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Strong convergence of a new iterative algorithm for fixed points of asymptotically nonexpansive mappings

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Abstract

In this paper, we investigate a new iterative implicit algorithm for fixed points of asymptotically nonexpansive mapping in Hilbert spaces. We also prove its strong convergence theorem under certain assumptions imposed on the parameters and extend some well-known results. As an application, we apply our main result to μ-inverse strongly monotone mapping.

Keywords: Asymptotically nonexpansive, strong convergence, μ-inverse strongly monotone mapping, Hilbert space.

2010 MSC: 47H09, 47H10, 54H25.

1. Introduction

In the theories of this paper, we suppose that H is a Hilbert space with the inner product $\langle \cdot\,,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty convex closed subset of H, and "$\rightharpoonup$" and "$\to$" denote the weak convergence and strong convergence, respectively. For a mapping $T : C \to C$, we use $F(T)$ and $N(T)$ to denote its fixed point set and zero point set, respectively.

Recall that the mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in C.$$

We call T is said to be attracting nonexpansive if it is nonexpansive and satisfies:

$$\|Tx - y\| < \|x - y\|, \quad \forall x \notin F(T), y \in F(T).$$

The mapping T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$, such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\|, \quad \forall x, y \in C.$$

The mapping $U : C \to H$ is said to be monotone if

$$\langle Ux - Uy, x - y\rangle \geqslant 0, \quad \forall x, y \in C.$$

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The mapping $U : C \to H$ is said to be μ-inverse strongly monotone if

$$\langle Ux - Uy, x - y\rangle \geqslant \mu\|Ux - Uy\|^2, \ \ \forall x, y \in C,$$

where $\mu > 0$ is a positive real number.

Lately, in order to find a common element of the fixed points of nonexpansive mappings or asymptotically nonexpansive mappings, many authors established many viscosity iterative algorithms, and they solved many practical problems, such as the equilibrium problems, variational inequality problems, and split feasibility problems and so on. See [1, 2, 4, 9–14, 21–24] and the references therein.

Recently, Moudafi [8] in Hilbert space introduced the viscosity iterative algorithm of nonexpansive. Xu [18] extended the Moudafi's results to more general uniformly smooth spaces. That is, he introduced the following explicit viscosity algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \tag{1.1}$$

where T is a nonexpansive mapping, f is a strict contraction mapping, and $\{\alpha_n\}$ is a real number sequence in $(0, 1)$. Then the sequence $\{x_n\}$ generated by (1.1) converges strongly to a fixed point of T.

In 2008, Lou et al. [7] showed the existence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n x_n,$$

converges strongly to the fixed point set of an asymptotically nonexpansive mapping T, which solves some variational inequality problems under suitable conditions. As an application, they proved that the iterative process defined by

$$x_0 \in C, \ \ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n x_n,$$

converges strongly to the same fixed point of T.

In 2015, Xu et al. [19] studied the following viscosity implicit midpoint rule:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}), \ \ n \geqslant 0. \tag{1.2}$$

They proved that the $\{x_n\}$ generated by (1.2) converges in norm to a fixed point of T.

Recently, Fan et al. [3] studied the following iterative algorithm:

$$\begin{cases} x_{n+1} = (1 - \beta_n)(\lambda_n x_n) + \beta_n y_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T x_n. \end{cases} \tag{1.3}$$

They proved that the $\{x_n\}$ generated by (1.3) converges strongly to a fixed point $\hat{x}$ of nonexpansive mapping T, where $\hat{x}$ is the minimum-norm element of $F(T)$.

In this paper, motivated and inspired by the above authors' results, we give a new implicit iterative algorithm, to approximate fixed points of asymptotically nonexpansive mapping and prove its strong convergence theorem. As applications, we can use our main results to a family of μ-inverse strongly monotone mappings for the common zeros.

2. Preliminaries

Definition 2.1. Let $C \subset H$ be a nonempty convex closed set of a Hilbert space H, we define $P_C : C \to H$ be the nearest point projection if

$$\|x - P_C x\| \leqslant \|x - y\|, \ \ \forall y \in C.$$

Lemma 2.2 ([6]). *In a real Banach space E, for $q > 1, \forall x, y \in E$, the following inequality holds*

$$\|x+y\|^q \leqslant \|x\|^q + q\langle y, j_q(x+y)\rangle, \quad \forall j_q(x+y) \in J_q(x+y). \tag{2.1}$$

When E is a Hilbert space and $q = 2$, (2.1) can be reduced to the following inequality

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 2.3 ([15]). *Let $C \subset H$ be a nonempty bounded convex closed set of a Hilbert space H. Let $T : C \to C$ be a asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $x_n \rightharpoonup x$ and $(I-T)x_n \to y$, then $(I-T)x = y$.*

Lemma 2.4 ([16, 17]). *Let $\{a_n\}_{n=0}^{\infty}$ be a nonnegative real number sequence satisfying the property:*

$$a_{n+1} \leqslant (1-\delta_n)a_n + \delta_n\gamma_n, \quad n \geqslant 0,$$

where $\{\delta_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\gamma_n\}_{n=0}^{\infty}$ satisfy:

(i) $\lim\limits_{n\to\infty} \delta_n = 0$ *and* $\sum\limits_{n=0}^{\infty} \delta_n = +\infty$;

(ii) *either* $\limsup\limits_{n\to\infty} \gamma_n \leqslant 0$ *or* $\sum\limits_{n=0}^{\infty} |\delta_n\gamma_n| < +\infty$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let $C \subset H$ be a nonempty bounded convex closed set of a Hilbert space H and $\theta \in C$. Let $T : C \to C$ be a asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $F(T) \neq \emptyset$. Suppose $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \{\xi_n\}$ are real number sequences in $(0,1)$. Let $\{x_n\}$ be generated by*

$$\begin{cases} x_1 \in C, \\ y_n = (1-\alpha_n)x_n + \alpha_n T^n(\xi_n x_n + (1-\xi_n)x_{n+1}), \\ x_{n+1} = (1-\beta_n)(\lambda_n x_n) + \beta_n y_n. \end{cases} \tag{3.1}$$

Suppose the following conditions are satisfied:

(i) $\lim\limits_{n\to\infty} \beta_n = 1$, $\lim\limits_{n\to\infty} \lambda_n = 1$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n < +\infty$, $\sum\limits_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < +\infty$, $\sum\limits_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < +\infty$;

(iii) $\sum\limits_{n=1}^{\infty} (1-\lambda_n)(1-\beta_n) = +\infty, |\lambda_{n+1} - \beta_n\lambda_n| + \beta_{n+1} \leqslant 1$;

(iv) $\sum\limits_{n=1}^{\infty} \sup\limits_{x\in C} \|T^{n+1}x - T^n x\| < +\infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point $x^* \in F(T)$.

Proof. We divide the proof into four steps.

Step 1. We show the sequence $\{x_n\}$ is bounded.

In fact, for $p \in F(T)$, we can easily know that

$$\begin{aligned} \|y_n - p\| &= \|(1-\alpha_n)x_n + \alpha_n T^n(\xi_n x_n + (1-\xi_n)x_{n+1}) - p\| \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n(\xi_n x_n + (1-\xi_n)x_{n+1}) - p)\| \\ &\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\|T^n(\xi_n x_n + (1-\xi_n)x_{n+1}) - p)\| \\ &\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n k_n\|\xi_n x_n + (1-\xi_n)x_{n+1} - p)\| \\ &\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n k_n\xi_n\|x_n - p\| + \alpha_n k_n(1-\xi_n)\|x_{n+1} - p\| \\ &= (1-\alpha_n + \alpha_n k_n\xi_n)\|x_n - p\| + \alpha_n k_n(1-\xi_n)\|x_{n+1} - p\|. \end{aligned} \tag{3.2}$$

And then from (3.2),

$$\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\beta_n)(\lambda_n x_n)+\beta_n y_n - p\| \\
&= \|(1-\beta_n)(\lambda_n x_n-\lambda_n p+\lambda_n p-p)+\beta_n(y_n-p)\| \\
&\leqslant (1-\beta_n)\lambda_n\|x_n-p\|+(1-\beta_n)(1-\lambda_n)\|p\|+\beta_n\|y_n-p\| \\
&\leqslant (1-\beta_n)\lambda_n\|x_n-p\|+(1-\beta_n)(1-\lambda_n)\|p\| \\
&\quad +(\beta_n-\alpha_n\beta_n+\alpha_n\beta_n k_n\xi_n)\|x_n-p\|+\alpha_n\beta_n k_n(1-\xi_n)\|x_{n+1}-p\| \\
&= (\lambda_n-\beta_n\lambda_n+\beta_n-\alpha_n\beta_n+\alpha_n\beta_n k_n\xi_n)\|x_n-p\| \\
&\quad +(1-\beta_n)(1-\lambda_n)\|p\|+\alpha_n\beta_n k_n(1-\xi_n)\|x_{n+1}-p\|,
\end{aligned}$$

which implies

$$\begin{aligned}
\|x_{n+1}-p\| &\leqslant \frac{\lambda_n-\beta_n\lambda_n+\beta_n-\alpha_n\beta_n+\alpha_n\beta_n k_n\xi_n}{1-\alpha_n\beta_n k_n(1-\xi_n)}\|x_n-p\|+\frac{(1-\beta_n)(1-\lambda_n)}{1-\alpha_n\beta_n k_n(1-\xi_n)}\|p\| \\
&= \Big(1-\frac{(1-\beta_n)(1-\lambda_n)-\alpha_n\beta_n(k_n-1)}{1-\alpha_n\beta_n k_n(1-\xi_n)}\Big)\|x_n-p\| \\
&\quad +\frac{(1-\beta_n)(1-\lambda_n)-\alpha_n\beta_n(k_n-1)+\alpha_n\beta_n(k_n-1)}{1-\alpha_n\beta_n k_n(1-\xi_n)}\|p\| \\
&= \Big(1-\frac{(1-\beta_n)(1-\lambda_n)-\alpha_n\beta_n(k_n-1)}{1-\alpha_n\beta_n k_n(1-\xi_n)}\Big)\|x_n-p\| \\
&\quad +\frac{(1-\beta_n)(1-\lambda_n)-\alpha_n\beta_n(k_n-1)}{1-\alpha_n\beta_n k_n(1-\xi_n)}\|p\|+\frac{\alpha_n\beta_n(k_n-1)}{1-\alpha_n\beta_n k_n(1-\xi_n)}\|p\|.
\end{aligned}$$

Let $M_1=\sup\limits_{n\in\mathbb{N}}\{k_n-1\}$, then by conditions (i) and (ii), there exists $N\in\mathbb{N}$ such that for all $n\geqslant N$,

$$\frac{(1-\beta_n)(1-\lambda_n)-\alpha_n\beta_n(k_n-1)}{1-\alpha_n\beta_n k_n(1-\xi_n)}<1,$$

and

$$1-\alpha_n\beta_n k_n(1-\xi_n)\geqslant\frac{1}{2}.$$

So we have

$$\begin{aligned}
\|x_{n+1}-p\| &\leqslant \Big(1-\frac{(1-\beta_n)(1-\lambda_n)-\alpha_n\beta_n(k_n-1)}{1-\alpha_n\beta_n k_n(1-\xi_n)}\Big)\|x_n-p\| \\
&\quad +\frac{(1-\beta_n)(1-\lambda_n)-\alpha_n\beta_n(k_n-1)}{1-\alpha_n\beta_n k_n(1-\xi_n)}\|p\|+2\alpha_n M_1\|p\| \\
&\leqslant \max\{\|x_n-p\|,\|p\|\}+2\alpha_n M_1\|p\| \\
&\ \ \vdots \\
&\leqslant \max\{\|x_1-p\|,\|p\|\}+2M_1\|p\|\sum_{n=1}^{\infty}\alpha_n<+\infty.
\end{aligned}$$

Therefore, by condition (ii), we obtain that $\{x_n\}$ is bounded. So are $\{y_n\}$ and $\{T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\}$.

Step 2. We prove that $\|x_{n+1}-x_n\|\to 0$, as $n\to\infty$.

Indeed, we observe that

$$\begin{aligned}
y_{n+1}-y_n &= (1-\alpha_{n+1})x_{n+1}+\alpha_{n+1}T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-(1-\alpha_n)x_n \\
&\quad -\alpha_n T^n(\xi_n x_n+(1-\xi_n)x_{n+1}) \\
&= (1-\alpha_{n+1})x_{n+1}-(1-\alpha_n)x_{n+1}+(1-\alpha_n)x_{n+1}-(1-\alpha_n)x_n \\
&\quad +\alpha_{n+1}T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-\alpha_n T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2}) \\
&\quad +\alpha_n T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-\alpha_n T^n(\xi_n x_n+(1-\xi_n)x_{n+1}) \\
&= (1-\alpha_n)(x_{n+1}-x_n)-(\alpha_{n+1}-\alpha_n)x_{n+1} \\
&\quad +\alpha_n(T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})) \\
&\quad +(\alpha_{n+1}-\alpha_n)T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2}) \\
&= (1-\alpha_n)(x_{n+1}-x_n)-(\alpha_{n+1}-\alpha_n)x_{n+1} \\
&\quad +\alpha_n(T^{n}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})) \\
&\quad +(\alpha_{n+1}-\alpha_n)T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2}) \\
&\quad +\alpha_n(T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-T^n(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})).
\end{aligned} \tag{3.3}$$

Then from (3.3)

$$\begin{aligned}
\|y_{n+1}-y_n\| &\leqslant (1-\alpha_n)\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|x_{n+1}\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\quad +\alpha_n\|T^{n}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\| \\
&\quad +\alpha_n\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-T^n(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\leqslant (1-\alpha_n)\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|x_{n+1}\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\quad +\alpha_n k_n\|(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-(\xi_n x_n+(1-\xi_n)x_{n+1})\| \\
&\quad +\alpha_n\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-T^n(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\leqslant (1-\alpha_n)\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|x_{n+1}\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\quad +\alpha_n\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-T^n(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\quad +\alpha_n k_n(1-\xi_{n+1})\|x_{n+2}-x_{n+1}\|+\alpha_n k_n\xi_n\|x_{n+1}-x_n\| \\
&= (1-\alpha_n+\alpha_n k_n\xi_n)\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|x_{n+1}\| \\
&\quad +|\alpha_{n+1}-\alpha_n|\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\quad +\alpha_n k_n(1-\xi_{n+1})\|x_{n+2}-x_{n+1}\| \\
&\quad +\alpha_n\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})-T^n(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\|.
\end{aligned} \tag{3.4}$$

And also

$$\begin{aligned}
x_{n+2}-x_{n+1} &= (1-\beta_{n+1})(\lambda_{n+1}x_{n+1})+\beta_{n+1}y_{n+1}-(1-\beta_n)(\lambda_n x_n)-\beta_n y_n \\
&= \lambda_{n+1}x_{n+1}-\beta_{n+1}\lambda_{n+1}x_{n+1}+\beta_{n+1}y_{n+1}-\lambda_n x_n+\beta_n\lambda_n x_n-\beta_n y_n \\
&= \lambda_{n+1}x_{n+1}-\lambda_{n+1}x_n+\lambda_{n+1}x_n-\lambda_n x_n-\beta_{n+1}\lambda_{n+1}x_{n+1}+\beta_n\lambda_n x_{n+1} \\
&\quad -\beta_n\lambda_n x_{n+1}+\beta_n\lambda_n x_n+\beta_{n+1}y_{n+1}-\beta_{n+1}y_n+\beta_{n+1}y_n-\beta_n y_n \\
&= \lambda_{n+1}(x_{n+1}-x_n)+(\lambda_{n+1}-\lambda_n)x_n-(\beta_{n+1}\lambda_{n+1}-\beta_n\lambda_n)x_{n+1} \\
&\quad -\beta_n\lambda_n(x_{n+1}-x_n)+\beta_{n+1}(y_{n+1}-y_n)+(\beta_{n+1}-\beta_n)y_n \\
&= (\lambda_{n+1}-\beta_n\lambda_n)(x_{n+1}-x_n)+(\lambda_{n+1}-\lambda_n)x_n \\
&\quad -(\beta_{n+1}\lambda_{n+1}-\beta_n\lambda_n)x_{n+1}+\beta_{n+1}(y_{n+1}-y_n)+(\beta_{n+1}-\beta_n)y_n.
\end{aligned} \tag{3.5}$$

Hence, it follows from (3.4) and (3.5),

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leqslant |\lambda_{n+1}-\beta_n\lambda_n|\|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|\|x_n\| \\
&\quad + |\beta_{n+1}\lambda_{n+1}-\beta_n\lambda_n|\|x_{n+1}\| + \beta_{n+1}\|y_{n+1}-y_n\| + |\beta_{n+1}-\beta_n|\|y_n\| \\
&\leqslant |\lambda_{n+1}-\beta_n\lambda_n|\|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|\|x_n\| \\
&\quad + |\beta_{n+1}-\beta_n|\|y_n\| + (\beta_{n+1}-\alpha_n\beta_{n+1}+\alpha_n\beta_{n+1}k_n\xi_n)\|x_{n+1}-x_n\| \\
&\quad + \beta_{n+1}|\alpha_{n+1}-\alpha_n|\|x_{n+1}\| + \beta_{n+1}|\alpha_{n+1}-\alpha_n|\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\quad + \alpha_n\beta_{n+1}\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2}) - T^n(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\quad + \alpha_n\beta_{n+1}k_n(1-\xi_{n+1})\|x_{n+2}-x_{n+1}\| + |\beta_{n+1}\lambda_{n+1}-\beta_n\lambda_n|\|x_{n+1}\| \\
&= (|\lambda_{n+1}-\beta_n\lambda_n| + \beta_{n+1}-\alpha_n\beta_{n+1}+\alpha_n\beta_{n+1}k_n\xi_n)\|x_{n+1}-x_n\| \\
&\quad + |\lambda_{n+1}-\lambda_n|\|x_n\| + (|\beta_{n+1}\lambda_{n+1}-\beta_n\lambda_n| + \beta_{n+1}|\alpha_{n+1}-\alpha_n|)\|x_{n+1}\| \\
&\quad + |\beta_{n+1}-\beta_n|\|y_n\| + \beta_{n+1}|\alpha_{n+1}-\alpha_n|\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\quad + \alpha_n\beta_{n+1}\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2}) - T^n(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\quad + \alpha_n\beta_{n+1}k_n(1-\xi_{n+1})\|x_{n+2}-x_{n+1}\|,
\end{aligned}$$

which implies

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leqslant \frac{|\lambda_{n+1}-\beta_n\lambda_n| + \beta_{n+1}-\alpha_n\beta_{n+1}+\alpha_n\beta_{n+1}k_n\xi_n}{1-\alpha_n\beta_{n+1}k_n(1-\xi_{n+1})}\|x_{n+1}-x_n\| \\
&\quad + \frac{|\lambda_{n+1}-\lambda_n|}{1-\alpha_n\beta_{n+1}k_n(1-\xi_{n+1})}\|x_n\| + \frac{|\beta_{n+1}-\beta_n|}{1-\alpha_n\beta_{n+1}k_n(1-\xi_{n+1})}\|y_n\| \\
&\quad + \frac{|\beta_{n+1}\lambda_{n+1}-\beta_n\lambda_n| + \beta_{n+1}|\alpha_{n+1}-\alpha_n|}{1-\alpha_n\beta_{n+1}k_n(1-\xi_{n+1})}\|x_{n+1}\| \\
&\quad + \frac{\beta_{n+1}|\alpha_{n+1}-\alpha_n|}{1-\alpha_n\beta_{n+1}k_n(1-\xi_{n+1})}\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\quad + \frac{\alpha_n\beta_{n+1}}{1-\alpha_n\beta_{n+1}k_n(1-\xi_{n+1})}\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2}) \\
&\quad - T^n(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\| \\
&\leqslant (1-\sigma_n)\|x_{n+1}-x_n\| + M_2(|\lambda_{n+1}-\lambda_n| + |\beta_{n+1}\lambda_{n+1}-\beta_n\lambda_n| \\
&\quad + 2\beta_{n+1}|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n| + \alpha_n\beta_{n+1}) \\
&\leqslant (1-\sigma_n)\|x_{n+1}-x_n\| + M_2(2|\lambda_{n+1}-\lambda_n| + 2|\alpha_{n+1}-\alpha_n| \\
&\quad + 2|\beta_{n+1}-\beta_n| + \alpha_n\beta_{n+1}),
\end{aligned} \tag{3.6}$$

where

$$\sigma_n = \frac{1-|\lambda_{n+1}-\beta_n\lambda_n| - \beta_{n+1}+\alpha_n\beta_{n+1}-\alpha_n\beta_{n+1}k_n(1-\xi_{n+1}+\xi_n)}{1-\alpha_n\beta_{n+1}k_n(1-\xi_{n+1})}$$

and $M_2 > 0$ is a constant such that for all $n \geqslant 1$,

$$\begin{aligned}
M_2 \geqslant \Big\{&\|x_n\|, \|y_n\|, \|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\|, \\
&\|T^{n+1}(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2}) - T^n(\xi_{n+1}x_{n+1}+(1-\xi_{n+1})x_{n+2})\|\Big\}.
\end{aligned}$$

By the assumptions (i)-(iv), we have

$$\lim_{n\to\infty}\sigma_n = 0, \quad \sum_{n=1}^{\infty}\sigma_n = \infty,$$

and

$$\sum_{n=1}^{\infty}(2|\lambda_{n+1}-\lambda_n|+2|\alpha_{n+1}-\alpha_n|+2|\beta_{n+1}-\beta_n|+\alpha_n\beta_{n+1})$$
$$\leqslant 2\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|+4\sum_{n=1}^{\infty}\alpha_n+2\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|+\sum_{n=1}^{\infty}\alpha_n<+\infty.$$

Therefore, using Lemma 2.4 and (3.6), we obtain

$$\|x_{n+1}-x_n\|\to 0, n\to\infty. \tag{3.7}$$

Step 3. We prove that $\|x_n-Tx_n\|\to 0, n\to\infty$.

From (3.1), we can easily have

$$\|x_{n+1}-y_n\|=\|(1-\beta_n)(\lambda_n x_n)+\beta_n y_n-y_n\|=(1-\beta_n)\|\lambda_n x_n-y_n\|,$$

then by the condition (i),

$$\|x_{n+1}-y_n\|\to 0,\quad \text{as } n\to\infty. \tag{3.8}$$

And

$$\begin{aligned}
&\|y_n-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|\\
&=\|(1-\alpha_n)x_n+\alpha_n T^n(\xi_n x_n+(1-\xi_n)x_{n+1})-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|\\
&=(1-\alpha_n)\|x_n-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|\\
&\leqslant(1-\alpha_n)\|x_n-x_{n+1}\|+(1-\alpha_n)\|x_{n+1}-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|.
\end{aligned}\tag{3.9}$$

So from (3.9),

$$\begin{aligned}
\|x_{n+1}-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|&=\|x_{n+1}-y_n+y_n-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|\\
&\leqslant\|x_{n+1}-y_n\|+\|y_n-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|\\
&\leqslant\|x_{n+1}-y_n\|+(1-\alpha_n)\|x_n-x_{n+1}\|\\
&\quad+(1-\alpha_n)\|x_{n+1}-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|,
\end{aligned}$$

which implies

$$\|x_{n+1}-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|\leqslant\frac{1}{\alpha_n}\|x_{n+1}-y_n\|+\frac{1-\alpha_n}{\alpha_n}\|x_n-x_{n+1}\|.$$

Then from (3.7) and (3.8),

$$\|x_{n+1}-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|\to 0,\quad \text{as } n\to\infty. \tag{3.10}$$

Moreover, we have

$$\begin{aligned}
\|x_n-T^n x_n\|&=\|x_n-x_{n+1}+x_{n+1}-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})+T^n(\xi_n x_n+(1-\xi_n)x_{n+1})-T^n x_n\|\\
&\leqslant\|x_n-x_{n+1}\|+\|x_{n+1}-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|\\
&\quad+\|T^n(\xi_n x_n+(1-\xi_n)x_{n+1})-T^n x_n\|\\
&\leqslant\|x_n-x_{n+1}\|+\|x_{n+1}-T^n(\xi_n x_n+(1-\xi_n)x_{n+1})\|+k_n(1-\xi_n)\|x_{n+1}-x_n\|.
\end{aligned}$$

Then from (3.7) and (3.10), we have

$$\|x_n-T^n x_n\|\to 0,\quad \text{as } n\to\infty. \tag{3.11}$$

Since T is asymptotically nonexpansive, we derive that

$$\begin{aligned}
|x_{n+1} - Tx_{n+1}\| &= \|x_{n+1} - T^{n+1}x_{n+1} + T^{n+1}x_{n+1} - Tx_{n+1}\| \\
&\leqslant \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\
&\leqslant \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\|T^n x_{n+1} - x_{n+1}\| \\
&= \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\|T^n x_{n+1} - T^n x_n + T^n x_n - x_n + x_n - x_{n+1}\| \\
&= \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\|T^n x_{n+1} - T^n x_n\| + k_1\|T^n x_n - x_n\| + k_1\|x_n - x_{n+1}\| \\
&= \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1 k_n\|x_{n+1} - x_n\| + k_1\|T^n x_n - x_n\| + k_1\|x_n - x_{n+1}\|.
\end{aligned}$$

So from (3.7) and (3.11), we obtain that

$$\|x_n - Tx_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.12}$$

Step 4. We show that $x_n \to x^* \in F(T)$. Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$ which converges weakly to x^*, we assume that

$$x_{n_i} \rightharpoonup x^* \in H, \tag{3.13}$$

from (3.12) and the demiclosedness principle in Lemma 2.3, we have $q \in F(T)$.

For $n \geqslant 1$, let $z_n = (1-\beta_n)x_n + \beta_n y_n$, then from (3.1),

$$x_{n+1} = z_n - (1-\beta_n)(1-\lambda_n)x_n.$$

By the boundedness of $\{x_n\}$, and the condition (i), we have

$$\|x_{n+1} - z_n\| = (1-\beta_n)(1-\lambda_n)\|x_n\|. \tag{3.14}$$

By the (3.13) and (3.14), we can conclude that $z_{n_i} \rightharpoonup q$. It follows that

$$\begin{aligned}
x_{n+1} &= z_n - (1-\beta_n)(1-\lambda_n)x_n \\
&= (1-(1-\beta_n)(1-\lambda_n))z_n - (1-\beta_n)(1-\lambda_n)(x_n - z_n) \\
&= (1-(1-\beta_n)(1-\lambda_n))z_n - (1-\beta_n)(1-\lambda_n)\beta_n(x_n - y_n).
\end{aligned} \tag{3.15}$$

Also from Lemma 2.2, we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|(1-\beta_n)x_n + \beta_n y_n - x^*\|^2 \\
&= \|(x_n - x^*) - \beta_n(x_n - y_n)\|^2 \leqslant \|x_n - x^*\|^2 - 2\beta_n\langle x_n - y_n, z_n - x^*\rangle.
\end{aligned} \tag{3.16}$$

By Lemma 2.2, (3.15), and (3.16), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-(1-\beta_n)(1-\lambda_n))(z_n - x^*) \\
&\quad - (1-\beta_n)(1-\lambda_n)\beta_n(x_n - y_n) - (1-\beta_n)(1-\lambda_n)x^*\|^2 \\
&\leqslant (1-(1-\beta_n)(1-\lambda_n))^2\|z_n - x^*\| \\
&\quad - 2(1-\beta_n)(1-\lambda_n)\langle\beta_n(x_n - y_n) + x^*, x_{n+1} - x^*\rangle \\
&\leqslant (1-(1-\beta_n)(1-\lambda_n))^2\|z_n - x^*\| \\
&\quad - 2(1-\beta_n)(1-\lambda_n)\beta_n\langle x_n - y_n, x_{n+1} - x^*\rangle - 2(1-\beta_n)(1-\lambda_n)\langle x^*, x_{n+1} - x^*\rangle \\
&\leqslant (1-(1-\beta_n)(1-\lambda_n))(\|x_n - x^*\|^2 - 2\beta_n\langle x_n - y_n, z_n - x^*\rangle) \\
&\quad - 2(1-\beta_n)(1-\lambda_n)\beta_n\langle x_n - y_n, x_{n+1} - x^*\rangle - 2(1-\beta_n)(1-\lambda_n)\langle x^*, x_{n+1} - x^*\rangle \\
&\leqslant (1-\delta_n)\|x_n - x^*\|^2 + \delta_n(-2\beta_n\langle x_n - y_n, z_n - x^*\rangle \\
&\quad - 2\beta_n\langle x_n - y_n, x_{n+1} - x^*\rangle - 2\langle x^*, x_{n+1} - x^*\rangle) \\
&= (1-\delta_n)\|x_n - x^*\|^2 + \delta_n\gamma_n,
\end{aligned} \tag{3.17}$$

where

$$\delta_n = (1-\beta_n)(1-\lambda_n), \quad \gamma_n = -2\beta_n\langle x_n - y_n, z_n - x^*\rangle - 2\beta_n\langle x_n - y_n, x_{n+1} - x^*\rangle - 2\langle x^*, x_{n+1} - x^*\rangle.$$

By the conditions (i) and (iii), we can easily have that

$$\lim_{n\to\infty} \delta_n = 0, \quad \sum_{n=1}^{\infty} \delta_n = +\infty.$$

Because

$$\limsup_{n\to\infty} -2\beta_n\langle x_n - y_n, z_n - x^*\rangle = 0, \quad \limsup_{n\to\infty} -2\beta_n\langle x_n - y_n, x_{n+1} - x^*\rangle = 0,$$

and also

$$\limsup_{n\to\infty} -2\langle x^*, x_{n+1} - x^*\rangle = \lim_{i\to\infty} -2\langle x^*, x_{n_i} - x^*\rangle = -2\langle x^*, q - x^*\rangle \leqslant 0.$$

So

$$\limsup_{n\to\infty} \gamma_n \leqslant 0.$$

Therefore, applying Lemma 2.4 to (3.17), we conclude that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0,$$

which completes this proof. □

Remark 3.2. In algorithm (3.1), the iterative coefficients $\alpha_n, \beta_n, \lambda_n$ are available.

For example, take $\alpha_n = \frac{1}{n^2}, \beta_n = \lambda_n = 1 - \frac{1}{\sqrt{n}}$, then

$$\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \quad \sum_{n=1}^{\infty} (1-\lambda_n)(1-\beta_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

From

$$|\lambda_{n+1} - \lambda_n| = |(1 - \frac{1}{\sqrt{n+1}}) - (1 - \frac{1}{\sqrt{n}})| = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{n(n+1)}(\sqrt{n} + \sqrt{n+1})} \leqslant \frac{1}{n^{\frac{3}{2}}},$$

we have

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| \leqslant \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < \infty,$$

and also

$$\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$$

Because

$$1 - \frac{1}{\sqrt{n+1}} \geqslant (1 - \frac{1}{\sqrt{n}})^2,$$

then

$$\left|(1 - \frac{1}{\sqrt{n+1}}) - (1 - \frac{1}{\sqrt{n}})^2\right| = (1 - \frac{1}{\sqrt{n+1}}) - (1 - \frac{1}{\sqrt{n}})^2.$$

We want to know that

$$|\lambda_{n+1} - \beta_n\lambda_n| + \beta_{n+1} = \left|(1 - \frac{1}{\sqrt{n+1}}) - (1 - \frac{1}{\sqrt{n}})^2\right| + 1 - \frac{1}{\sqrt{n+1}} \leqslant 1,$$

which is equivalent to

$$1-\frac{1}{\sqrt{n+1}}-1+\frac{2}{\sqrt{n}}-\frac{1}{n}+1-\frac{1}{\sqrt{n+1}}-1\leqslant 0, \qquad \frac{2}{\sqrt{n}}-\frac{1}{n}-\frac{2}{\sqrt{n+1}}\leqslant 0,$$

$$\frac{2}{\sqrt{n}}-\frac{2}{\sqrt{n+1}}\leqslant\frac{1}{n}, \qquad \frac{2n}{\sqrt{n(n+1)}(\sqrt{n+1}+\sqrt{n})}\leqslant 1.$$

Therefore

$$\frac{2n}{\sqrt{n(n+1)}(\sqrt{n+1}+\sqrt{n})}\leqslant\frac{2n}{2n\sqrt{n}}=\frac{1}{\sqrt{n}}\leqslant 1.$$

Corollary 3.3. *Let $C\subset H$ be a nonempty bounded convex closed set of a Hilbert space H and $\theta\in C$. Let $T:C\to C$ be a asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $F(T)\neq\emptyset$. Suppose $\{\alpha_n\},\{\beta_n\},\{\lambda_n\},\{\xi_n\}$ are real number sequences in $(0,1)$. Let $\{x_n\}$ be generated by*

$$\begin{cases} x_1\in C,\\ y_n=(1-\alpha_n)x_n+\alpha_n T^n(\frac{x_n+x_{n+1}}{2}),\\ x_{n+1}=(1-\beta_n)(\lambda_n x_n)+\beta_n y_n. \end{cases} \tag{3.18}$$

Suppose the following conditions are satisfied:

(i) $\lim\limits_{n\to\infty}\beta_n=1,\ \lim\limits_{n\to\infty}\lambda_n=1;$

(ii) $\sum\limits_{n=1}^{\infty}\alpha_n<+\infty,\ \sum\limits_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<+\infty,\ \sum\limits_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<+\infty;$

(iii) $\sum\limits_{n=1}^{\infty}(1-\lambda_n)(1-\beta_n)=+\infty, |\lambda_{n+1}-\beta_n\lambda_n|+\beta_{n+1}\leqslant 1;$

(iv) $\sum\limits_{n=1}^{\infty}\sup\limits_{x\in C}\|T^{n+1}x-T^n x\|<+\infty.$

Then the sequence $\{x_n\}$ converges strongly to a fixed point $x^*\in F(T)$.

Proof. Let $\xi_n=\frac{1}{2}$, then by the proof of Theorem 3.1, we can easily prove the sequence $\{x_n\}$ generated by (3.18) converges strongly to a point $x^*\in F(T)$. □

Corollary 3.4. *Let $C\subset H$ be a nonempty bounded convex closed set of a Hilbert space H and $\theta\in C$. Let $T:C\to C$ be a asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $F(T)\neq\emptyset$. Suppose $\{\alpha_n\},\{\beta_n\},\{\xi_n\}$ are real number sequences in $(0,1)$. Let $\{x_n\}$ be generated by*

$$\begin{cases} x_1\in C,\\ y_n=(1-\alpha_n)x_n+\alpha_n T^n(\xi_n x_n+(1-\xi_n)x_{n+1}),\\ x_{n+1}=(1-\beta_n)u+\beta_n y_n. \end{cases} \tag{3.19}$$

Suppose the following conditions are satisfied:

(i) $\lim\limits_{n\to\infty}\beta_n=1,\ \lim\limits_{n\to\infty}\xi_n=1;$

(ii) $\sum\limits_{n=1}^{\infty}\alpha_n<+\infty,\ \sum\limits_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<+\infty;$

(iii) $\sum\limits_{n=1}^{\infty}\sup\limits_{x\in C}\|T^{n+1}x-T^n x\|<+\infty.$

Then the sequence $\{x_n\}$ converges strongly to a fixed point $x^*\in F(T)$.

Proof. By the proof of Theorem 3.1, we can easily prove the sequence $\{x_n\}$ generated by the (3.19) converges strongly to a point $x^*\in F(T)$. □

Remark 3.5. From our theorem and corollaries we know the algorithm (3.1) is implicit iterative algorithm. When $\xi_n = \frac{1}{2}$, and T is a nonexpansive mapping, Theorem 3.1 can be reduced to the main result of Xu (see [19]). When $\xi_n = 1$, and T is a nonexpansive mapping, Theorem 3.1 can be reduced to the main result of Fan et al. (see [3]).

4. Applications

Lemma 4.1 ([20]). *Let $C \subset H$ be a nonempty convex closed set of a Hilbert space H and $\theta \in C$. Let $U_i : C \to H$ be μ_i-inverse strongly monotone mapping. Let P_C denote the orthogonal projection onto the set C. Let $0 < \gamma_i < \mu_i, i = 1, 2, \dots, t$, where t is a positive integer. Let*

$$S \triangleq P_C(I - \gamma_t U_t)P_C(I - \gamma_{t-1}U_{t-1}) \cdots P_C(I - \gamma_1 U_1).$$

Then S is a attracting mapping and $F(S) = \bigcap_{i=1}^{t} N(U_i)$.

Theorem 4.2. *Let $C \subset H$ be a nonempty bounded convex closed set of a Hilbert space H and $\theta \in C$. Let $U_i : C \to H$ be μ_i-inverse strongly monotone. Suppose $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \{\xi_n\}$ are real number sequences in $(0, 1)$. Let $\{x_n\}$ be generated by*

$$\begin{cases} x_1 \in C, \\ y_n = (1 - \alpha_n)x_n + \alpha_n S(\xi_n x_n + (1 - \xi_n)x_{n+1}), \\ x_{n+1} = (1 - \beta_n)(\lambda_n x_n) + \beta_n y_n. \end{cases}$$

Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \beta_n = 1$, $\lim_{n\to\infty} \lambda_n = 1$, $\lim_{n\to\infty} \xi_n = 1$;

(ii) $\sum_{n=1}^{\infty} \alpha_n < +\infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < +\infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < +\infty$;

(iii) $\sum_{n=1}^{\infty} (1 - \lambda_n)(1 - \beta_n) = +\infty, |\lambda_{n+1} - \beta_n \lambda_n| + \beta_{n+1} \leqslant 1$.

Then the sequence $\{x_n\}$ converges strongly to a common element $\hat{x} \in \bigcap_{i=1}^{t} N(U_i)$.

Proof. From Lemma 4.1 and Theorem 3.1, we can easily prove the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \bigcap_{i=1}^{t} N(U_i)$. □

Theorem 4.3. *Let $C \subset H$ be a nonempty bounded convex closed set of a Hilbert space H and $\theta \in C$. Let $U_i : C \to H$ be μ_i-inverse strongly monotone. Suppose $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \{\xi_n\}$ are real number sequences in $(0, 1)$. Let $\{x_n\}$ be generated by*

$$\begin{cases} x_1 \in C, \\ y_n = (1 - \alpha_n)x_n + \alpha_n S(\xi_n x_n + (1 - \xi_n)x_{n+1}), \\ x_{n+1} = (1 - \beta_n)u + \beta_n y_n. \end{cases}$$

Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \beta_n = 1$, $\lim_{n\to\infty} \lambda_n = 1$, $\lim_{n\to\infty} \xi_n = 1$;

(ii) $\sum_{n=1}^{\infty} \alpha_n < +\infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < +\infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < +\infty$;

(iii) $\sum_{n=1}^{\infty} (1 - \lambda_n)(1 - \beta_n) = +\infty, |\lambda_{n+1} - \beta_n \lambda_n| + \beta_{n+1} \leqslant 1$.

Then the sequence $\{x_n\}$ *converges strongly to a common element* $\hat{x} \in \bigcap_{i=1}^{t} N(U_i)$.

Proof. From Theorem 3.1, Lemma 4.1, and the Theorem 1 of [5], we can easily prove the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \bigcap_{i=1}^{t} N(U_i)$. □

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