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The uniqueness of solution for initial value problems for fractional differential equation involving the Caputo-Fabrizio derivative

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Abstract

In this paper, we study some results about the expression of solutions to some linear differential equations for the Caputo-Fabrizio fractional derivative. Furthermore, by the Banach contraction principle, the unique existence of the solution to an initial value problem for nonlinear differential equation involving the Caputo-Fabrizio fractional derivative is obtained.

Keywords: The Caputo-Fabrizio fractional derivative, initial value problem, fractional differential equations, Banach contraction principle, uniqueness.

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1. Introduction

In this paper, we consider the following initial value problems for fractional differential equation

$$\begin{cases} {}^{CF}D^{\alpha}x(t) = g(t, x, {}^{CF}D^{\beta}x) - g(0, x(0), {}^{CF}D^{\beta}x(0)), & 0 < t < +\infty, \\ x(0) = x_0 \in R, \end{cases} \tag{1.1}$$

where $0 < \beta < \alpha < 1$, ${}^{CF}D^{\alpha}x, {}^{CF}D^{\beta}x$ are the Caputo-Fabrizio fractional derivative defined in [14], by

$$\begin{aligned} {}^{CF}D^{\alpha}x(t) &= \frac{M(\alpha)}{1-\alpha}\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s))x'(s)ds, \quad t \geqslant 0, \\ {}^{CF}D^{\beta}x(t) &= \frac{M(\beta)}{1-\beta}\int_0^t \exp(-\frac{\beta}{1-\beta}(t-s))x'(s)ds, \quad t \geqslant 0, \end{aligned} \tag{1.2}$$

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where $M(\alpha), M(\beta)$ are normalization functions depending on α and β such that $M(0) = M(1) = 1$. Also, in [4] authors obtained the Laplace transform of the Caputo-Fabrizio fractional derivative

$$L[{}^{CF}D^{\alpha}f(t)](s) = \frac{M(\alpha)}{s+\alpha(1-s)}(sL[f(t)](s) - f(0)), \quad s > 0,$$

where $L[f(t)]$ denotes the Laplace transform of function f.

In [1, 14], authors studied some properties of Caputo-Fabrizio fractional derivative defined in (1.2). In [22], authors consider the expression of solutions of some class of linear differential equations for the fractional derivative defined as following

$$^{CF}D_{*}^{\alpha}x(t) = \frac{(2-\alpha)M_1(\alpha)}{2(1-\alpha)}\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s))x'(s)ds, t \geqslant 0, \tag{1.3}$$

where $M_1(\alpha)$ is a normalization function depending on α. Obviously, if let $M_1(\alpha) = \frac{2M(\alpha)}{2-\alpha}$, here $M(\alpha)$ is the normalization function such that $M(0) = M(1) = 1$, then fractional derivative defined in (1.3) is the Caputo-Fabrizio fractional derivative defined in (1.2).

In [5, 11], authors considered the existence of solutions of some differential and integro-differential equations involving with the Caputo-Fabrizio derivative. There are many works concerning with the applications and some partial differential equations of Caputo-Fabrizio derivative, for the details, please see [2–4, 12, 13, 15, 17, 21, 23, 24].

Fractional calculus (integrals and derivatives) are the generalizations of integer-order differential and integral operators. It is well known that the motivation for those works rises from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology, etc. For details, see [6, 18, 19]. There are many papers which discussed the existence of solutions to initial or boundary value problems for fractional differential equations denoted by the Rieamnn-Liouville fractional derivative or the Caputo fractional derivative, see [7–10, 16, 20].

Motivated by these, in this paper, we explore the existence of the solution of the problem (1.1).

This paper is organized as follows. In Section 2, we study the expression of the solution of an initial value problem studied in [22] for linear differential equation for the fractional derivative defined in (1.3). In Section 3, using results obtained in Section 2, we consider the unique existence of solutions of an initial value problem for nonlinear differential equation denoted by the Caputo-Fabrizio fractional derivative.

2. Linear equations

In [22], authors obtained the following result.

Lemma 2.1 ([22])**.** *Let $0 < \alpha < 1$. Then the unique solution of the following initial value problem*

$$\begin{cases} {}^{CF}D_{*}^{\alpha}f(t) = \sigma(t), & t > 0, \\ f(0) = f_0 \in \mathbb{R}, \end{cases}$$

is given by

$$f(t) = f_0 + a_\alpha(\sigma(t) - \sigma(0)) + b_\alpha I^1\sigma(t), t \geqslant 0,$$

where I^1 denotes a primitive of σ and

$$a_\alpha = \frac{2(1-\alpha)}{(2-\alpha)M_1(\alpha)}, \quad b_\alpha = \frac{2\alpha}{(2-\alpha)M_1(\alpha)}.$$

According to the definition of ${}^{CF}D_{*}^{\alpha}$ defined in (1.3), we find out that Lemma 2.1 is incorrect, except that $\sigma(0) = 0$. With careful deduction, we obtain the following result.

Theorem 2.2. *Let $0 < \alpha < 1$, $\sigma : [0,+\infty) \to R$ be a continuous differential function, then the solution of the following initial value problem*

$$\begin{cases} {}^{CF}D_*^{\alpha} f(t) = \sigma(t) - \sigma(0)e^{-\frac{\alpha}{1-\alpha}t}, & t > 0, \\ f(0) = f_0 \in R, \end{cases} \tag{2.1}$$

is given by

$$f(t) = f_0 + a_\alpha(\sigma(t) - \sigma(0)) + b_\alpha I^1\sigma(t), \quad t \geqslant 0, \tag{2.2}$$

where I^1 denotes a primitive of σ and

$$a_\alpha = \frac{2(1-\alpha)}{(2-\alpha)M_1(\alpha)}, \quad b_\alpha = \frac{2\alpha}{(2-\alpha)M_1(\alpha)}.$$

Proof. We verify that function defined in (2.2) is the solution of the problem (2.1). By (2.2), we get

$$f'(t) = a_\alpha \sigma'(t) + b_\alpha \sigma(t), t \geqslant 0. \tag{2.3}$$

Hence, multiplied by function $\exp(\frac{\alpha}{1-\alpha}t)$ on both sides of (2.3), for $t \geqslant 0$, we have

$$\exp(\frac{\alpha}{1-\alpha}t)f'(t) = \exp(\frac{\alpha}{1-\alpha}t)(a_\alpha\sigma'(t) + b_\alpha\sigma(t)) = a_\alpha(\exp(\frac{\alpha}{1-\alpha}t)\sigma(t))'. \tag{2.4}$$

Integrating from 0 to t on both sides of (2.4), for $t \geqslant 0$, we have

$$\int_0^t \exp(\frac{\alpha}{1-\alpha}s)f'(s)ds = \frac{2(1-\alpha)}{(2-\alpha)M_1(\alpha)}(\exp(\frac{\alpha}{1-\alpha}t)\sigma(t) - \sigma(0)). \tag{2.5}$$

From (2.5), for $t \geqslant 0$, we get

$$\frac{(2-\alpha)M_1(\alpha)}{2(1-\alpha)}\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s))f^{'}(s)ds = \sigma(t) - \sigma(0)\exp(-\frac{\alpha}{1-\alpha}t). \tag{2.6}$$

As well, from (2.2), we know that $f(0) = f_0$. With (2.6) combined, we get that $f(t)$ is the solution of the problem (2.1). Thus, we complete the proof. □

We also obtain the following result.

Theorem 2.3. *Let $0 < \alpha < 1$, $\sigma : [0,+\infty) \to R$ be a continuous differential function, then the solution of the following initial value problem*

$$\begin{cases} {}^{CF}D_*^{\alpha} f(t) = \sigma(t) - \sigma(0), & t > 0, \\ f(0) = f_0 \in R, \end{cases} \tag{2.7}$$

is given by

$$f(t) = f_0 + a_\alpha(\sigma(t) - \sigma(0)) + b_\alpha I^1\sigma(t) - b_\alpha\sigma(0)t, \quad t \geqslant 0, \tag{2.8}$$

where I^1 denotes a primitive of σ and

$$a_\alpha = \frac{2(1-\alpha)}{(2-\alpha)M_1(\alpha)}, \quad b_\alpha = \frac{2\alpha}{(2-\alpha)M_1(\alpha)}.$$

Proof. Similar to Theorem 2.2, here, we test that function defined in (2.8) is the solution of the problem (2.7). By (2.8), we get

$$f'(t) = a_\alpha\sigma'(t) + b_\alpha\sigma(t) - b_\alpha\sigma(0), t \geqslant 0. \tag{2.9}$$

Multiplying by function $\exp(\frac{\alpha}{1-\alpha}t)$ on both sides of (2.9), we get

$$\begin{aligned}\exp(\frac{\alpha}{1-\alpha}t)f'(t) =& \exp(\frac{\alpha}{1-\alpha}t)(a_\alpha\sigma'(t)+b_\alpha\sigma(t)) - b_\alpha\sigma(0)\exp(\frac{\alpha}{1-\alpha}t)\\ =& \frac{2(1-\alpha)}{(2-\alpha)M_1(\alpha)}(\exp(\frac{\alpha}{1-\alpha}t)\sigma(t))' - \exp(\frac{\alpha}{1-\alpha}t)\frac{2\alpha}{(2-\alpha)M_1(\alpha)}\sigma(0).\end{aligned}$$

Integrating now from 0 to t on both sides of the equality above, we have

$$\begin{aligned}\int_0^t \exp(\frac{\alpha}{1-\alpha}s)f'(s)ds =& \frac{2(1-\alpha)}{(2-\alpha)M_1(\alpha)}(\exp(\frac{\alpha}{1-\alpha}t)\sigma(t)-\sigma(0)) - \frac{2(1-\alpha)\sigma(0)}{(2-\alpha)M_1(\alpha)}(\exp(\frac{\alpha}{1-\alpha}t)-1)\\ =& \frac{2(1-\alpha)}{(2-\alpha)M_1(\alpha)}\exp(\frac{\alpha}{1-\alpha}t)\sigma(t) - \frac{2(1-\alpha)}{(2-\alpha)M_1(\alpha)}\exp(\frac{\alpha}{1-\alpha}t)\sigma(0).\end{aligned}$$

According to the equality above, we get

$$\frac{(2-\alpha)M_1(\alpha)}{2(1-\alpha)}\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s))f'(s)ds = \sigma(t)-\sigma(0), t \geqslant 0.$$

By (2.8), we know that $f(0) = f_0$. Hence, $f(t)$ defined in (2.8) is the solution of the problem (2.7). □

From Theorem 2.3, for the problem (1.1), we have the following result.

Lemma 2.4. *Let* $0 < \alpha < 1$, $h : [0, +\infty) \to R$ *be a continuous differential function, then the solution of the following initial value problem*

$$\begin{cases} {}^{CF}D^\alpha f(t) = h(t) - h(0), & t > 0,\\ f(0) = f_0 \in R, \end{cases} \tag{2.10}$$

is given by

$$f(t) = f_0 + a_\alpha(h(t)-h(0)) + b_\alpha I^1 h(t) - b_\alpha h(0)t, \ t \geqslant 0, \tag{2.11}$$

where I^1 *denotes a primitive of* σ *and*

$$a_\alpha = \frac{1-\alpha}{M(\alpha)}, \quad b_\alpha = \frac{\alpha}{M(\alpha)}.$$

Proof. From (1.2), (1.3), and Theorem 2.3, let $M_1(\alpha) = \frac{2M(\alpha)}{2-\alpha}$, here $M(\alpha)$ is the normalization function such that $M(0) = M(1) = 1$, then we could get that the function in (2.11) is the solution of the problem (2.10). □

3. Nonlinear equations

In this section, we consider the existence of solution of the problem (1.1). Our main result is the following.

Theorem 3.1. *Let* $0 < \beta < \alpha < 1$, $g : [0, +\infty) \times R \times R \to R$ *be a continuous differentiable function, there exist* $L_1 > 0, L_2 > 0$ *satisfying* $L_1 < \frac{M(\alpha)}{4}$, $L_2 < \frac{M(\alpha)}{4(A+B)}$ *and* $\lambda > 2$, *such that for all* $t \in [0, +\infty), x, y \in R$,

$$|g(t,(1+t^\lambda)x,(1+t^\lambda){}^{CF}D^\beta x) - g(t,(1+t^\lambda)y,(1+t^\lambda){}^{CF}D^\beta y)| \leqslant L_1|x-y| + L_2|{}^{CF}D^\beta x - {}^{CF}D^\beta y| \tag{3.1}$$

holds, and that

$$\lim_{t\to+\infty}\frac{|g(t,0,0)| + \int_0^t |g(s,0,0)|ds}{1+t^\lambda} = 0, \tag{3.2}$$

where

$$A = \frac{2M(\beta)}{1-\beta}, B = \frac{\beta M(\beta)}{(1-\beta)^2}. \tag{3.3}$$

Then the problem (1.1) *has one unique solution.*

Proof. Let

$$E = \left\{ x(t) \middle| \, x(t) \in C[0,+\infty), \sup_{t \geqslant 0} \frac{x(t)}{1+t^{\lambda}} < \infty \right\}$$

with the norm

$$\|x\|_E = \sup_{t \geqslant 0} \frac{|x(t)|}{1+t^{\lambda}},$$

then, by the same arguments as in [2, Lemma 2.2], we could know that $(E, \|\cdot\|_E)$ is a Banach space, here we omit this proof.

For $x \in E$, we have

$$|x(0)| = \frac{|x(t)|}{1+t^{\lambda}}|_{t=0} \leqslant \sup_{t \geqslant 0} \frac{|x(t)|}{1+t^{\lambda}} = \|x\|_E. \tag{3.4}$$

From Lemma 2.4, in order to obtain the existence result of solution of the problem (1.1), it is sufficient to consider the existence of fixed point $T : E \to E$, defined as

$$\begin{aligned} Tx(t) &= x_0 + \frac{1-\alpha}{M(\alpha)}(g(t, x(t), {}^{CF}D^{\beta}x(t)) - g(0, x(0), {}^{CF}D^{\beta}x(0))) \\ &\quad + \frac{\alpha}{M(\alpha)} \int_0^t g(s, x(s), {}^{CF}D^{\beta}x(s))ds - \frac{\alpha g(0, x(0), {}^{CF}D^{\beta}x(0))}{M(\alpha)}t, \quad 0 \leqslant t < +\infty. \end{aligned}$$

Operator $Tx(t) \in C[0,+\infty)$ for $x \in E$ is obtained through the continuity assumption of function g.

By the definition of the Caputo-Fabrizio fractional derivative, we know that

$$\begin{aligned} {}^{CF}D^{\beta}x(t) &= \frac{M(\beta)}{1-\beta} \int_0^t \exp(-\frac{\beta}{1-\beta}(t-s))x'(s)ds \\ &= \frac{M(\beta)}{1-\beta}x(t) - \frac{M(\beta)}{1-\beta}\exp(-\frac{\beta}{1-\beta}t)x(0) - \frac{\beta M(\beta)}{(1-\beta)^2}\int_0^t \exp(-\frac{\beta}{1-\beta}(t-s))x(s)ds, \quad 0 \leqslant t < +\infty. \end{aligned}$$

For $x \in E$, from the equality above, we have

$$\begin{aligned} |\frac{{}^{CF}D^{\beta}x(t)}{1+t^{\lambda}}| &= |\frac{M(\beta)}{1-\beta}\frac{x(t)}{1+t^{\lambda}} - \frac{M(\beta)x(0)}{(1-\beta)(1+t^{\lambda})}\exp(-\frac{\beta}{1-\beta}t) - \frac{\beta M(\beta)}{(1-\beta)^2(1+t^{\lambda})}\int_0^t \exp(-\frac{\beta}{1-\beta}(t-s))x(s)ds| \\ &\leqslant \frac{2M(\beta)}{1-\beta}\|x\|_E + \frac{\beta M(\beta)\|x\|_E}{(1-\beta)^2(1+t^{\lambda})}\int_0^t (1+s^{\lambda})ds \\ &\leqslant \frac{2M(\beta)}{1-\beta}\|x\|_E + \frac{\beta M(\beta)\|x\|_E}{(1-\beta)^2(1+t^{\lambda})}\int_0^t (1+t^{\lambda})ds \\ &= (\frac{2M(\beta)}{1-\beta} + \frac{\beta M(\beta)t}{(1-\beta)^2})\|x\|_E = (A+Bt)\|x\|_E, \quad 0 \leqslant t < +\infty, \end{aligned}$$

where A, B are the constants defined in (3.3).

For $x \in E$, by (3.1) and (3.4), we have

$$\begin{aligned} |g(t, x(t), {}^{CF}D^{\beta}x(t))| &= |g(t, x(t), {}^{CF}D^{\beta}x(t)) - g(t, 0, 0) + g(t, 0, 0)| \\ &\leqslant L_1|\frac{x(t)}{1+t^{\lambda}}| + L_2|\frac{{}^{CF}D^{\beta}x(t)}{1+t^{\lambda}}| + |g(t, 0, 0)| \\ &\leqslant (L_1 + L_2(A+Bt))\|x\|_E + |g(t, 0, 0)| \\ &= (L_1 + L_2A + L_2Bt)\|x\|_E + |g(t, 0, 0)|, \\ |g(0, x(0), {}^{CF}D^{\beta}x(0))| &= |g(0, x(0), {}^{CF}D^{\beta}x(0)) - g(0, 0, 0) + g(0, 0, 0)| \\ &\leqslant L_1|x(0)| + L_2|{}^{CF}D^{\beta}x(0)| + |g(0, 0, 0)| \end{aligned}$$

$$
\begin{aligned}
&\leqslant (L_1+L_2(A+Bt))\|x\|_E+|g(0,0,0)| \\
&= (L_1+L_2A+L_2Bt)\|x\|_E+|g(0,0,0)|,
\end{aligned}
$$

$$
\begin{aligned}
|\frac{Tx(t)}{1+t^\lambda}| &\leqslant \frac{1-\alpha}{M(\alpha)(1+t^\lambda)}|g(t,x(t),^{CF}D^\beta x(t))-g(0,x(0),^{CF}D^\beta x(0))| \\
&+\frac{\alpha}{M(\alpha)(1+t^\lambda)}\int_0^t |g(s,x(s),^{CF}D^\beta x(s))|ds \\
&+\frac{\alpha}{M(\alpha)}|g(0,x(0),^{CF}D^\beta x(0))|\frac{t}{1+t^\lambda}+\frac{|x_0|}{1+t^\lambda} \\
&\leqslant \frac{|x_0|}{1+t^\lambda}+\frac{(1-\alpha)((2L_1+2L_2A+2L_2Bt)\|x\|_E+|g(t,0,0)|+|g(0,0,0)|)}{M(\alpha)(1+t^\lambda)} \\
&+\frac{\alpha((L_1+L_2A)t+\frac{L_2Bt^2}{2})\|x\|_E+\int_0^t|g(s,0,0)|ds)}{M(\alpha)(1+t^\lambda)} \\
&+\frac{\alpha((L_1+L_2A)t+L_2Bt^2)\|x\|_E+|g(0,0,0)|t)}{M(\alpha)(1+t^\lambda)} \\
&\leqslant \frac{|x_0|}{1+t^\lambda}+\frac{(1-\alpha)((2L_1+2L_2A+2L_2Bt)\|x\|_E+|g(t,0,0)|+|g(0,0,0)|)}{M(\alpha)(1+t^\lambda)} \\
&+\frac{2\alpha((L_1+L_2A)t+L_2Bt^2)\|x\|_E+\alpha\int_0^t|g(s,0,0)|ds+\alpha|g(0,0,0)|t)}{M(\alpha)(1+t^\lambda)} \\
&\leqslant \frac{(2(1-\alpha)(L_1+L_2A)+(2L_2B(1-\alpha)+2\alpha(L_1+L_2A))t+2\alpha L_2Bt^2)\|x\|_E}{M(\alpha)(1+t^\lambda)} \\
&+\frac{|x_0|}{1+t^\lambda}+\frac{1}{M(\alpha)}\max\{\alpha,1-\alpha\}\frac{\int_0^t|g(s,0,0)|ds+|g(t,0,0)|}{1+t^\lambda} \\
&+\frac{(1-\alpha)|g(0,0,0)|+\alpha|g(0,0,0)|t}{M(\alpha)(1+t^\lambda)},
\end{aligned}
$$

by (3.2), we get that $\lim_{t\to+\infty}\frac{Tx(t)}{1+t^\lambda}=0$. Hence, we obtain that $T:E\to E$ is well defined.

Now, we verify that $T:E\to E$ is a contraction operator.

For $x,y\in E$, from the previous arguments, we obtain

$$
\left|\frac{^{CF}D^\beta x(t)}{1+t^\lambda}-\frac{^{CF}D^\beta y(t)}{1+t^\lambda}\right|\leqslant (A+Bt)\|x-y\|_E,\quad 0\leqslant t<+\infty. \tag{3.5}
$$

For $x,y\in E$, by (3.1) and (3.5), we get

$$
\begin{aligned}
\left|\frac{Tx(t)}{1+t^\lambda}-\frac{Ty(t)}{1+t^\lambda}\right| &\leqslant \frac{1-\alpha}{M(\alpha)(1+t^\lambda)}\Bigg[L_1\left|\frac{x(t)}{1+t^\lambda}-\frac{y(t)}{1+t^\lambda}\right|+L_1|x(0)-y(0)| \\
&+L_2\left|\frac{^{CF}D^\beta x(t)}{1+t^\lambda}-\frac{^{CF}D^\beta y(t)}{1+t^\lambda}\right|+L_2|^{CF}D^\beta x(0)-^{CF}D^\beta y(0)|\Bigg] \\
&+\frac{\alpha}{M(\alpha)(1+t^\lambda)}\int_0^t L_1\left|\frac{x(s)}{1+t^\lambda}-\frac{y(s)}{1+t^\lambda}\right|+L_2\left|\frac{^{CF}D^\beta x(s)}{1+t^\lambda}-\frac{^{CF}D^\beta y(s)}{1+t^\lambda}\right|ds \\
&+\frac{\alpha t}{M(\alpha)(1+t^\lambda)}(L_1|x(0)-y(0)|+L_2|^{CF}D^\beta x(0)-^{CF}D^\beta y(0)|) \\
&\leqslant \left[\frac{2(1-\alpha)L_1}{M(\alpha)(1+t^\lambda)}+\frac{2(1-\alpha)L_2(A+Bt)}{M(\alpha)(1+t^\lambda)}\right]\|x-y\|_E \\
&+\frac{\alpha}{M(\alpha)(1+t^\lambda)}\int_0^t(L_1+L_2A+L_2Bs)ds\|x-y\|_E
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\alpha t}{M(\alpha)(1+t^\lambda)}(L_1 + L_2 A + L_2 B t)\|x-y\|_E \\
&\leqslant \left[\frac{2(1-\alpha)L_1}{M(\alpha)} + \frac{2(1-\alpha)L_2(A+B)}{M(\alpha)}\right]\|x-y\|_E \\
&+ \left[\frac{L_1\alpha}{M(\alpha)} + \frac{L_2\alpha(A+B)}{M(\alpha)}\right]\|x-y\|_E + \left[\frac{L_1\alpha}{M(\alpha)} + \frac{L_2\alpha(A+B)}{M(\alpha)}\right]\|x-y\|_E \\
&= \left[\frac{2L_1}{M(\alpha)} + \frac{2L_2(A+B)}{M(\alpha)}\right]\|x-y\|_E.
\end{aligned}
$$

It follows from $L_1 < \frac{M(\alpha)}{4}$ and $L_2 < \frac{M(\alpha)}{4(A+B)}$ that $\frac{2L_1}{M(\alpha)} + \frac{2L_2(A+B)}{M(\alpha)} < 1$, which implies that $T : E \to E$ is a contraction operator. Then, the Banach contraction principle assures that operator T has a unique fixed point $x \in E$, which means that there exists a unique solution $x \in E$ for problem (1.1). Thus, we complete this proof. □

4. An example

Example 4.1. Consider the following initial value problem of differential equation involving the Caputo-Fabrizio fractional derivative

$$
\begin{cases}
{}^{CF}D^{\frac{3}{4}}x(t) = \dfrac{t^{\frac{1}{2}}}{2} + \dfrac{5M(\frac{3}{4})x^2 \sin t}{68(1+t^2)^2(1+x^2)} + \dfrac{5M(\frac{3}{4})({}^{CF}D^{\frac{1}{2}}x)^2 \sin t^2}{290(1+t^2)^2 M(\frac{1}{2})(1+({}^{CF}D^{\frac{1}{2}}x)^2)}, \quad 0 \leqslant t < \infty, \\
x(0) = 0.
\end{cases} \tag{4.1}
$$

Let $g(t, x, {}^{CF}D^{\frac{1}{2}}x) = \frac{t^{\frac{1}{2}}}{2} + \frac{5M(\frac{3}{4})x^2 \sin t}{68(1+t^2)^2(1+x^2)} + \frac{5M(\frac{3}{4})({}^{CF}D^{\frac{1}{2}}x)^2 \sin t^2}{290(1+t^2)^2 M(\frac{1}{2})(1+({}^{CF}D^{\frac{1}{2}}x)^2)}$. Obviously, $g : [0,+\infty) \times R \times R \to R$ is a continuous differentiable function, $g(0, x(0), {}^{CF}D^{\frac{1}{2}}x(0)) = 0$. Let

$$
\begin{aligned}
G(t,x,y) &= g(t,(1+t^2)x,(1+t^2)y) \\
&= \frac{t^{\frac{1}{2}}}{2} + \frac{5M(\frac{3}{4})x^2 \sin t}{68x^2} + \frac{5M(\frac{3}{4})(1+t^2)^2 y^2 \sin t^2}{290M(\frac{1}{2})(1+(1+t^2)^2 y^2)}, \quad 0 \leqslant t < \infty, x, y \in R.
\end{aligned}
$$

For all $0 \leqslant t < \infty, x, y \in R$, with simple calculation, we obtain

$$
G_x(t,x,y) = \frac{5M(\frac{3}{4})\sin t}{34} \frac{(1+t^2)^2 x}{\left(1+(1+t^2)^2x^2\right)^2}, \quad G_y(t,x,y) = \frac{5M(\frac{3}{4})\sin t^2}{145M(\frac{1}{2})} \frac{(1+t^2)^2 y}{\left(1+(1+t^2)^2 y^2\right)^2}.
$$

Obviously, for all $0 \leqslant t < \infty, x, y \in R$, we have

$$
|G_x(t,x,y)| \leqslant \frac{5M(\frac{3}{4})}{34}, \quad |G_y(t,x,y)| \leqslant \frac{5M(\frac{3}{4})}{145M(\frac{1}{2})}.
$$

Hence, for all $t \in [0,+\infty), x \in R$, it holds

$$
\begin{aligned}
&|g(t,(1+t^2)x,(1+t^2)\,{}^{CF}D^{\frac{1}{2}}x) - g(t,(1+t^2)y,(1+t^2)\,{}^{CF}D^{\frac{1}{2}}y)| \\
&= |G(t,x,{}^{CF}D^{\frac{1}{2}}x) - G(t,y,{}^{CF}D^{\frac{1}{2}}y)| \leqslant \frac{5M(\frac{3}{4})}{34}|x-y| + \frac{5M(\frac{3}{4})}{145M(\frac{1}{2})}|{}^{CF}D^{\frac{1}{2}}x - {}^{CF}D^{\frac{1}{2}}y|,
\end{aligned}
$$

which implies that g satisfies (3.1) of Theorem 3.1, with $L_1 = \frac{5M(\frac{3}{4})}{34}$, $L_2 = \frac{5M(\frac{3}{4})}{145M(\frac{1}{2})}$, $\lambda = 2$. Meanwhile, by calculating, we obtain

$$
A + B = \frac{2M(\beta)}{1-\beta} + \frac{\beta M(\beta)}{(1-\beta)^2} = 4M(\frac{1}{2}) + 2M(\frac{1}{2}) = 6M(\frac{1}{2}).
$$

Hence, we have

$$L_1 = \frac{5M(\frac{3}{4})}{34} < \frac{M(\frac{3}{4})}{4} = \frac{M(\alpha)}{4}, \quad L_2 = \frac{5M(\frac{3}{4})}{145M(\frac{1}{2})} < \frac{M(\frac{3}{4})}{24M(\frac{1}{2})} = \frac{M(\alpha)}{4(A+B)}.$$

And that

$$\lim_{t\to+\infty} \frac{|g(t,0,0)| + \int_0^t |g(s,0,0)|ds}{1+t^2} = \lim_{t\to+\infty} \frac{t^{\frac{1}{2}} + \int_0^t s^{\frac{1}{2}} ds}{2(1+t^2)} = 0.$$

Then, we see that all assumption conditions in Theorem 3.1 are satisfied. Hence, by Theorem 3.1, the problem (4.1) have a unique solution $x \in E$.

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