\kappa^{M_{2j}}(I_2-\kappa L_{2j}))B_1))(y_n - \frac{\xi}{3}(y_n-x_n-z_n))$ *for all* $n\in\mathbb{N}$*;*

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_1-\mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k(I_1 - \frac{1}{\|B_2\|^2}B_2^*(I_3-G_{k\eta_k})B_2)(z_n - \frac{\xi}{3}(2z_n-x_n-y_n))$ *for all* $n\in\mathbb{N}$*.*

Then $\lim_{n\to\infty}(x_n,y_n,z_n) \in VI(\mu F - \gamma V, \Lambda)$*.*

Proof. Theorem 3.5 shows that for $\eta_k \in (0,1-\delta_k)$, $G_{k\eta_k}$ is a demiclosed $(1-\delta_k-\eta_k)$-strongly quasi-nonexpansive mapping. In Theorem 4.1, we show that

(i) $J_\kappa^{M_{1i}}(I_1-\kappa L_{1i})$ is a λ_i-strongly quasi-nonexpansive mapping for some $\lambda_i > 0$;

(ii) $J_\kappa^{M_{2j}}(I_2-\kappa L_{2j})$ is a β_j-strongly quasi-nonexpansive mapping for some $\beta_j > 0$.

Let

(i) $U_j = (I_1 - \frac{1}{\|B_1\|^2}B_1^*(I_2 - J_\kappa^{M_{2j}}(I_2-\kappa L_{2j}))B_1)$;

(ii) $W_{k\eta_k} = (I_1 - \frac{1}{\|B_2\|^2}B_2^*(I_3-G_{k\eta_k})B_2)$.

By Proposition 2.8, for $\eta_k \in (0,1-\delta_k)$,

(i) $\mathrm{Fix}(U_j) = B_1^{-1}\mathrm{Fix}(J_\kappa^{M_{2j}}(I_2-\kappa L_{2j}))$ and $\mathrm{Fix}(W_{k\eta_k}) = B_2^{-1}\mathrm{Fix}(G_k)$;

(ii) $U_j : H_1 \to H_1$ is a demiclosed β_j-strongly quasi-nonexpansive mapping, and $W_{k\eta_k} : H_1 \to H_1$ is a demiclosed $(1-\delta_k-\eta_k)$-strongly quasi-nonexpansive mapping.

Let

$$\Omega = \{(x,x,x) : x \in H_1, x \in \bigcap_{i=1}^{m}\mathrm{Fix}(J_\kappa^{M_{1i}}(I_1-\kappa L_{1i})), x \in \bigcap_{j=1}^{\ell}\mathrm{Fix}(U_j),\ x \in \bigcap_{k=1}^{s}\mathrm{Fix}(W_{k\eta_k})\}.$$

It is easy to see that $\Omega = \Lambda \neq \emptyset$. Then Theorem 4.2 follows from Theorem 3.3. □

Remark 4.3. In [21] the authors introduced an iteration to study the following problem:

$$\text{Find } x \in M^{-1}0 \text{ such that } Lx \in \mathrm{Fix}(T),$$

where $M : H_1 \multimap H_1$ is a maximum monotone operator, and $T : H_2 \to H_2$ is a nonexpansive operator. Theorems 4.1 and 4.2 improve and generalize [21, Theorems 4.2 and 4.3]. In [21, Theorems 4.2 and 4.3], the authors established weak convergence theorems of this problem .

Theorem 4.4. *Let* $(\zeta_1,\zeta_2,\dots,\zeta_m) \in \Delta_m, (\theta_1,\theta_2,\dots,\theta_\ell) \in \Delta_\ell, (\omega_1,\omega_2,\dots,\omega_s) \in \Delta_s$*, let* $V_i : H_1 \to H_1$ *be* L_i*-Lipschitz continuous,* $F_i : H_1 \to H_1$ *be* κ_i*-Lipschitz continuous and* η_i*-strongly monotone with* $\kappa_i > 0$*, and* $\eta_i > 0$*, and let* $\kappa > 0$*. Suppose that*

$$\Lambda = \{(x,x,x) : x \in H_1, x \in \bigcap_{i=1}^{m}(M_{1i}+L_{1i})^{-1}0, B_1(x) \in \bigcap_{j=1}^{\ell}(M_{2j}+L_{2j})^{-1}0, B_2(x) \in \bigcap_{k=1}^{s}(M_{3k}+L_{3k})^{-1})\} \neq \emptyset.$$

Let $x_1 \in H_1$*,* $y_1 \in H_1$*,* $z_1 \in H_1$*, and let the sequences* $\{(x_n,y_n,z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i J_\kappa^{M_{1i}}(I_1 - \kappa L_{1i})(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ *for all* $n \in \mathbb{N}$;
(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_1 - \mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_{2j}(I_1 - \frac{1}{\|B_1\|^2}B_1^*(I_2 - (J_\kappa^{M_{2j}}(I_2 - \kappa L_{2j}))B_1))(y_n - \frac{\xi}{3}(y_n - x_n - z_n))$ *for all* $n \in \mathbb{N}$;
(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_1 - \mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k(I_1 - \frac{1}{\|B_2\|^2}B_2^*(I_3 - J_\kappa^{M_{3k}}(I_3 - \kappa L_{3k}))B_2)(z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Theorem 4.1 shows that

(i) $J_\kappa^{M_{1i}}(I_1 - \kappa L_{1i})$ is a λ_i-strongly quasi-nonexpansive mapping for some $\lambda_i > 0$;
(ii) $J_\kappa^{M_{2j}}(I_2 - \kappa L_{2j})$ is a β_j-strongly quasi-nonexpansive mapping for some $\beta_j > 0$;
(iii) $J_\kappa^{M_{3k}}(I_3 - \kappa L_{3k}^k)$ is a δ_k-strongly quasi-nonexpansive mapping for some $\delta_k > 0$.

Let

(i) $M_i = J_\kappa^{M_{1i}}(I_1 - \kappa L_{1i})$;
(ii) $Q_j = (I_1 - \frac{1}{\|B_1\|^2}B_1^*(I_2 - J_\kappa^{M_{2j}}(I_2 - \kappa L_{2j}))B_1)$;
(iii) $W_k = (I_1 - \frac{1}{\|B_2\|^2}B_2^*(I_3 - J_\kappa^{M_{3k}}(I_3 - \kappa L_{3k}))B_2)$.

By Proposition 2.8,

(i) $\mathrm{Fix}(M_i) = H_1 \to H_1$ is a demiclosed λ_i-strongly quasi-nonexpansive mapping, and $\mathrm{Fix}(M_i) = ((M_{1i} + L_{1i})^{-1}0)$;
(ii) $Q_j : H_1 \to H_1$ is a demiclosed β_j-strongly quasi-nonexpansive mapping, and $\mathrm{Fix}(Q_j) = (B_1^{-1}(M_{2j} + L_{2j})^{-1}0)$;
(iii) and $W_k : H_1 \to H_1$ is a demiclosed δ_k-strongly quasi-nonexpansive mapping, and $\mathrm{Fix}(W_k) = (B_2^{-1}(M_{3k} + L_{3k})^{-1}0)$.

Let

$$\Omega = \{(x,x,x) : x \in H_1, x \in \bigcap_{i=1}^{m}\mathrm{Fix}(M_i), x \in \bigcap_{j=1}^{\ell}\mathrm{Fix}(Q_j), x \in \bigcap_{k=1}^{s}\mathrm{Fix}(W_k)\}.$$

It is easy to see that $\Omega = \Lambda \neq \emptyset$. Then Theorem 4.4 follows from Corollary 3.4. □

Theorem 4.5. *Let* $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$, *and let* $\kappa > 0$. *Suppose that*

$$\Lambda = \{(x,y,z) \in \bigotimes_{1\leqslant i\leqslant 3} H_i : x \in H_1, x \in \bigcap_{i=1}^{m}(M_{1i} + L_{1i})^{-1}0, z \in \bigcap_{k=1}^{s}(M_{3k} + L_{3k})^{-1}0, y \in \bigcap_{j=1}^{\ell}(M_{2j} + L_{2j})^{-1}0,$$
$$A_1(x) = A_2(y) = A_3(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i J_\kappa^{M_{1i}}(I_1 - \kappa L_{1i})(x_n - \frac{\xi}{3}A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;
(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_2 - \mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_j J_\kappa^{M_{2j}}(I_2 - \kappa L_{2j})(y_n - \frac{\xi}{3}A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n)))$ *for all* $n \in \mathbb{N}$;
(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_3 - \mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k J_\kappa^{M_{3k}}(I_3 - \kappa L_{3k}^k)(z_n - \frac{\xi}{3}A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n)))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. We apply Theorem 4.1 and argue as in Theorem 4.4, we can prove Theorem 4.5. □

Remark 4.6. Theorems 4.1 and 4.5 improve and generalize [23, Theorem 4.2].

Corollary 4.7. *In Theorem 4.4, let* $H_1 = H_2 = H_3 = H_4, I_1 = I_2 = I_3$, $(\zeta_1, \zeta_2, \dots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \dots, \theta_\ell) \in \Delta_\ell$, *and* $(\omega_1, \omega_2, \dots, \omega_s) \in \Delta_s$. *Suppose that*

$$\Lambda = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^{m}(M_{1i} + L_{1i})^{-1}0 \bigcap \bigcap_{j=1}^{\ell}(M_{2j} + L_{2j})^{-1}0 \bigcap \bigcap_{k=1}^{s}(M_{3k} + L_{3k})^{-1}0)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i J_\kappa^{M_{1i}}(I_1 - \kappa L_{1i})(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_1 - \mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_j J_\kappa^{M_{2j}}(I_1 - \kappa L_{2j})(y_n - \frac{\xi}{3}(y_n - x_n - z_n))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_1 - \mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k J_\kappa^{M_{3k}}(I_1 - \kappa L_{3k})(z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Let $B_1 = I_1 = B_2 = I_2 = I_3$ in Theorem 4.4, then $\|B_1\| = \|B_2\| = 1$ and Corollary 4.7 follows from Theorem 4.4. □

Theorem 4.8. *Let* $(\zeta_1, \zeta_2, \dots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \dots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \dots, \omega_s) \in \Delta_s$, *let* $V_i : H_1 \to H_1$ *be* L_i-*Lipschitz continuous,* $F_i : H_1 \to H_1$ *be* κ_i-*Lipschitz continuous and* η_i-*strongly monotone with* $\kappa_i > 0$, *and* $\eta_i > 0$. *Suppose that*

$$\Lambda = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^{m} C_i, B_1(x) \in \bigcap_{j=1}^{\ell} D_j, B_2(x) \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_k)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i P_{C_i}(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_1 - \mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_j(I_1 - \frac{1}{\|B_1\|^2}B_1^*(I_2 - P_{D_j})B_1)(y_n - \frac{\xi}{3}(y_n - x_n - z_n))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_1 - \mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k(I_1 - \frac{1}{\|B_2\|^2}B_2^*(I_3 - G_{k\eta_k})B_2)(z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Let $L_{1i} = 0, L_{2j} = 0, M_{1i} = \partial\iota_{C_i}, M_{2j} = \partial\iota_{D_j}$ in Theorem 4.2, then $J_\kappa^{\partial\iota_{C_i}} = P_{C_i}$, $J_\kappa^{\partial\iota_{D_j}} = P_{D_j}$, and theorem 4.8 follows from Theorem 4.2. □

Remark 4.9. Theorem 4.8 improves and generalizes [4, Theorem 3.1]. In [4, Theorem 3.1], the authors established a strongly convergence theorem for split feasibility problem and fixed point problem of k-strictly pseudo-contractive mapping.

Corollary 4.10. *Lett* $(\zeta_1, \zeta_2, \dots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \dots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \dots, \omega_s) \in \Delta_s$, *let* $V_i : H_1 \to H_1$ *be* L_i-*Lipschitz continuous,* $F_i : H_1 \to H_1$ *be* κ_i-*Lipschitz continuous and* η_i-*strongly monotone with* $\kappa_i > 0$, *and* $\eta_i > 0$. *Suppose that*

$\Lambda = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^{m} C_i, B_1(x) \in \bigcap_{j=1}^{\ell} D_j, B_2(x) \in \bigcap_{k=1}^{s} E_k\} \neq \emptyset$.

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i P_{C_i}(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_1 - \mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_j(I_1 - \frac{1}{\|B_1\|^2}B_1^*(I_1 - P_{D_j})B_1)(y_n - \frac{\xi}{3}(y_n - x_n - z_n))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_1 - \mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k(I_1 - \frac{1}{\|B_2\|^2}B_2^*(I_1 - P_{E_k})B_2)(z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. For each $k \in \{1, 2, \ldots, s\}$, let $G_{3k} = P_{E_k}$. Since P_{E_k} is a firmly nonexpansive mapping, P_{E_k} is averaged and P_{E_k} is demiclosed. By Proposition 2.2, G_k is a δ_k-strongly quasi-nonexpansive mapping for some $\delta_k > 0$. Hence G_k is a δ_k-demicontractive mapping for some $\delta_k > 0$. Then Corollary 4.10 follows from Theorem 4.8. □

Theorem 4.11. *Let* $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$, *let* $V_i : H_1 \to H_1$ *be* L_i*-Lipschitz continuous,* $F_i : H_1 \to H_1$ *be* κ_i*-Lipschitz continuous and* η_i*-strongly monotone with* $\kappa_i > 0$*, and* $\eta_i > 0$ *and let* $\kappa > 0$*. Suppose that*

$$\Lambda = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^{m} \arg\min_{u\in H_1}(h_{1i} + g_{1i}(u)),\ B_1(x) \in \bigcap_{j=1}^{\ell} \arg\min_{u\in H_2}(h_{2j} + g_{2j})(u),$$

$$B_2(x) \in \bigcap_{k=1}^{s} \mathrm{Fix}(G_k)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i \mathrm{prox}_{\kappa h_{1i}}(I_1 - \kappa\nabla g_{1i})(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_1 - \mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_j(I_1 - \frac{1}{\|B_1\|^2}B_1^*(I_2 - (\mathrm{prox}_{\kappa h_{2j}}(I_2 - \kappa\nabla g_{2j}))B_1))(y_n - \frac{\xi}{3}(y_n - x_n - z_n))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_1 - \mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k(I_1 - \frac{1}{\|B_2\|^2}B_2^*(I_3 - G_{k\eta_k})B_2)(z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Apply Lemma 2.3 and argue as the proof II of [25, Theorem 4.2], we see that for each $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}$,

(i) ∂h_i, $\partial h_j'$ are maximum monotone operator;

(ii) ∇g_{1i} is a $\frac{1}{\sigma_{1i}}$-inverse strongly monotone operator, ∇g_{2j} is a $\frac{1}{\sigma_{2j}}$-inverse strongly monotone operator;

(iii) $\mathrm{prox}_{\kappa h_{1i}}(I_1 - \kappa\nabla g_{1i}) = J_\kappa^{\partial h_{1i}}(I_1 - \kappa\nabla g_{1i})$, $\mathrm{prox}_{\kappa h_{2j}}(I_1 - \kappa\nabla g_{2j}) = J_\kappa^{\partial h_{2j}}(I_1 - \kappa\nabla g_{2j})$;

(iv) $\arg\min_{x\in H_1}(h_{1i} + g_{1i})(x) = (\partial h_{1i} + \nabla g_{1i})^{-1}0$, $\arg\min_{x\in H_2}(h_{2j} + g_{2j})(x) = (\partial h_{2j} + \nabla g_{2j})^{-1}0$.

Then Theorem 4.11 follows from Theorem 4.2. □

Remark 4.12. Since a strictly pseudo-contractive mapping is a demi-contractive mapping. It is easy to see that Theorem 4.11 extends [4, Theorem 2.9].

Theorem 4.13. *Let* $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$, *let* $V_i : H_1 \to H_1$ *be* L_i*-Lipschitz continuous,* $F_i : H_1 \to H_1$ *be* κ_i*-Lipschitz continuous and* η_i*-strongly monotone with* $\kappa_i > 0$*, and* $\eta_i > 0$ *and let* $\kappa > 0$*. Suppose that*

$$\Lambda = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^{m} \arg\min_{u\in H_1}(h_{1i} + g_{1i}(u)), B_1(x) \in \bigcap_{j=1}^{\ell} \arg\min_{y\in H_2}(h_{2j} + g_{2j})(y),$$

$$B_2(x) \in \bigcap_{k=1}^{s} \arg\min_{z\in H_3}(h_{3k} + g_{3k})(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i \text{prox}_{\kappa h_{1i}}(I_1 - \kappa\nabla g_{1i})(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_1 - \mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_j(I_1 - \frac{1}{\|B_1\|^2}B_1^*(I_2 - \text{prox}_{\kappa h_{2j}}(I_2 - \kappa\nabla g_{2j})B_1))(y_n - \frac{\xi}{3}(y_n - x_n - z_n))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_1 - \mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k(I_1 - \frac{1}{\|B_2\|^2}B_2^*(I_3 - \text{prox}_{\kappa h_{3k}}(I_3 - \kappa\nabla g_{3k})B_2))(z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. For each $k \in \{1, 2, \ldots, s\}$, let $G_k = \text{prox}_{\kappa h_{3k}}(I_2 - \kappa\nabla g_{3k})$. We show in Theorems 4.1 and 4.11 that

(i) $\text{prox}_{\kappa h_{3k}}(I_1 - \kappa\nabla g_{3k}) = J_\kappa^{\partial h_{3k}}(I_1 - \kappa\nabla g_{3k})$;

(ii) $\arg\min_{x\in H_1}(h_{3k} + g_{3k})(x) = (\partial h_{3k} + \nabla g_{3k})^{-1}0$;

(iii) $J_\kappa^{\partial h_{3k}}(I_1 - \kappa\nabla g_{3k})$ is a δ_k-strongly quasi-nonexpansive mapping for some $\delta_k > 0$.

Then Theorem 4.13 follows from Theorem 4.4. □

Corollary 4.14. *In Theorem* 4.13*, let* $H_1 = H_2 = H_3 = H_4$, $I_1 = I_2 = I_3$, *and* $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m$, $(\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell$, $(\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$, $\kappa > 0$. *Suppose that*

$$\Lambda = \{(x,x,x) : x \in H_1, x \in \bigcap_{i=1}^{m}\arg\min_{u\in H_1}(h_{1i} + g_{1i}(u)) \bigcap \bigcap_{j=1}^{\ell}\arg\min_{y\in H_2}(h_{2j} + g_{2j})(y) \bigcap \bigcap_{k=1}^{s}\arg\min_{z\in H_3}(h_{3k} + g_{3k})(z)\} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, *and let the sequences* $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1} = \alpha_n\gamma V_1(x_n) + (I_1 - \mu\alpha_n F_1)\sum_{i=1}^{m}\zeta_i \text{prox}_{\kappa h_{1i}}(I_1 - \kappa\nabla g_{1i})(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ *for all* $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n\gamma V_2(y_n) + (I_1 - \mu\alpha_n F_2)\sum_{j=1}^{\ell}\theta_j(I_2 - \text{prox}_{\kappa h_{2j}}(I_2 - \kappa\nabla g_{2j})(y_n - \frac{\xi}{3}(y_n - x_n - z_n))$ *for all* $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n\gamma V_3(z_n) + (I_1 - \mu\alpha_n F_3)\sum_{k=1}^{s}\omega_k(I_3 - \text{prox}_{\kappa h_{3k}}(I_2 - \kappa\nabla g_{3k})(z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ *for all* $n \in \mathbb{N}$.

Then $\lim_{n\to\infty}(x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Let $B_1 = I_1 = B_2$ in Theorem 4.13, then Corollary 4.14 follows from Theorem 4.13. □

Remark 4.15. Corollaries 3.4, 3.12, 4.7, 4.10, and 4.14 have real applications in the large scale of nonlinear problems and optimization problems. Indeed if the scale of nonlinear problems is large, we can group these problems into finite families of nonlinear problems, then we use simultaneous iteration to find the solutions of these problems.

Theorem 4.16. *For each* $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, \ell\}$, $k \in \{1, 2, \ldots, s\}$, *let* $\kappa > 0$, *let*

(i) $f_i : C_i \times C_i \to \mathbb{R}$ *be a bifunction which satisfies conditions* (A1)-(A4);

(ii) $Q_j : H_2 \to H_2$ *be a hemicontinuous, locally bounded monotone mapping;*

(iii) $h_{3k} \in \Gamma_0(H_3)$, $g_{3k} \in \Gamma_0(H_3)$, g_{3k} *be Fréchet differentiable with* σ_{3k}*-Lipschitz continuous Fréchet derivative* ∇g_{3k}.

For $r > 0$, $x \in H_1$, *and* $u \in H_2$, *let*

(i) $M_i : H_1 \to C_i$ *be defined by* $M_i(x) = \{z \in C_i : f_i(z, u) + \frac{1}{r}\langle u - z, z - x\rangle \geqslant 0,\ \forall\, u \in C_i\}$;

(ii) $P_j : H_2 \to D_j$ *be defined by* $P_j(u) = \{z \in D_j : \langle y - z, Q_j(z)\rangle + \frac{1}{r}\langle y - z, z - u\rangle \geqslant 0,\ \forall\, y \in D_j\}$.

Let $(\zeta_1,\zeta_2,\dots,\zeta_m)\in\Delta_m,(\theta_1,\theta_2,\dots,\theta_\ell)\in\Delta_\ell,(\omega_1,\omega_2,\dots,\omega_s)\in\Delta_s,\eta>0$. *Suppose that*

$$\Lambda=\{(x,y,z)\in\bigotimes_{1\leqslant i\leqslant 3}H_i : x\in\bigcap_{i=1}^{m}EP(f_i), y\in\bigcap_{j=1}^{\ell}VI(Q_j,D_j),\ z\in\bigcap_{k=1}^{s}\arg\min_{w\in H_3}(h_{3k}+g_{3k})(w),$$
$$A_1(x)=A_2(y)=A_3(z)\}\neq\emptyset.$$

Let $x_1\in H_1$, $y_1\in H_2$, $z_1\in H_3$, *and let the sequences* $\{(x_n,y_n,z_n)\}_{n\in\mathbb{N}}$ *be defined by*

(i) $x_{n+1}=\alpha_n\gamma V_1(x_n)+(I_1-\mu\alpha_nF_1)\sum_{i=1}^{m}\zeta_iM_i(x_n-\frac{\xi}{3}A_1^*(2A_1(x_n)-A_2(y_n)-A_3(z_n)))$ *for all* $n\in\mathbb{N}$;
(ii) $y_{n+1}=\alpha_n\gamma V_2(y_n)+(I_2-\mu\alpha_nF_2)\sum_{j=1}^{\ell}\theta_jP_j(y_n-\frac{\xi}{3}A_2^*(2A_2(y_n)-A_1(x_n)-A_3(z_n)))$ *for all* $n\in\mathbb{N}$;
(iii) $z_{n+1}=\alpha_n\gamma V_3(z_n)+(I_3-\mu\alpha_nF_3)\sum_{k=1}^{s}\omega_k(I_3-\text{prox}_{\kappa h_{3k}}(I_3-\kappa\nabla g_{3k})(z_n-\frac{\xi}{3}A_3^*(2A_3(z_n)-A_1(x_n)-A_2(y_n)))$ *for all* $n\in\mathbb{N}$.

Then $\lim_{n\to\infty}(x_n,y_n,z_n)\in VI(\mu F-\gamma V,\Lambda)$.

Proof. It follows from Theorem 2.5 that for each $i\in\{1,2,\dots,m\}$,

(i) M_i is single-valued;
(ii) M_i is firmly nonexpansive;
(iii) $\{x\in H_1:\ M_ix=x\}=\{x\in C_i:\ f_i(x,u)\geqslant 0,\ \forall u\in C_i\}$.

By Theorem 2.6,

(i) P_j is single-valued;
(ii) P_j is a firmly nonexpansive mapping;
(iii) $\{x\in H:\ P_jx=x\}=VI(Q_j,D_j)$.

As in the proof of Theorem 3.18, we see that

(i) M_i is a λ_i-strongly quasi-nonexpansive mapping for some $\lambda_i>0$;
(ii) P_j is a demiclosed β_j-strongly quasi-nonexpansive mapping for some $\beta_j>0$.

We show in Theorems 4.13 that

(i) $\text{prox}_{\kappa h_{3k}}(I_1-\kappa\nabla g_{3k})=J_\kappa^{\partial h_{3k}}(I_1-\kappa\nabla g_{3k})$;
(ii) $\arg\min_{x\in H_1}(h_{3k}+g_{3k})(x)=(\partial h_{3k}+\nabla g_{3k})^{-1}0$;
(iii) $J_\kappa^{\partial h_{3k}}(I_1-\kappa\nabla g_{3k})$ is a δ_k-strongly quasi-nonexpansive mapping for some $\delta_k>0$.

Then Theorem 4.16 follows from Theorem 3.3. □

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