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Fixed point belonging to the zero-set of a given function

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Abstract

We prove the existence and uniqueness of fixed point belonging to the zero-set of a given function. The results are established in the setting of metric spaces and partial metric spaces. Our approach combines the recent notions of (F, φ)-contraction and $\mathcal{Z}$-contraction. The main result allows to deduce, as a particular case, some of the most known results in the literature. An example supports the theory.

Keywords: Fixed point, metric space, partial metric space, nonlinear contraction, simulation function.

2010 MSC: 47H10, 54H25.

1. Introduction

Let (Z, ρ) be a metric space, $h : Z \to Z$ be a mapping and $\theta : Z \to [0, +\infty[$ be a given function. In this paper, we prove the existence and uniqueness of fixed point of the function h, which belongs to the zero-set of a given function θ (see [10, 11]). The notion of contraction used in this paper merges the notions of (F, φ)-contraction (see [3]) and $\mathcal{Z}$-contraction (see [4]). The main result allows to deduce, as a particular case, some of the most known results in the literature. An example supports the theory.

2. Preliminaries

We recall the notions and notation which we will use later on. In [3], Jleli et al. take into account the family $\mathcal{H}$ of functions $H : [0, +\infty[^3 \to [0, +\infty[$ satisfying the following conditions:

(H_1) $\max\{\alpha, \beta\} \leqslant H(\alpha, \beta, \gamma)$ for all $\alpha, \beta, \gamma \in [0, +\infty[$;

(H_2) $H(0, 0, 0) = 0$;

(H_3) H is continuous.

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They use the family $\mathcal{H}$ to introduce a new type of contraction useful to get the existence and uniqueness of fixed point belonging to the zero-set of a particular function.

Example 2.1. The following are examples of functions which belong to $\mathcal{H}$:

(i) $H(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$;

(ii) $H(\alpha, \beta, \gamma) = \max\{\alpha, \beta\} + \gamma$;

(iii) $H(\alpha, \beta, \gamma) = \alpha + \beta + \alpha\beta + \gamma$.

Let us denote by $\mathcal{S}$ the family of functions $S : [0, +\infty[^2 \to \mathbb{R}$ satisfying the following conditions (see [4]):

(S_1) $S(z, w) < w - z$ for all $z, w > 0$;

(S_2) if $\{z_n\}$, $\{w_n\}$ are sequences in $]0, +\infty[$ such that $\lim_{n\to+\infty} z_n = \lim_{n\to+\infty} w_n = \lambda \in]0, +\infty[$ then $\limsup_{n\to+\infty} S(z_n, w_n) < 0$.

Example 2.2. The following are examples of functions in $\mathcal{S}$:

(j) Let $\mu : [0, +\infty[\to [0, +\infty[$ be a lower semi-continuous function with $\mu^{-1}(0) = \{0\}$. So, $S \in \mathcal{S}$ if $S(z, w) = w - \mu(w) - z$ for all $z, w \geqslant 0$.

(jj) Let $\mu : [0, +\infty[\to [0, 1[$ be such that $\limsup_{z\to r^+} \mu(z) < 1$ for all $r > 0$. So, $S \in \mathcal{S}$ if $S(z, w) = w\,\mu(w) - z$ for all $z, w \geqslant 0$.

(jj) Let $\mu : [0, +\infty[\to [0, +\infty[$ be an upper semi-continuous function with $\mu(w) < w$ for all $w > 0$ and $\mu(0) = 0$. So, $S \in \mathcal{S}$ if $S(z, w) = \mu(w) - z$ for all $z, w \geqslant 0$.

3. Main result

In this section, we prove our first result (see Theorem 3.2). Here, we use a notion of contraction involving the families $\mathcal{S}$ and $\mathcal{H}$. Let $Z \neq \emptyset$, $h : Z \to Z$, $u_0 \in Z$ and $u_n = hu_{n-1}$ for all $n \in \mathbb{N}$. We call $\{u_n\}$ a sequence of Picard starting at u_0.

Lemma 3.1. *Let (Z, ρ) be a metric space and let $h : Z \to Z$ be a mapping. Assume that there exist a function $S \in \mathcal{S}$, a function $H \in \mathcal{H}$ and a function $\theta : Z \to [0, +\infty[$ such that*

$$S\left(H(\rho(hu, hv), \theta(hu), \theta(hv)), H(\rho(u, v), \theta(u), \theta(v))\right) \geqslant 0 \quad \textit{for all } u, v \in Z. \tag{3.1}$$

If $\{u_n\}$ is a sequence of Picard starting at $u_0 \in Z$ such that $u_{n-1} \neq u_n$ for all $n \in \mathbb{N}$, then $\{u_n\}$ is a Cauchy sequence.

Proof. Let $u_0 \in Z$ and let $\{u_n\}$ be a sequence of Picard starting at $u_0 \in Z$ such that $u_{n-1} \neq u_n$ for all $n \in \mathbb{N}$. Firstly, we prove that

$$\lim_{n\to+\infty} \rho(u_{n-1}, u_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \theta(u_n) = 0. \tag{3.2}$$

Taking into account that $u_{n-1} \neq u_n$ for all $n \in \mathbb{N}$, by (H_1) we obtain

$$H(\rho(u_{n-1}, u_n), \theta(u_{n-1}), \theta(u_n)) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Using (3.1) and (S_1), with $u = u_{n-1}$ and $v = v_n$, we deduce that

$$0 \leqslant S(H(\rho(u_n, u_{n+1}), \theta(u_n), \theta(u_{n+1})), H(\rho(u_{n-1}, u_n), \theta(u_{n-1}), \theta(u_n)))$$

$$< H(\rho(u_{n-1}, u_n), \theta(u_{n-1}), \theta(u_n)) - H(\rho(u_n, u_{n+1}), \theta(u_n), \theta(u_{n+1}))$$

for all $n \in \mathbb{N}$. The previous inequality leads to

$$H(\rho(u_n, u_{n+1}), \theta(u_n), \theta(u_{n+1})) < H(\rho(u_{n-1}, u_n), \theta(u_{n-1}), \theta(u_n)) \quad \text{for all } n \in \mathbb{N}.$$

So, $\{H(\rho(u_{n-1}, u_n), \theta(u_{n-1}), \theta(u_n))\} \subset]0, +\infty[$ is a decreasing sequence and hence there is some $\lambda \geqslant 0$ such that

$$\lim_{n\to+\infty} H(\rho(u_{n-1}, u_n), \theta(u_{n-1}), \theta(u_n)) = \lambda. \tag{3.3}$$

We remark that if $\lambda > 0$, by using condition (S_2) with $z_n = H(\rho(u_n, u_{n+1}), \theta(u_n), \theta(u_{n+1}))$ and $w_n = H(\rho(u_{n-1}, u_n), \theta(u_{n-1}), \theta(u_n))$, we get

$$0 \leqslant \limsup_{n\to+\infty} S\left(H(\rho(u_n, u_{n+1}), \theta(u_n), \theta(u_{n+1})), H(\rho(u_{n-1}, u_n), \theta(u_{n-1}), \theta(u_n))\right) < 0.$$

Clearly, this is a contradiction and hence $\lambda = 0$. Thus, from (H_1) we get

$$\max\{\rho(u_{n-1}, u_n), \theta(u_{n-1})\} \leqslant H(\rho(u_{n-1}, u_n), \theta(u_{n-1}), \theta(u_n)) \quad \text{for all } n \in \mathbb{N}$$

and hence

$$\lim_{n\to+\infty} \rho(u_{n-1}, u_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \theta(u_{n-1}) = 0.$$

Now, we establish the Cauchyness of $\{u_n\}$. By contradiction, we assume that $\{u_n\}$ is not Cauchy. Then, there exist a positive real number σ and two sequences $\{j_k\}$ and $\{i_k\}$ such that $i_k > j_k \geqslant k$ and $\rho(u_{j_k}, u_{i_k}) \geqslant \sigma > \rho(u_{j_k}, u_{i_k-1})$ for all $k \in \mathbb{N}$. This implies

$$\lim_{k\to+\infty} \rho(u_{j_k}, u_{i_k}) = \lim_{k\to+\infty} \rho(u_{j_k-1}, u_{i_k-1}) = \sigma.$$

So, it is not restrictive to assume that $\rho(u_{j_k-1}, u_{i_k-1}) > 0$ for all $k \in \mathbb{N}$. Consequently, we deduce that also

$$H(\rho(u_{j_k-1}, u_{i_k-1}), \theta(u_{j_k-1}), \theta(u_{i_k-1})) > 0 \quad \text{and} \quad H(\rho(u_{j_k}, u_{i_k}), \theta(u_{j_k}), \theta(u_{i_k})) > 0.$$

Now, taking into account that H is a continuous function, we have

$$\begin{aligned}\lim_{k\to+\infty} H(\rho(u_{j_k-1}, u_{i_k-1}), \theta(u_{j_k-1}), \theta(u_{i_k-1})) &= \lim_{k\to+\infty} H(\rho(u_{j_k}, u_{i_k}), \theta(u_{j_k}), \theta(u_{i_k})) \\ &= H(\sigma, 0, 0) > 0.\end{aligned}$$

So, by using (S_2) with

$$z_k = H(\rho(u_{j_k}, u_{i_k}), \theta(u_{j_k}), \theta(u_{i_k})) \text{ and } w_k = H(\rho(u_{j_k-1}, u_{i_k-1}), \theta(u_{j_k-1}), \theta(u_{i_k-1})),$$

we obtain

$$0 \leqslant \limsup_{k\to+\infty} S\left(H(\rho(u_{j_k}, u_{i_k}), \theta(u_{j_k}), \theta(u_{i_k})), H(\rho(u_{j_k-1}, u_{i_k-1}), \theta(u_{j_k-1}), \theta(u_{i_k-1}))\right) < 0.$$

Thus, we get a contradiction and hence $\{u_n\}$ is a Cauchy sequence. □

Now, we can demonstrate our main result.

Theorem 3.2. *Let (Z, ρ) be a complete metric space and let $h : Z \to Z$ be a mapping. Suppose that there exist a function $S \in \mathcal{S}$, a function $H \in \mathcal{H}$ and a lower semi-continuous function $\theta : Z \to [0, +\infty[$ such that (3.1) holds, that is,*

$$S\left(H(\rho(hu, hv), \theta(hu), \theta(hv)), H(\rho(u, v), \theta(u), \theta(v))\right) \geqslant 0 \quad \textit{for all } u, v \in Z.$$

Then h has a unique fixed point w such that $\theta(w) = 0$.

Proof. Firstly, we establish uniqueness of the fixed point, that is, w is the unique fixed point of h. Suppose that there exists $z \in Z$ such that $z = hz$ and $z \neq w$. Using (3.1) and (S_1) with $u = w$ and $v = z$ we get

$$\begin{aligned} 0 &\leqslant S(H(\rho(hw, hz), \theta(hw), \theta(hz)), H(\rho(w, z), \theta(w), \theta(z))) \\ &< H(\rho(w, z), \theta(w), \theta(z)) - H(\rho(w, z), \theta(w), \theta(z)) = 0. \end{aligned}$$

Clearly, this is a contradiction. Hence, we have $w = z$ and so we obtain the claim.

In order to establish the existence of a fixed point, we consider a point $u_0 \in Z$. Let $\{u_n\}$ be a sequence of Picard starting at u_0. We stress that if $u_k = u_{k+1}$ for some $k \in \mathbb{N}$ then $u_k = u_{k+1} = hu_k$, that is, u_k is a fixed point of h. Clearly $\theta(u_k) = 0$ (in order to show this, it is sufficient to proceed as in the first part of the proof of Lemma 3.1). Taking into account that if $u_k = u_{k+1}$ we have existence of a fixed point, it is not restrictive to assume that $u_n \neq u_{n+1}$ for every $n \in \mathbb{N}$.

Now, Lemma 3.1 ensures that the sequence $\{u_n\}$ is Cauchy. Since (Z, ρ) is complete, there exists some $w \in Z$ such that

$$\lim_{n \to +\infty} u_n = w. \tag{3.4}$$

By (3.2), taking into account that θ is a lower semi-continuous function, we get

$$0 \leqslant \theta(w) \leqslant \liminf_{n \to +\infty} \theta(u_n) = 0,$$

that is, $\theta(w) = 0$. We have that w is a fixed point of h. Clearly, w is a fixed point of h if there exists a subsequence $\{u_{j_k}\}$ of $\{u_n\}$ such that $u_{j_k} = w$ or $hu_{j_k} = hw$, for all $k \in \mathbb{N}$. If there is no such subsequence, we can assume that $u_n \neq w$ and $hu_n \neq hw$ for all $n \in \mathbb{N}$. So, using (3.1) and (S_1) with $u = u_n$ and $v = w$, we have

$$\begin{aligned} 0 &\leqslant S(H(\rho(hu_n, hw), \theta(hu_n), \theta(hw)), H(\rho(u_n, w), \theta(u_n), \theta(w))) \\ &< H(\rho(u_n, w), \theta(u_n), \theta(w)) - H(\rho(hu_n, hw), \theta(hu_n), \theta(hw)). \end{aligned}$$

Consequently

$$H(\rho(hu_n, hw), \theta(hu_n), \theta(hw)) < H(\rho(u_n, w), \theta(u_n), \theta(w)) \quad \text{for all } n \in \mathbb{N}$$

and so

$$\begin{aligned} \rho(w, hw) &\leqslant \rho(w, u_{n+1}) + \rho(hu_n, hw) \\ &\leqslant \rho(w, u_{n+1}) + H(\rho(hu_n, hw), \theta(hu_n), \theta(hw)) \\ &< \rho(w, u_{n+1}) + H(\rho(u_n, w), \theta(u_n), \theta(w)) \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in the previous inequality, the continuity of the function H in $(0, 0, 0)$ ensures that $\rho(w, hw) \leqslant H(0, 0, 0) = 0$. So, $w = hw$ and hence w is a fixed point of h. □

4. Consequences

Now, we point out the unifying force of the contractive condition (3.1). In fact, using Theorem 3.2, we recover various contractive conditions in the existing literature.

We start by a result of Jleli et al. type (see [3, Theorem 2.1]).

Corollary 4.1. *Let (Z, ρ) be a complete metric space and let $h : Z \to Z$ be a mapping. Suppose that there exist $\delta \in]0, 1[$, $H \in \mathcal{H}$ and a lower semi-continuous function $\theta : Z \to [0, +\infty[$ such that*

$$H(\rho(hu, hv), \theta(hu), \theta(hv)) \leqslant \delta\, H(\rho(u, v), \theta(u), \theta(v)) \quad \textit{for all } u, v \in Z.$$

Then h has a unique fixed point w such that $\theta(w) = 0$.

Proof. The claim follows by Theorem 3.2 if we choose the function $S \in \mathcal{S}$ given by $S(z,w) = \lambda w - z$ for all $z, w \geqslant 0$. □

Clearly, if $H(\alpha,\beta,\gamma) = \alpha+\beta+\gamma$ for all $\alpha,\beta,\gamma \in [0,+\infty[$ and $\theta(u) = 0$ for all $u \in Z$, then we obtain the Banach contraction principle [1].

Now, we give a result of Rhoades type (see [9]).

Corollary 4.2. *Let (Z,ρ) be a complete metric space and let $h : Z \to Z$ be a mapping. Suppose that there exist $H \in \mathcal{H}$ and two lower semi-continuous functions $\mu : [0,+\infty[\to [0,+\infty[$ with $\mu^{-1}(0) = \{0\}$ and $\theta : Z \to [0,+\infty[$ such that*

$$H(\rho(hu,hv),\theta(hu),\theta(hv)) \leqslant H(\rho(u,v),\theta(u),\theta(v)) - \mu(H(\rho(u,v),\theta(u),\theta(v))) \quad \textit{for all } u,v \in Z.$$

Then h has a unique fixed point w such that $\theta(w) = 0$.

Proof. Again, we deduce the claim by Theorem 3.2 if we choose $S \in \mathcal{S}$ defined by $S(z,w) = w - \mu(w) - z$, for all $z, w \geqslant 0$ (see Example 2.2 (j)). □

As consequence of Theorem 3.2, we get also the following result (see [8]).

Corollary 4.3. *Let (Z,ρ) be a complete metric space and let $h : Z \to Z$ be a mapping. Suppose that there exist $H \in \mathcal{H}$, a function $\mu : [0,+\infty[\to [0,1[$ with $\limsup_{z\to r^+} \mu(z) < 1$ for all $r > 0$ and a lower semi-continuous function $\theta : Z \to [0,+\infty[$ such that*

$$H(\rho(hu,hv),\theta(hu),\theta(hv)) \leqslant \mu(H(\rho(u,v),\theta(u),\theta(v))\, H(\rho(u,v),\theta(u),\theta(v))) \quad \textit{for all } u,v \in Z.$$

Then h has a unique fixed point w such that $\theta(w) = 0$.

Proof. Using Theorem 3.2 with respect to the function $S \in \mathcal{S}$ defined by $S(z,w) = w\,\mu(w) - z$ for all $z, w \geqslant 0$ (see Example 2.2 (jj)), we obtain the claim. □

The following is a result of Boyd-Wong type (see [2]).

Corollary 4.4. *Let (Z,ρ) be a complete metric space and let $h : Z \to Z$ be a mapping. Suppose that there exist a function $H \in \mathcal{H}$, an upper semi-continuous function $\mu : [0,+\infty[\to [0,+\infty[$ with $\mu(z) < z$ for all $z > 0$ and $\mu(0) = 0$ and a lower semi-continuous function $\theta : Z \to [0,+\infty[$ such that*

$$H(\rho(hu,hv),\theta(hu),\theta(hv)) \leqslant \mu(H(\rho(u,v),\theta(u),\theta(v))) \quad \textit{for all } u,v \in Z.$$

Then h has a unique fixed point w such that $\theta(w) = 0$.

Proof. In order to obtain the claim it is sufficient to apply Theorem 3.2 and to take the function $S \in \mathcal{S}$ given by $S(z,w) = \mu(w) - z$ for all $z, w \geqslant 0$ (see Example 2.2 (jjj)). □

We stress that if we suppose $H(\alpha,\beta,\gamma) = \alpha+\beta+\gamma$ for all $\alpha,\beta,\gamma \in [0,+\infty[$ and $\theta(u) = 0$ for all $u \in Z$, then we obtain the Boyd-Wong result.

The following example shows that Theorem 3.2 is a proper generalization of both Banach contraction principle and Boyd-Wong result in the setting of metric spaces.

Example 4.5 (see [11, Example 4])**.** We consider $Z = [0,1]$ and we endow Z with the usual metric $\rho(u,v) = |u-v|$ for all $u,v \in Z$. Clearly, (Z,ρ) is a complete metric space. Fix $\delta \in [0,1[$ and consider the function $h : Z \to Z$ defined by

$$hu = \begin{cases} 0 & \text{if } u = 0, \\[2ex] \dfrac{\delta}{2m} - \delta\dfrac{2m-1}{2m}(2mu-1) & \text{if } \dfrac{1}{2m} \leqslant u \leqslant \dfrac{1}{2m-1}, \\[2ex] \dfrac{\delta}{2m} + \delta\dfrac{2m+1}{2m}(2mu-1) & \text{if } \dfrac{1}{2m+1} \leqslant u \leqslant \dfrac{1}{2m}. \end{cases}$$

We remark that h is not a nonexpansive function (if δ is chosen appropriately close to 1). In fact, if for odd $m > 1$ we choose $u = \frac{1}{2m-1}$ and $v = \frac{1}{m-1}$, we have

$$\rho(hu, hv) = \frac{\delta}{m-1} \leqslant \rho(u, v) = \frac{m}{(m-1)(2m-1)} \leqslant \frac{3}{5(m-1)}.$$

This inequality is not satisfied for $\delta > 3/5$. This ensures that both the Banach contraction principle and Boyd-Wong result cannot be applied to show that h has a fixed point.

Now, if we consider the function $\theta : Z \to [0, +\infty[$ defined by $\theta(u) = u$ for all $u \in Z$ and the function $H(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$ for all $\alpha, \beta, \gamma \in [0, +\infty[$, then we have

$$\begin{aligned} H(\rho(hu, hv), \theta(hu), \theta(hv)) &= \rho(hu, hv) + \theta(hu) + \theta(hv) \\ &= 2\max\{hu, hv\} \leqslant 2\max\{\delta u, \delta v\} \\ &= 2\delta \max\{u, v\} \\ &= \delta[\rho(u, v) + \theta(u) + \theta(v)] \\ &= \delta H(\rho(u, v), \theta(u), \theta(v)) \end{aligned}$$

for all $u, v \in Z$. So, all the conditions of Corollary 4.1 are satisfied and this implies that h has a unique fixed point in Z, we say $w = 0$ with $\theta(w) = 0$.

5. Fixed points in partial metric spaces

Let us observe that we can use Theorem 3.2 to get some results of fixed point on the setting of partial metric spaces.

For other details on partial metric spaces, we refer the reader to [5–7] and the references therein.

Definition 5.1. Let Z be a non-empty set. A function $\pi : Z \times Z \to [0, +\infty[$ such that

(π_1) $u = v \Leftrightarrow \pi(u, u) = \pi(u, v) = \pi(v, v)$ for all $u, v \in Z$;

(π_2) $\pi(u, u) \leqslant \pi(u, v)$ for all $u, v \in Z$;

(π_3) $\pi(u, v) = \pi(v, u)$ for all $u, v \in Z$;

(π_4) $\pi(u, v) \leqslant \pi(u, z) + \pi(z, v) - \pi(z, z)$ for all $u, v, z \in Z$

is a partial metric on Z. The pair (Z, π) is called partial metric space.

We stress that $u = v$ does not imply $\pi(u, v) = 0$. The function $\pi : [0, +\infty[\times [0, +\infty[\to [0, +\infty[$ defined by $\pi(u, v) = \max\{u, v\}$ for all $u, v \in [0, +\infty[$ is a partial metric on $[0, +\infty[$.

Let (Z, π) be a partial metric space and let $\{u_n\} \subset Z$ be a sequence. Then

(a) $\{u_n\}$ converges to a point $u \in Z$ if and only if $\pi(u, u) = \lim_{n\to+\infty} \pi(u, u_n)$.

(b) $\{u_n\}$ is called a Cauchy sequence if there exists (and it is finite) $\lim_{n,m\to+\infty} \pi(u_n, u_m)$.

A partial metric space (Z, π) is complete if every Cauchy sequence $\{u_n\} \subset Z$ converges to a point $u \in Z$ such that $\pi(u, u) = \lim_{n,m\to+\infty} \pi(u_n, u_m)$.

We recall that to every partial metric π on a set Z we can associate one metric on Z, precisely, the function $\rho_\pi : Z \times Z \to [0, +\infty[$ given by

$$\rho_\pi(u, v) = 2\pi(u, v) - \pi(u, u) - \pi(v, v) \tag{5.1}$$

is a metric on Z. We say that $\lim_{n\to+\infty} \rho_\pi(u_n, u) = 0$ if and only if

$$\pi(u, u) = \lim_{n\to+\infty} \pi(u_n, u) = \lim_{n,m\to+\infty} \pi(u_n, u_m).$$

Lemma 5.2. *Let (Z,π) be a partial metric space and let $\theta : Z \to [0,+\infty[$ be given by $\theta(u) = \pi(u,u)$ for all $u \in Z$. Then θ is a lower semi-continuous function in the metric space (Z,ρ_π).*

Proof. Let $\{u_m\} \subset Z$ and $u \in Z$. If $u_m \to u$ as $m \to +\infty$ in the metric space (Z,ρ_π), then

$$\theta(u) = \pi(u,u) = \lim_{m\to+\infty} \pi(u_m,u_m) = \liminf_{m\to+\infty} \theta(u_m),$$

consequently, θ is lower semi-continuous in u and hence in (Z,ρ_π). □

Lemma 5.3. *Let (Z,π) be a partial metric space and $\{u_n\} \subset Z$. Then*

(a) *if $\{u_n\}$ is a Cauchy sequence in (Z,π), then it is a Cauchy sequence in (Z,ρ_π) and vice versa.*

(b) *If (Z,π) is a complete partial metric space, then (Z,ρ_π) is a complete metric space and vice versa.*

Now, we produce our main result in the context of partial metric spaces.

Theorem 5.4. *Let (Z,π) be a complete partial metric space and let $h : Z \to Z$ be a mapping. Suppose that there exists $S \in \mathcal{S}$ such that*

$$S\left(\pi(hu,hv),\pi(u,v)\right) \geqslant 0 \quad \text{for all } u,v \in Z. \tag{5.2}$$

Then h has a unique fixed point $w \in Z$ such that $\pi(w,w) = 0$.

Proof. We remark that by (5.1), we get

$$\pi(u,v) = \frac{\rho_\pi(u,v) + \pi(u,u) + \pi(v,v)}{2} \quad \text{for all } u,v \in Z. \tag{5.3}$$

Now, we equip Z with the metric $\rho = 2^{-1}\rho_\pi$. Since (Z,π) is complete, by Lemma 5.3 we infer that the metric space (Z,ρ) is complete. By Lemma 5.2, we say that the function $\theta : Z \to [0,+\infty[$ given by $\theta(u) = 2^{-1}\pi(u,u)$ is lower semi-continuous in (Z,ρ). Thus, from (5.2) and (5.3), we have that the mapping h verifies the following condition

$$S\left(H(\rho(hu,hv),\theta(hu),\theta(hv)),H(\rho(u,v),\theta(u),\theta(v))\right) \geqslant 0 \quad \text{for all } u,v \in Z,$$

where $H \in \mathcal{H}$ is defined by $H(\alpha,\beta,\gamma) = \alpha+\beta+\gamma$ for all $\alpha,\beta,\gamma \in [0,+\infty[$. So, the mapping h satisfies all the conditions of Theorem 3.2 with respect to the metric space (Z,ρ). By Theorem 3.2, we can affirm that h has a unique fixed point w such that $\theta(w) = 0$ and so $\pi(w,w) = 0$. □

We observe that the Matthews fixed point theorem (see [5]) follows from Theorem 5.4. It is sufficient to choose the function $S \in \mathcal{S}$ given by $S(u,v) = \sigma v - u$ for all $u,v \in [0,+\infty[$ with $\sigma \in [0,1[$.

Corollary 5.5. *Let (Z,π) be a complete partial metric space and let $h : Z \to Z$ be a mapping. Suppose that there exists $\sigma \in [0,1[$ such that*

$$\pi(hu,hv) \leqslant \sigma\,\pi(u,v) \quad \text{for all } u,v \in Z. \tag{5.4}$$

Then h has a unique fixed point $w \in Z$ such that $\pi(w,w) = 0$.

Proceeding as in the proof of Theorem 5.4, from Corollary 4.2 we infer the following corollary.

Corollary 5.6. *Let (Z,π) be a complete partial metric space and let $h : Z \to Z$ be a mapping. Suppose that there exists a lower semi-continuous function $\mu : [0,+\infty[\to [0,+\infty[$ with $\mu^{-1}(0) = \{0\}$ such that*

$$\rho(hu,hv) \leqslant \rho(u,v) - \mu(\rho(u,v)) \quad \text{for all } u,v \in Z.$$

Then h has a unique fixed point $w \in Z$ such that $\pi(w,w) = 0$.

Again proceeding as in the proof of Theorem 5.4, from Corollary 4.3 we infer the following corollary.

Corollary 5.7. *Let* (Z, π) *be a complete partial metric space and let* $h : Z \to Z$ *be a mapping. Suppose that there exists* $\mu : [0, +\infty[\to [0, 1[$ *with* $\limsup_{z\to r^+} \mu(z) < 1$ *for all* $r > 0$ *such that*

$$\rho(hu, hv) \leqslant \mu(\rho(u, v))\, \rho(u, v) \quad \textit{for all } u, v \in Z.$$

Then h *has a unique fixed point* $w \in Z$ *such that* $\pi(w, w) = 0$.

By Corollary 4.4, proceeding as in the proof of Theorem 5.4, we deduce the following corollary.

Corollary 5.8. *Let* (Z, π) *be a complete partial metric space and let* $h : Z \to Z$ *be a mapping. Suppose that there exists an upper semi-continuous function* $\mu : [0, +\infty[\to [0, +\infty[$ *with* $\mu(z) < z$ *for all* $z > 0$ *and* $\mu(0) = 0$ *such that*

$$\rho(hu, hv) \leqslant \mu(\rho(u, v)) \quad \textit{for all } u, v \in Z.$$

Then h *has a unique fixed point* $w \in Z$ *such that* $\pi(w, w) = 0$.

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