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On mixed complex intersection bodies

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Abstract

In 2013, the mixed complex intersection bodies of star bodies was introduced. Following this, in the paper, we establish Aleksandrov-Fenchel and Brunn-Minkowski type inequalities for the mixed complex intersection bodies, which in special case yield some of the recent results.

Keywords: Dual mixed volumes, intersection bodies, mixed intersection bodies, complex intersection bodies, mixed complex intersection bodies.

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1. Introduction

The intersection operator in $\mathbb{R}^n$ and the class of intersection bodies were defined by Lutwak [22].

For a star body K, there is a unique star body $\mathbf{I}K$ which is called the intersection bodies. Whose radial function satisfies, for $u \in S^{n-1}$

$$\rho(\mathbf{I}K, u) = v(K \cap E_u),$$

where S^{n-1} denotes the surface of the unit ball, v denotes the $(n-1)$-dimensional dual volume and E_u denotes the hyperplane through that is orthogonal to u.

The closure of the class of intersection bodies was studied by Goodey et al. [12]. The intersection operator and the class of intersection bodies played a critical role in Gardner [5] and Zhang [25] solution of the famous Busemann-Petty problem in three dimensions and four dimensions, respectively (see also Gardner et al. [11]). Please see Koldobsky's book [19] for the details about the solution of the famous Busemann-Petty problem. During the past 30 years significant advances have been made in our understanding of the intersection operator and the class of intersection bodies by Koldobsky, Campi, Goodey, Gardner, Grinberg, Lutwak, Fallert, Weil, Zhang, Ludwig and others (see, e.g., [1–4, 6–10, 12–18, 21, 23, 26–30]).

For star bodies $K_1, \ldots, K_{n-1}$, there is a unique star body $\mathbf{I}(K_1, \ldots, K_{n-1})$ which is called the mixed intersection bodies, whose radial function satisfies (see [22])

$$\rho(\mathbf{I}(K_1, \ldots, K_{n-1}), u) = \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)$$

for $u \in S^{n-1}$, where $\tilde{v}$ denotes the $(n-1)$-dimensional dual mixed volume.

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The problem of real domain is extended to complex space, which is an important research direction and interest of mathematics. Hence, a very natural question is proposed: can the mixed intersection of real domain space be extended to the complex domain space?

Recently, Koldobsky et al. [20] first introduced the mixed complex intersection bodies of complex star bodies. Let $K_1, \ldots, K_{2n-2}$ be star bodies in $\mathbb{C}^n$. The mixed complex intersection body $\mathbf{I}_C(K_1, \ldots, K_{2n-2})$ is defined by ([24])

$$\rho(\mathbf{I}_C(K_1, \ldots, K_{2n-2}), \xi)^2 = \frac{1}{(2n-2)\pi} \int_{S^{2n-1} \cap H_\xi} \rho(K_1, \omega) \cdots \rho(K_{2n-2}, \omega) dS(\omega), \tag{1.1}$$

where H_ξ denotes a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vector ξ and $dS(\omega)$ is the standard spherical Lebesgue measure on S^{2n-1}. Putting $K_1 = \cdots = K_{2n-2} = K$ in (1.1), it reduces to

$$\rho(\mathbf{I}_C K, \xi)^2 = \frac{1}{(2n-2)\pi} \int_{S^{2n-1} \cap H_\xi} \rho(K, \omega)^{2n-2} dS(\omega),$$

and call $\mathbf{I}_C K$ as the complex intersection bodies of K.

In 2015, Wang et al. [24] established Minkowski inequality and Brunn-Minkowski inequality for the mixed complex intersection bodies, respectively.

Theorem 1.1. *If K and L are star bodies in $\mathbb{C}^n$, then*

$$V(\mathbf{I}_C(K, 2n-3; L))^{2n-2} \leqslant V(\mathbf{I}_C K)^{2n-3} V(\mathbf{I}_C L), \tag{1.2}$$

with equality if and only if K and L are dilates.

Here, $\mathbf{I}_C(K, 2n-3; L)$ denotes the mixed complex intersection body $\mathbf{I}_C(\underbrace{K, \ldots, K}_{2n-3}, L)$, and $V(K)$ for the n-dimensional volume of the body K.

The sum $+_C$ denotes the complex radial sum defined by Koldobsky et al. [20]: The complex radial sum is a complex star body that has radial function

$$\rho(K +_C L, \cdot)^2 = \rho(K, \cdot)^2 + \rho(L, \cdot)^2.$$

Theorem 1.2. *If K and L are star bodies in $\mathbb{C}^n$, then*

$$V(\mathbf{I}_C(K +_C L))^{1/n(n-1)} \leqslant V(\mathbf{I}_C K)^{1/n(n-1)} + V(\mathbf{I}_C L)^{1/n(n-1)}, \tag{1.3}$$

with equality if and only if K and L are dilates.

According to the classical dual theory in convex geometry, on getting (1.2) and (1.3), a natural conjecture is whether the Aleksandrov-Fenchel inequality for the complex mixed intersection body exists? In the paper, we establish Aleksandrov-Fenchel inequality and Brunn-Minkowski type inequality for complex mixed intersection bodies, respectively.

Theorem 1.3. *If $K_1, \ldots, K_{2n-2}$ are star bodies in $\mathbb{C}^n$ and $1 \leqslant r \leqslant 2n-2$, then*

$$V(\mathbf{I}_C(K_1, \ldots, K_{2n-2}))^r \leqslant \prod_{j=1}^{r} V(\mathbf{I}_C(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2})), \tag{1.4}$$

with equality if and only if $K_1, \ldots, K_r$ are dilates of each other.

Here, $\mathbf{I}_C(K_j, r; K_{r+1}, \ldots; K_{2n-2})$ denotes the complex mixed intersection body of star bodies

$$\mathbf{I}_C(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{2n-2}).$$

Taking $r = 2n-2$, $K_1 = \cdots = K_{2n-3} = K$ and $K_{2n-2} = L$ in (1.4), it becomes (1.2).

Theorem 1.3 is just a special case of Theorem 3.4 established in Section 3.

Theorem 1.4. *Let* $C = (M_1, \ldots, M_i)$ *and* $K, L, M_1, \ldots, M_i$ *be star bodies in* $\mathbb{C}^n$. *If* $0 \leqslant i < 2n-2$, *then*

$$\begin{aligned} &V(\mathbf{I}_C(K +_C L, 2n-2-i; C)^{2/n(2n-i-2)} \\ &\quad \leqslant V(\mathbf{I}_C(K, 2n-2-i; C)^{2/n(2n-i-2)} + V(\mathbf{I}_C(L, 2n-2-i; C)^{2/n(2n-i-2)}, \end{aligned} \tag{1.5}$$

with equality if and only if K *and* L *are dilates.*

Here, $\mathbf{I}_C(K, 2n-2-i; C)$ denotes the complex mixed intersection body $\mathbf{I}_C(\underbrace{K, \ldots, K}_{2n-2-i}, M_1, \ldots, M_i)$.

Let B denote a ball in $\mathbb{C}^n$. Taking for $C = (M_1, \ldots, M_i) = (B, \ldots, B)$ and $i = 0$ in (1.5), it becomes (1.3). Theorem 1.4 is just a special case of Theorem 4.2 established in Section 4.

2. Notations and preliminaries

2.1. Complex star bodies

It is well known that origin symmetric convex bodies in $\mathbb{C}^n$ are the unit balls of norms on $\mathbb{C}^n$. We denote by $\|\cdot\|_K$ the norm corresponding to the body K:

$$K = \{z \in \mathbb{C}^n : \|z\|_K \leqslant 1\}.$$

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard mapping

$$\xi = (\xi_1, \ldots, \xi_n) = (\xi_{11} + i\xi_{12}, \ldots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}).$$

Since norms on $\mathbb{C}^n$ satisfy the equality

$$\|\lambda z\| = |\lambda|\|z\|, \quad \forall z \in \mathbb{C}^n, \lambda \in \mathbb{C},$$

origin symmetric complex convex bodies correspond to those origin symmetric convex bodies K in $\mathbb{R}^{2n}$ that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each $\theta \in [0, 2\pi]$ and each $\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

$$\|\xi\|_K = \|R_\theta(\xi_{11}, \xi_{12}), \ldots, R_\theta(\xi_{n1}, \xi_{n2})\|_K, \tag{2.1}$$

where R_θ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle θ with respect to the origin. The Minkowski functional of complex star body K, $\|x\|_K$, defined by $\|x\|_K = \min\{\lambda \geqslant 0 : x \in \lambda K\}, \forall x \in \mathbb{R}^{2n}$ is a continuous function.

We shall say that K is a complex convex body in $\mathbb{R}^{2n}$ if K is a convex body and satisfies equation (2.1). If the Minkowski functional of a star body K in $\mathbb{R}^{2n}$ is R_θ-invariant (i.e., satisfies equations (2.1)), we say that K is a complex star body in $\mathbb{R}^{2n}$.

2.2. Dual mixed volumes

The unit ball B in $\mathbb{C}^n$ is given by

$$B = \{\xi \in \mathbb{C}^n : \sum_{i=1}^{n} (\xi_{i1}^2 + \xi_{i2}^2) \leqslant 1\}.$$

Its unit sphere can be denoted by S^{2n-1}. The volume of B is denoted by ω_{2n}.

A compact set $K \subset \mathbb{C}^n$ is called a star body if its radial function $\rho(K, \cdot)$ defined by

$$\rho(K, \xi) = \max\{\lambda : \lambda\xi \in K, \}, \quad \xi \in S^{2n-1}$$

is positive and continuous.

Let $K_1,\dots,K_{2n}$ be star bodies in $\mathbb{C}^n$, the dual mixed volume $\tilde{V}(K_1,\dots,K_{2n})$ has the following integral representation [25]:

$$\tilde{V}(K_1,\dots,K_{2n})=\frac{1}{2n}\int_{S^{2n-1}}\rho(K_1,u)\cdots\rho(K_{2n},u)dS(u), \tag{2.2}$$

where $dS(u)$ is the standard spherical Lebesgue measure on S^{2n-1}.

We write $\tilde{V}(K_1,\dots,K_{2n-2};L,2)$ for $\tilde{V}(K_1,\dots,K_{2n-2},L,L)$, where the $K_i(i=1,\dots,2n-2)$ appear once and L appears twice. For $i\geqslant 0, j\geqslant 0$ and $i+j\leqslant 2n$, we write $\tilde{W}_i(K,2n-i-j;L,j)$ for the dual mixed volume $\tilde{V}(K,\dots,K,B,\dots,B,L,\dots,L)$, where K appears $(2n-i-j)$ times, B appears i times, and L appears j times. The dual mixed volume $\tilde{W}_i(K,2n-i-j;K,j)$ will be written as $\tilde{W}_i(K)$. Moreover, $\tilde{V}(\underbrace{K,\dots,K}_{2n-i},M_1,\dots,M_i)$ is written as $\tilde{V}_i(K,C)$ and where $C=(M_1,\dots,M_i)$ and $\tilde{V}(\underbrace{K,\dots,K}_{2n-i-2},D_1,\dots,D_i,L,L)$ is written as $\tilde{V}(K,2n-2-i;D;L,2)$ and where $D=(D_1,\dots,D_i)$.

2.3. Complex intersection bodies

For $\xi\in\mathbb{C}^n$, $\xi=1$, denote by

$$H_\xi=\{z\in\mathbb{C}^n:(z,\xi)=\sum_{k=1}^{n}z_k\bar{\xi}_k=0\},$$

the complex hyperplane through the origin, perpendicular to ξ. Under the standard mapping from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$, the hyperplane H_ξ turns into a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi=(\xi_{11},\xi_{12},\dots,\xi_{n1},\xi_{n2}) \text{ and } \xi^\perp=(-\xi_{12},\xi_{11},\dots,-\xi_{n2},\xi_{n1}).$$

A direct conclusion of (1.1) is the following:

$$\mathbf{I}_C B=\left(\frac{\omega_{2n-2}}{\pi}\right)^{1/2}B. \tag{2.3}$$

If $K_1=\cdots=K_{2n-i-2}=K$ and $K_{2n-i-1}=\cdots=K_{2n-2}=L$, the mixed complex intersection body $\mathbf{I}_C(K_1,\dots,K_{2n-2})$ is written as $\mathbf{I}_C(K,2n-2-i;L,i)$. If $L=B$, $\mathbf{I}_C(K,2n-2-i;B,i)$ is written as $\mathbf{I}_C(K)_i$. We simply write $\mathbf{I}_C K$ rather than $\mathbf{I}_C(K)_0$ and is called complex intersection body of K, which was defined by Koldobsky et al. [20]. Moreover, $\mathbf{I}_C(K,M_1,\dots,M_j)$, where K appears $(2n-2-j)$ times, which is written as $\mathbf{I}_C(K,2n-2-j;M)$ and $M=(M_1,\dots,M_j)$.

3. Aleksandrov-Fenchel inequality for complex mixed intersection bodies

In order to establish the Aleksandrov-Fenchel inequality for complex mixed intersection bodies, we need Lemmas 3.1-3.3.

Lemma 3.1 (Aleksandrov-Fenchel inequality)**.** *If* $K_1,\dots,K_{2n}$ *are star bodies in* $\mathbb{C}^n$ *and* $1\leqslant r\leqslant 2n$, *then*

$$\tilde{V}(K_1,\dots,K_{2n})^r\leqslant\prod_{j=1}^{r}\tilde{V}(\underbrace{K_j,\dots,K_j}_{r},K_{r+1},K_{r+2},\dots,K_{2n}). \tag{3.1}$$

with equality if and only if $K_1,\dots,K_r$ *are all dilations of each other.*

A special case of (3.1) is the following useful result.

Lemma 3.2. *Let* K *and* L *be star bodies in* $\mathbb{C}^n$. *If* $0\leqslant i<2n-2$ *and* $1\leqslant j\leqslant 2n-i$, *then*

$$\tilde{W}_i(K,2n-2-j;L,j)^{2n-i}\leqslant\tilde{W}_i(K)^{2n-i-j}\tilde{W}_i(L)^j, \tag{3.2}$$

with equality if and only if K *and* L *are dilates.*

Inequality (3.2) is also proved by Wang et al. in [24].

Lemma 3.3 ([24])**.** *If* $K_1, \ldots, K_{2n-2}, L_1, \ldots, L_{2n-2}$ *are star bodies in* $\mathbb{C}^n$*, then*

$$\tilde{V}(K_1, \ldots, K_{2n-2}; \mathbf{I}_C(L_1, \ldots, L_{2n-2}), 2) = \tilde{V}(L_1, \ldots, L_{2n-2}; \mathbf{I}_C(K_1, \ldots, K_{2n-2}), 2). \tag{3.3}$$

Noting that if $K, L_1, \ldots, L_{2n-2}$ are star bodies in $\mathbb{C}^n$, and $0 \leqslant i < 2n-2$, then

$$\tilde{V}(K, 2n-i-2; B, i; \mathbf{I}_C(L_1, \ldots, L_{2n-2}), 2) = \tilde{V}(L_1, \ldots, L_{2n-2}; \mathbf{I}_C(K)_i, 2). \tag{3.4}$$

From (2.3), we have that if $K_1, \ldots, K_{2n-2}$ are star bodies in $\mathbb{C}^n$, then

$$\tilde{W}_{2n-2}(\mathbf{I}_C(K_1, \ldots, K_{2n-2})) = \frac{\omega_{2n-2}}{\pi}\tilde{V}(K_1, \ldots, K_{2n-2}; B, 2). \tag{3.5}$$

The Aleksandrov-Fenchel inequality for complex mixed intersection bodies stated in the introduction will be established: If $K_1, \ldots, K_{2n-2}$ are star bodies in $\mathbb{C}^n$ and $1 \leqslant r \leqslant 2n-2$, then

$$V(\mathbf{I}_C(K_1, \ldots, K_{2n-2}))^r \leqslant \prod_{j=1}^{r} V(\mathbf{I}_C(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2})),$$

with equality if and only if $K_1, \ldots, K_r$ are dilates of each other.

In fact a general version of the Aleksandrov-Fenchel inequality for complex mixed intersection bodies holds as the following.

Theorem 3.4. *If* $K_1, \ldots, K_{2n-2}$ *are star bodies in* $\mathbb{C}^n$, $0 \leqslant i \leqslant 2n-2$ *and* $1 \leqslant r \leqslant 2n-2$*, then*

$$\tilde{W}_i(\mathbf{I}_C(K_1, \ldots, K_{2n-2}))^r \leqslant \prod_{j=1}^{r} \tilde{W}_i(\mathbf{I}_C(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2})), \tag{3.6}$$

with equality if and only if $K_1, \ldots, K_r$ *are dilates of each other.*

Proof. First consider the case $i = 2n-2$. Taking for $K_{2n-1} = K_{2n} = B$ in (3.1), we have

$$\tilde{V}(K_1, \ldots, K_{2n-2}; B, 2)^r \leqslant \prod_{j=1}^{r} \tilde{V}(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2}; B, 2),$$

with equality if and only if $K_1, \ldots, K_r$ are dilates of each other.

From (3.5), the above inequality reduces to

$$\tilde{W}_{2n-2}(\mathbf{I}_C(K_1, \ldots, K_{2n-2}))^r \leqslant \prod_{j=1}^{r} \tilde{W}_{2n-2}(\mathbf{I}_C(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2})),$$

with equality if and only if $K_1, \ldots, K_r$ are dilates of each other.

This shows that Theorem 3.4 is correct for the case $i = 2n-2$.

In the following we assume $i < 2n-2$, from (3.3), obtain that for star body M in $\mathbb{C}^n$

$$\tilde{V}(M, 2n-2-i; \mathbf{I}_C(K_1, \ldots, K_{2n-2}), 2) = \tilde{V}(K_1, \ldots, K_{2n-2}; \mathbf{I}_C(M)_i, 2). \tag{3.7}$$

From Lemma 3.1, it follows that

$$[\tilde{V}(K_1, \ldots, K_{2n-1}; B, i; \mathbf{I}_C(M)_i, 2)]^r \leqslant \prod_{j=1}^{r} \tilde{V}(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2}; \mathbf{I}_C(M)_i, 2), \tag{3.8}$$

with equality if and only if $K_1, \ldots, K_r$ are all dilations of each other.

Moreover, from (3.4), we have

$$\tilde{V}(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2}; \mathbf{I}_C(M)_i, 2) = \tilde{V}(M, 2n-i-2; B, i; \mathbf{I}_C(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2}), 2). \quad (3.9)$$

From (3.9) and in view of Lemma 3.2, we obtain that

$$\begin{aligned}&[\tilde{V}(M, 2n-i-2; B, i; \mathbf{I}_C(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2}), 2)]^{2n-i}\\ &\quad\leqslant \tilde{W}_i(M)^{2n-2-i}\tilde{W}_i(\mathbf{I}_C(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2}))^2,\end{aligned} \quad (3.10)$$

with equality if and only if M is a dilation of $\mathbf{I}_C(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2})$.

Combine this with (3.7), (3.8), (3.9), and (3.10), it follows

$$\begin{aligned}&\tilde{V}(M, 2n-i-2; B, i; \mathbf{I}_C(K_1, \ldots, K_{2n-2}), 2)^{r(2n-i)}\\ &\quad\leqslant \tilde{W}_i(M)^{r(2n-i-2)}\prod_{j=1}^{r}\tilde{W}_i(\mathbf{I}_C(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2}))^2.\end{aligned} \quad (3.11)$$

Now taking $M = \mathbf{I}_C(K_1, \ldots, K_{2n-2})$ to (3.11), it changes to

$$\tilde{W}_i(\mathbf{I}_C(K_1, \ldots, K_{2n-2}))^r \leqslant \prod_{j=1}^{r}\tilde{W}_i(\mathbf{I}_C(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{2n-2})). \quad (3.12)$$

From equality conditions of (3.8) and (3.10), it follows that the equality in (3.12) holds if and only if $K_1, \ldots, K_r$ are all dilations of each other. The proof is completed. □

Remark 3.5. Let us point out the complete analogy with the real case: replacing formally $\mathbb{C}$ by $\mathbb{R}^n$ in Theorem 3.4 gives the following result (see [28]):

Let $K_1, \ldots, K_n$ be star bodies in $\mathbb{R}^n$. If $0 \leqslant i \leqslant n$ and $1 \leqslant r \leqslant n$, then

$$\tilde{W}_i(\mathbf{I}(K_1, \ldots, K_{n-1}))^r \leqslant \prod_{j=1}^{r}\tilde{W}_i(\mathbf{I}(K_j, r; K_{r+1}, K_{r+2}, \ldots, K_{n-1})),$$

with equality if and only if $K_1, \ldots, K_r$ are all dilations of each other.

Taking for $r = 2n-2$, $K_1 = \cdots = K_{2n-j-2} = K$ and $K_{2n-j-1} = \cdots = K_{2n-2} = L$ in (3.6), it becomes to the following result.

Corollary 3.6. *If K, L are star bodies in $\mathbb{C}^n$, $0 \leqslant i \leqslant 2n-2$ and $1 \leqslant j \leqslant 2n-2$, then*

$$\tilde{W}_i(\mathbf{I}_C(K, 2n-2-j; L, j))^{2n-2} \leqslant \tilde{W}_i(\mathbf{I}_C K)^{2n-j-2}\tilde{W}_i(\mathbf{I}_C L)^j,$$

with equality if and only if K and L are dilates.

This is just a new result established by Wang et al. [24].

On the other hand, taking for $r = 2n-2$, $K_1 = \cdots = K_{2n-j-2} = K$, and $K_{2n-j-1} = \cdots = K_{2n-2} = B$ in (3.6), it becomes to the following result.

Corollary 3.7. *If K is star body in $\mathbb{C}^n$, $0 \leqslant i \leqslant 2n-2$, and $1 \leqslant j \leqslant 2n-2$, then*

$$\tilde{W}_i(\mathbf{I}_C(K)_j)^{2n-2} \leqslant \omega_{2n}^j\left(\frac{\omega_{2n-2}}{\pi}\right)^{j(2n-i)/2}\tilde{W}_i(\mathbf{I}_C K)^{2n-j-2}, \quad (3.13)$$

with equality if and only if K is a ball.

Let us point out the complete analogy with the real case: replacing formally $\mathbb{C}^n$ by $\mathbb{R}^n$ in (3.13) gives the following result (see [28]):

If K is star body in $\mathbb{R}^n$, and $0 \leqslant i < n$ and $0 < j < n-1$, then

$$\tilde{W}_i(\mathbf{I}_j K)^{n-1} \leqslant \omega_{n-1}^{j(n-i)}\omega_n^j\tilde{W}_i(\mathbf{I}K)^{n-j-1},$$

with equality if and only if K is a ball.

Here $\mathbf{I}_j K$ denotes the mixed projection body $\mathbf{I}(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B, \ldots, B}_{i})$ in $\mathbb{R}^n$.

4. Brunn-Minkowski inequality for complex mixed intersection bodies

In order to prove Theorem 4.2, in addition to using Lemma 3.3, we need the following lemma.

Lemma 4.1. *Let* $K, L, N, M_1, \ldots, M_i$ *be star bodies in* $\mathbb{C}^n$ *and* $M = (M_1, \ldots, M_i)$. *If* $0 \leqslant i \leqslant 2n-2$, *then*

$$\begin{aligned}&[\tilde{V}(K +_C L, 2n-i-2; M; N, 2)]^{2/(2n-i-2)} \\ &\quad \leqslant [\tilde{V}(K, 2n-i-2; M; N, 2)]^{2/(2n-i-2)} + [\tilde{V}(L, 2n-i-2; M; N, 2)]^{2/(2n-i-2)},\end{aligned} \tag{4.1}$$

with equality if and only if K *and* L *are dilates.*

Proof. By (2.2) and Minkowski integral inequality, we obtain

$$\begin{aligned}&[\tilde{V}(K +_C L, 2n-i-2; M; N, 2)]^{2/(2n-i-2)} \\ &= \left(\frac{1}{2n}\int_{S^{2n-1}} (\rho(K,\xi)^2 + \rho(L,\xi)^2)^{(2n-i-2)/2} \rho(N,\xi)^2 \prod_{k=1}^{i} \rho(M_k,\xi) d\xi\right)^{2/(2n-i-2)} \\ &\leqslant \left(\frac{1}{2n}\int_{S^{2n-1}} \rho(K,\xi)^{2n-i-2} \rho(N,\xi)^2 \prod_{k=1}^{i} \rho(M_k,\xi) d\xi\right)^{2/(2n-i-2)} \\ &\quad + \left(\frac{1}{2n}\int_{S^{2n-1}} \rho(L,\xi)^{2n-i-2} \rho(N,\xi)^2 \prod_{k=1}^{i} \rho(M_k,\xi) d\xi\right)^{2/(2n-i-2)} \\ &= [\tilde{V}(K, 2n-i-2; M; N, 2)]^{2/(2n-i-2)} + [\tilde{V}(L, 2n-i-2; M; N, 2)]^{2/(2n-i-2)}.\end{aligned}$$

From the equality condition of Minkowski integral inequality, it follows the equality in (4.1) holds if and only if K and L are dilates. □

Theorem 4.2. *Let* $0 \leqslant i \leqslant 2n-2$, *while* $0 \leqslant j \leqslant 2n-3$, *and* $K, L, M_1, \ldots, M_i, M'_1, \ldots, M'_j$ *be star bodies in* $\mathbb{C}^n$. *If* $C = (M_1, \ldots, M_i)$ *and* $D = (M'_1, \ldots, M'_j)$, *then*

$$\begin{aligned}&\tilde{V}_i(\mathbf{I}_C(K +_C L, 2n-2-j; D), C)^{4/(2n-i)(2n-j-2)} \\ &\quad \leqslant \tilde{V}_i(\mathbf{I}_C(K, 2n-2-j; D), C)^{4/(2n-i)(2n-j-2)} + \tilde{V}_i(\mathbf{I}_C(L, 2n-2-j; D), C)^{4/(2n-i)(2n-j-2)},\end{aligned} \tag{4.2}$$

with equality if and only if K *and* L *are dilates.*

Proof. Suppose Q be star body in $\mathbb{C}^n$, from the identity (3.3),

$$\tilde{V}(Q, 2n-2-i; C; \mathbf{I}_C(K +_C L, D)_j, 2) = \tilde{V}(K +_C L, 2n-j-2; D; \mathbf{I}_C(Q, C)_i, 2). \tag{4.3}$$

Inequality (4.1) in Lemma 4.1 shows that

$$\begin{aligned}&\tilde{V}(K +_C L, 2n-j-2; D; \mathbf{I}_C(Q, C)_i, 2)^{2/(2n-j-2)} \\ &\quad \leqslant \tilde{V}(K, 2n-j-2; D; \mathbf{I}_C(Q, C)_i, 2)^{2/(2n-j-2)} + \tilde{V}(L, 2n-j-2; D; \mathbf{I}_C(Q, C)_i, 2)^{2/(2n-j-2)},\end{aligned} \tag{4.4}$$

with equality if and only if K and L are dilates.

But from (3.3), it follows

$$\tilde{V}(K, 2n-j-2; D; \mathbf{I}_C(Q, 2n-i-2; C), 2)^{2/(2n-j-2)} = \tilde{V}(Q, 2n-i-2; C; \mathbf{I}_C(K, 2n-j-2; D), 2)^{2/(2n-j-2)},$$

and hence, inequality (3.1) gives

$$\begin{aligned}&\tilde{V}(K, 2n-j-2; D; \mathbf{I}_C(Q, 2n-i-2; C), 2)^{2/(2n-j-2)} \\ &\quad \leqslant \tilde{V}_i(Q, C)^{2(2n-i-2)/(2n-i)(2n-j-2)} \tilde{V}_i(\mathbf{I}_C(K, 2n-j-2; D), C)^{4/(2n-i)(2n-j-2)},\end{aligned} \tag{4.5}$$

with equality if and only if K, D, and $\mathbf{I}_C(Q, 2n-i-2; C)$ are dilates.

In exactly the same way, it can be seen that

$$\begin{aligned}&\tilde{V}(L, 2n-j-2; D; \mathbf{I}_C(Q, 2n-i-2; C), 2)^{2/(2n-j-2)}\\&\quad\leqslant \tilde{V}_i(Q, C)^{2(2n-i-2)/(2n-i)(2n-j-2)}\tilde{V}_i(\mathbf{I}_C(L, 2n-j-2; D), C)^{4/(2n-i)(2n-j-2)},\end{aligned} \tag{4.6}$$

with equality if and only if L, D, and $\mathbf{I}_C(Q, 2n-i-2; C)$ are dilates.

Combine (4.3), (4.4), (4.5), and (4.6), and the result is

$$\begin{aligned}&\tilde{V}(Q, 2n-i-2; C; \mathbf{I}_C(K+_C L, 2n-2-j; D), 2)^{2/(2n-j-2)} \leqslant \tilde{V}_i(Q, C)^{2(2n-i-2)/(2n-i)(2n-j-2)}\\&\times\left(\tilde{V}_i(\mathbf{I}_C(K, 2n-2-j; D), C)^{4/(2n-i)(2n-j-2)} + \tilde{V}_i(\mathbf{I}_C(L, 2n-2-j; D), C)^{4/(2n-i)(2n-j-2)}\right).\end{aligned} \tag{4.7}$$

Take $\mathbf{I}_C(K+_C L, 2n-2-j; D)$ for Q, and noting that

$$\tilde{V}(Q, 2n-i-2; C; \mathbf{I}_C(K+_C L, 2n-2-j; D), 2) = \tilde{V}_i(\mathbf{I}_C(K+_C L, 2n-2-j; D), C)$$

and

$$\tilde{V}_i(Q, C) = \tilde{V}_i(\mathbf{I}_C(K+_C L, 2n-2-j; D), C).$$

This shows that the last inequality is the inequality of the theorem.

From the conditions of equality of inequalities (4.4), (4.5), and (4.6), it follows the equality in (4.7) holds if and only if K and L are dilates. □

Remark 4.3. The most interesting case of the inequality of Theorem 4.2 is the special case where $D = (B, \ldots, B)$. In this case (4.2) reads

$$\begin{aligned}&\tilde{V}_i(\mathbf{I}_C(K+_C L)_j, C)^{4/(2n-i)(2n-j-2)}\\&\quad\leqslant \tilde{V}_i(\mathbf{I}_C(K)_j, C)^{4/(2n-i)(2n-j-2)} + \tilde{V}_i(\mathbf{I}_C(L)_j, C)^{4/(2n-i)(2n-j-2)},\end{aligned} \tag{4.8}$$

with equality if and only if K and L are dilates.

Taking for $C = (B, \ldots, B)$ in (4.8), it reduces to the following interesting result.

$$\tilde{W}_i(\mathbf{I}_C(K+_C L)_j)^{4/(2n-i)(2n-j-2)} \leqslant \tilde{W}_i(\mathbf{I}_C(K)_j)^{4/(2n-i)(2n-j-2)} + \tilde{W}_i(\mathbf{I}_C(L)_j)^{4/(2n-i)(2n-j-2)}, \tag{4.9}$$

with equality if and only if K and L are dilates.

This is just a new inequality established by Wang et al. [24].

Let us point out the complete analogy with the real case: replacing formally $\mathbb{C}^n$ by $\mathbb{R}^n$ in (4.9), it gives the following result: If K and L are star bodies in $\mathbb{R}^n$ and $0 \leqslant i < n$, $0 \leqslant j < n-2$, then

$$\tilde{W}_i(\mathbf{I}_j(K\tilde{+}L))^{1/(n-i)(n-j-1)} \leqslant \tilde{W}_i(\mathbf{I}_j K)^{1/(n-i)(n-j-1)} + \tilde{W}_i(\mathbf{I}_j L)^{1/(n-i)(n-j-1)},$$

with equality if and only if K and L are dilates.

Here $K\tilde{+}L$ denotes the usual radial sum of star bodies K and L in $\mathbb{R}^n$.

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