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Existence of solutions for Schrödinger-Poisson system with asymptotically periodic terms

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Abstract

In this paper, we consider the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $V, K \in L^\infty(\mathbb{R}^3)$ and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is continuous. We prove that the problem has a nontrivial solution under asymptotically periodic case of V, K, and f at infinity. Moreover, the nonlinear term f does not satisfy any monotone condition.

Keywords: Schrödinger-Poisson system, asymptotically periodic, variational method.

2010 MSC: 34C25, 58E50.

1. Introduction and main result

For past decades, much attention has been paid to the nonlinear Schrödinger-Poisson system

$$\begin{cases} i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + \phi(x)\Psi - |\Psi|^{q-1}\Psi, & x \in \mathbb{R}^3, \ t \in \mathbb{R}, \\ -\Delta\phi = |\Psi|^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\hbar$ is the Planck constant. System (1.1) derived from quantum mechanics. For this system, the existence of stationary wave solutions is often sought, that is, the following form of solutions

$$\Psi(x,t) = e^{it}u(x), x \in \mathbb{R}^3, \ t \in R.$$

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Therefore, the existence of the standing wave solutions of the system (1.1) is equivalent to finding the solutions of the following system

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \hbar u + \phi u = |u|^{q-1}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

Let $m = \frac{1}{2}$ and $\hbar = 1$, system (1.2) becomes the following system

$$\begin{cases} -\Delta u + u + \phi u = |u|^{q-1}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

There was a series of work to discuss the existence, non existence, radially symmetric solutions, non-radially symmetry solutions, ground states, semiclassical states and sign-changing solutions to Schrödinger-Poisson system (1.3) by using the variational method [1, 2, 5–7, 9–13, 17–19, 21–24, 28, 29, 32, 34, 37, 38, 40–42, 44–46].

In case $3 < q < 5$, Coclite [10] considered the nontrivial radially symmetric solutions for system (1.3). In [11], when $3 \leqslant q < 5$, D'Aprile and Mugnai obtained similar results. By using Pohozaev's identity, in [12], D'Aprile and Mugnai considered the non existence of nontrivial solution to system (1.3) in case $q \leqslant 1$ or $q \geqslant 5$.

In [32], Ruiz studied the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda\phi u = u^p, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\lambda > 0$ is parameter and $1 < p < 5$. Using the mountain pass theorem and Ekeland variational principle, Ruiz proved that system (1.4) has at least two (one) positive radial solutions when $1 < p < 2$ ($p = 2$) and $\lambda > 0$ sufficiently small and system (1.4) has no nontrival solution when $1 < p \leqslant 2$ and $\lambda \geqslant \frac{1}{4}$. Moreover, by applying the method of finding the minimal sequence on a manifold associated with the Nehari manifold and the Pohozaev's identity, Ruiz proved that the system (1.4) has a positive radial solution in case $2 < p < 5$.

In [5], Ambrosetti and Ruiz obtained the existence of infinitely many radially symmetric solutions to system (1.4) when $2 < p \leqslant 5$.

Using Lyapunov-Schmidt reduction method, D'Aprile and Wei [13] obtained the bound state solution for system (1.3), and the concentration of the solution is also studied. With regard to other relevant results, please see [23, 24, 40].

In [2], Alves et al. studied Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.5}$$

where V is bounded, local Hölder continuous, and satisfies:

(1) $V(x) \geqslant \alpha > 0, x \in \mathbb{R}^3$;
(2) $V(x) = V(x+y)$, $\forall x \in \mathbb{R}^3$, $\forall y \in \mathbb{Z}^3$;
(3) $\lim_{|x|\to\infty} |V(x) - V_0(x)| = 0$;
(4) $V(x) \leqslant V_0(x)$, $\forall x \in \mathbb{R}^3$, and there exists $\Omega \subset \mathbb{R}^3$ such that

$$V(x) \leqslant V_0(x), \forall x \in \Omega$$

where V_0 satisfies (2).

Alves studied the ground sates solutions to system (1.5) in case the periodic condition under (1)-(2) and in case the asymptotically periodic condition under (1), (3), and (4), respectively.

In case $p \in (3, 5)$, Cerami and Vaira [9] studied the existence of positive solutions for the following non-autonomous Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + K(x)\phi(x)u = a(x)|u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

where a, K are nonnegative functions such that $\lim_{|x|\to\infty} a(x) = a_\infty > 0,\ \lim_{|x|\to\infty} K(x) = 0$.

In [45], Zhang et al. studied existence of positive ground state solutions for the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.7}$$

where V, K, and f are asymptotically periodic at infinity. Moreover, the nonlinear term f satisfies the monotone condition: $\forall t \neq 0$, $s \mapsto \frac{f(x,st)t}{s^3}$ is nondecreasing on $(0, \infty)$.

On the other hand, when $K = 0$, the Schrödinger-Poisson equation (1.7) becomes the standard Schrödinger equation (replace $\mathbb{R}^3$ with $\mathbb{R}^N$)

$$-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N. \tag{1.8}$$

The Schrödinger equation (1.8) has been widely investigated by many authors in the last decades, see [3, 8, 14–16, 20, 26, 30, 31] and reference thein.

Especially, in [14], Marchi studied the nontrivial solutions and ground state solutions for problem (1.8) in which V, f satisfies the asymptotic periodic condition. In the context about asymptotic periodic, we refer the reader to [25, 27, 35, 36].

Motivated by above results, especially by [2, 14, 45], in this paper we study nontrivial solutions and ground state solutions to system (1.7) under asymptotically periodic case of V, K, and f at infinity.

Let $\mathfrak{I}$ be the functions $h \in L^\infty(\mathbb{R}^3, \mathbb{R})$ such that, for every $\varepsilon > 0$, the set $\{x \in \mathbb{R}^3 : |h(x)| \geqslant \varepsilon\}$ has finite Lebesgue measure. To state our main result, we assume that:

(H_1) $V, K \in L^\infty(\mathbb{R}^3), \inf_{x\in\mathbb{R}^3} V(x) > 0, \inf_{x\in\mathbb{R}^3} K(x) > 0$;
(H_2) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}),\ |f(x, u)| \leqslant C(1 + |u|^p), 3 < p < 5$;
(H_3) $f(x, u) = o(u)$ $u \to 0$ uniformly in $x \in \mathbb{R}^3$;
(H_4) $f(x, u)u - 4F(x, u) \geqslant 0$ for all $(x, u) \in (\mathbb{R}^3, \mathbb{R})$;
(H_5) $\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^4} = +\infty$ uniformly in $x \in \mathbb{R}^3$;
(H_6) there exist $V_0, K_0 \in L^\infty(\mathbb{R}^3), f_0 \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ satisfies:
  (i) V_0, K_0, and f_0 are $1-$periodic in $x_i, 1 \leqslant i \leqslant 3$;
  (ii) $V - V_0, K - K_0 \in \mathfrak{I}, |f(x, u) - f_0(x, u)| \leqslant |h(x)|(|u| + |u|^p), x \in \mathbb{R}^3, h \in \mathfrak{I}$;
  (iii) $V \leqslant V_0, K \leqslant K_0, F(x, t) \geqslant F_0(x, t) = \int_0^t f_0(x, s)ds$ for all $(x, t) \in (\mathbb{R}^3, \mathbb{R})$;
  (iv) $\forall u \neq 0, s \mapsto \frac{f_0(x,su)}{s^3}$ is nondecreasing on $(-\infty, 0)$ and $(0, \infty)$.

Our main results of this paper is as follows.

Theorem 1.1. *Assume* (H_1)-(H_6) *are satisfied, then system* (1.7) *has at least one solution.*

Theorem 1.2. *Suppose that* $V(x)$, $K(x)$, *and* $f(x, t)$ *are* 1*-periodic in* $x_i, 1 \leqslant i \leqslant 3$, *and* $V(x) \geqslant a_0 > 0$ *for all* $x \in \mathbb{R}^3$. *If* f *satisfies* (H_2), (H_3), (H_5), *and*

(H_4)* $f(x, u)u - 4F(x, u) > 0$ *for all* $u \neq 0$,

then system (1.7) *has a ground-state solution.*

Remark 1.3.

(1) In this paper, the condition (H_6) means asymptotically periodic case of V, K, and f at infinity. This condition was introduced by Lins and Silva [27] in the study of a Schrödinger equation.

(2) In our paper, f does not satisfy any monotone condition, that is $\frac{f(x,t)}{t}$ is oscillatory, and therefore the method of Nehari manifold [39] used in [45] is not applicable.

(3) In Theorem 1.1, in case of (H_4) being replaced by

$$f(x,u)u - 4F(x,u) \geqslant -\sigma u^2 \text{ uniformly in } x \in \mathbb{R}^3,$$

where $0 < \sigma < \inf_{\mathbb{R}^3} V$, then the result will still hold.

2. Notation and preliminaries

The scalar product and norm in Sobolev space $H^1(\mathbb{R}^3)$ is defined by

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)dx, \quad \|u\|^2 = \langle u, u\rangle.$$

Set

$$\|u\|_0^2 = \int_{\mathbb{R}^3} (|\nabla u| + V_0(x)u^2)dx,$$

$\|u\|_0$ is an equivalent norm in $H^1(\mathbb{R}^3)$ since condition (H_1).

$D^{1,2}(\mathbb{R}^3)$ is the Sobolev space endowed with the scalar product and norm

$$\langle u, v\rangle_{D^{1,2}} = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v dx, \quad \|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

Since $K \in L^\infty(\mathbb{R}^3), \inf_{\mathbb{R}^3} K > 0$, $\forall u \in H^1(\mathbb{R}^3)$, by Lax-Milgram theorem, there exists unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi = K(x)u^2.$$

Functional ϕ_u satisfies the following properties.

Lemma 2.1 ([9, 11, 32, 45, 46])**.** $\forall u \in H^1(\mathbb{R}^3)$,

(i) *there exists* $C > 0$ *such that* $\|\phi_u\|_{D^{1,2}} \leqslant C\|u\|^2$ *and*

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx \leqslant \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \leqslant C\|u\|^4, \ \forall u \in H^1(\mathbb{R}^3);$$

(ii) $\phi_u \geqslant 0, \forall u \in H^1(\mathbb{R}^3)$;

(iii) $\phi_{tu} = t^2\phi_u, \forall\, t > 0, \forall u \in H^1(\mathbb{R}^3)$;

(iv) *If* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$.

Lemma 2.2. *Suppose that* f *satisfies* (H_2) *and* (H_3). *Then, for any given* $\varepsilon > 0$ *there exist* C_ε *such that*

$$|f(x,t)| \leqslant \varepsilon|t| + C_\varepsilon|t|^p, \ |F(x,t)| \leqslant \varepsilon|t|^2 + C_\varepsilon|t|^{p+1} \textit{ for all } (x,t) \in (\mathbb{R}^3, \mathbb{R}).$$

The energy functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ corresponding to system (1.7) is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx.$$

In fact,

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx.$$

In view Lemma of 2.2, the functional I is well defined. Furthermore, under our condition, $I \in C^1(H^1(\mathbb{R}^3))$ and $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of system (1.7) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of I and $\phi = \phi_u$.

$\forall u \in H^1(\mathbb{R}^3)$, let $\tilde{\phi}_u \in D^{1,2}(\mathbb{R}^3)$ is unique solution of the following equation

$$-\Delta\phi = K_0(x)u^2.$$

Then $I_0(u) = \frac{1}{2}\|u\|_0^2 + \frac{1}{4}\int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_u u^2 dx - \int_{\mathbb{R}^3} F_0(x,u)dx$ is the energy functional corresponding to the following system

$$\begin{cases} -\Delta u + V_0(x)u + K_0(x)\phi u = f_0(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K_0(x)u^2, & x \in \mathbb{R}^3. \end{cases}$$

Lemma 2.3 ([45])**.** *If (i) of* (H_6) *holds, then*

$$G(u(\cdot + y)) = G(u), \forall y \in \mathbb{Z}^3, u \in H^1(\mathbb{R}^3),$$

where $G(u) = \int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_u u^2 dx$.

Let $u_n \subset H^1(\mathbb{R}^3)$, we said u_n is a Cerami sequence for the functional I at level $c \in \mathbb{R}$ if

$$I(u_n) \to c, (1 + \|u_n\|)I'(u_n) \to 0, n \to \infty.$$

The following result is a version of the classical mountain pass theorem [4, 43]. For the proof, please see [33].

Theorem 2.4. *Let* E *be a real Banach space. Assume* $I \in C'(E, \mathbb{R})$ *satisfies* $I(0) = 0$ *and*

(I_1) *there exist* $\rho, \alpha > 0$ *such that* $I(u) \geqslant \alpha > 0$ *for all* $\|u\| = \rho$;
(I_2) *there exist* $e \in E$ *with* $\|e\| > \rho$ *such that* $I(e) \leqslant 0$.

Then I *possesses a Cerami sequence at level*

$$c = \inf_{\Theta} \max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Theta = \gamma \in C([0,1], E) : \gamma(0) = 0, \|\gamma(1)\| > \rho, I(\gamma(1)) \leqslant 0.$$

Theorem 2.5 (local mountain pass theorem [27])**.** *Let* E *be a real Banach space. Assume* $I \in C'(E, \mathbb{R})$ *satisfies* $I(0) = 0$, (I_1) *and* (I_2). *If there exists* $\gamma_0 \in \Theta$, Θ *defined as in Theorem 2.4, such that*

$$c = \max_{t\in[0,1]} I(\gamma_0(t)) > 0,$$

then I *possesses a non-trivial critical point* $u \in \gamma_0([0,1])$ *at the level* c.

Lemma 2.6. *Suppose that* f *satisfies* (H_1), (H_2), (H_3), *and* (H_5). *Then* I *satisfies* (I_1) *and* (I_2).

Proof. By Lemma 2.2 and Sobolev's inequality, we have

$$\int_{\mathbb{R}^N} F(x,u)dx \leqslant \varepsilon|u|_2^2 + C_\varepsilon|u|_{p+1}^{p+1} \leqslant \varepsilon C_1\|u\|^2 + C\|u\|^{p+1}$$

for some $C_1 > 0$. By $\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \geqslant 0$, we have

$$I(u) \geqslant \frac{1}{2}\|u\|^2 - C_1\varepsilon\|u\|^2 - C\|u\|^{p+1} = \left(\frac{1}{2} - C_1\varepsilon\right)\|u\|^2 - C\|u\|^{p+1}.$$

Since $p > 2$, we have

$$I(u) \geqslant \left(\frac{1}{2} - C_1\varepsilon\right)\|u\|^2 + o(\|u\|^p) \geqslant \alpha$$

for $\|u\| = \rho$ small enough. This proves (I_1).

Next we prove $\exists e \in H^1(\mathbb{R}^3)$ such that $I(e) < 0$. By (H_3) and (H_5), for any $0 \neq v \in H^1(\mathbb{R}^3)$ that satisfies

$$M\int_{\mathbb{R}^3} v^4 dx > \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_v v^2 dx,$$

there exists $C > 0$ such that

$$F(x,u) \geqslant Mu^4 - Cu^2, \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

Hence

$$\begin{aligned} I(tv) &= \frac{t^2}{2}\|v\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_v v^2 dx - \int_{\mathbb{R}^3} F(x,tv)dx \\ &\leqslant \frac{t^2}{2}\|v\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_v v^2 dx - Mt^4\int_{\mathbb{R}^3} v^4 dx + Ct^2\int_{\mathbb{R}^3} v^2 dx \\ &= (C + \frac{1}{2})t^2\|v\|^2 - \left(M\int_{\mathbb{R}^3} v^4 dx - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_v v^2 dx\right)t^4 \\ &\to -\infty \end{aligned}$$

as $t \to \infty$. So, for t sufficient large, choose $e = tv$. □

Lemma 2.7. *Suppose that* f *satisfies* (H_1)-(H_5). *Then any Cerami sequence for* I *is bounded.*

Proof. Let $u_n \subset H^1(\mathbb{R}^3)$ be such that

$$I(u_n) \to c, (1 + \|u_n\|)I'(u_n) \to 0, n \to \infty.$$

Since

$$c + o_n(1) = 4I(u_n) - I'(u_n)u_n = \|u_n\|^2 + \int_{\mathbb{R}^3}(f(x,u_n)u_n - 4F(x,u_n))dx \geqslant \|u_n\|^2.$$

From above inequality, u_n is bounded. □

Lemma 2.8. *Suppose that* f *satisfies* (H_1)-(H_5). *Let* $u_n \subset H^1(\mathbb{R}^3)$ *be Cerami sequence for* I *at level* $c > 0$. *If* $u_n \rightharpoonup 0$ *in* $H^1(\mathbb{R}^3)$, *then there exist a sequence* $\{y_n\} \subset \mathbb{R}^3$ *and* $R > 0, \beta > 0$ *such that* $y_n \to \infty$ *and*

$$\lim_{n\to\infty} \sup \int_{B_R(y_n)} |u_n|^2 \geqslant \beta > 0.$$

Proof. Suppose by contradiction, that the Lemma fails. Then, for any $R > 0$, we have that

$$\lim_{n\to\infty} \sup \int_{B_R(y)} |u_n|^2 = 0$$

for all $R > 0$. By Lions Lemma [43], we have that $|u_n|_{L^s} \to 0$ for any $s \in (2, 2^*)$.

By Lemma 2.2, we have $\int_{\mathbb{R}^3} f(x,u_n)u_n \to 0$.

Since $I'(u_n)u_n \to 0$ as $n \to \infty$, we get

$$\|u_n\|^2 \leqslant \|u_n\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} f(x,u_n)u_n dx + o_n(1).$$

So, $u_n \to 0$ in $H^1(\mathbb{R}^3)$. Therefore, $\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 dx \to 0$.

From above facts, we get $I(u_n) \to 0$ as $n \to \infty$, which contradicts with $I(u_n) \to c > 0$. □

Lemma 2.9 ([45]). *Suppose that* (ii) *of* (H_6) *holds. If* $\{u_n\} \in H^1(\mathbb{R}^3)$ *such that* $u_n \rightharpoonup 0$ *in* $H^1(\mathbb{R}^3)$, $\{\varphi_n\} \in H^1(\mathbb{R}^3)$ *is bounded, then*

$$\int_{\mathbb{R}^3} [V(x) - V_0(x)] u_n \varphi_n dx \to 0,$$
$$\int_{\mathbb{R}^3} [K(x)\phi_{u_n} u_n \varphi_n - K_0(x)\tilde{\phi}_{u_n} u_n \varphi_n] dx \to 0,$$
$$\int_{\mathbb{R}^3} [f(x,u_n) - f_0(x,u_n)]\varphi_n dx \to 0.$$

3. Proof of main result

In this section we are ready to prove our main theorems.

Proof of Theorem 1.1. In view of Lemma 2.6 and Theorem 2.4, there exists a sequence $(u_n) \subset H^1(\mathbb{R}^3)$ such that

$$I'(u_n) \to c \geqslant \alpha > 0 \quad \text{and} \quad (1 + \|u_n\|)I'(u_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.1}$$

From Lemma 2.7, $\{u_n\}$ is bounded. So, without loss of generality, one assumes that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$.

Now we prove $I'(u) = 0$. Indeed, since $C_0^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$, it suffices to show that $I'(u)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. $\forall\ \varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}
I'(u_n)\varphi - I'(u)\varphi &= \int_{\mathbb{R}^3} (\nabla u_n \nabla \varphi + V(x)u_n\varphi)\, dx + \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n \varphi dx - \int_{\mathbb{R}^3} f(x,u_n)\varphi dx \\
&\quad - \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(x)u\varphi)\, dx - \int_{\mathbb{R}^3} K(x)\phi_u u \varphi dx + \int_{\mathbb{R}^3} f(x,u)\varphi dx \\
&= \langle u_n - u, \varphi\rangle - \int_{\mathbb{R}^3} K(x)\left(\phi_{u_n} u_n - \phi_u u\right)\varphi dx \\
&\quad - \int_{\mathbb{R}^3} (f(x,u_n) - f(x,u))\,\varphi dx.
\end{aligned}$$

Since $u_n \rightharpoonup u$, by Lemmas 2.1 and 2.2, we obtain

$$I'(u)\varphi = \lim_{n\to\infty} I'(u_n)\varphi = 0,$$

which implies that $I'(u) = 0$.

If $u \neq 0$, the theorem is proved.

If $u = 0$, from Lemma 2.8, there exists a sequence $(y_n) \subset \mathbb{R}^3$, $R > 0$, $\beta > 0$ such that $|y_n| \to \infty$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \int_{B_{R(y_n)}} |u_n|^2 \geqslant \beta > 0. \tag{3.2}$$

Let $(y_n) \subset \mathbb{Z}^3$ and $\tilde{u}_n(x) = u_n(x + y_n)$, and observing that $\|\tilde{u}_n\| = \|u_n\|_0$, up to a subsequence we have that $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^3)$, $\tilde{u}_n \to \tilde{u}$ in $L^2_{loc}(\mathbb{R}^3)$ and for almost every $x \in \mathbb{R}^3$. From (3.2), we have $\tilde{u} \neq 0$.

Next we prove $I_0'(\tilde{u}) = 0$. $\forall \varphi \in C_0^\infty(\mathbb{R}^3)$, for each $n \in \mathbb{N}$, let $\varphi_n(x) = \varphi(x - y_n)$, we get that

$$I_0'(\tilde{u})\varphi = I_0'(\tilde{u}_n)\varphi + o_n(1) = I_0'(u_n)\varphi_n + o_n(1).$$

On the other hand, by Lemma 2.9, we get that

$$\begin{aligned}
I_0'(u_n)\varphi_n &= I'(u_n)\varphi_n + \int_{\mathbb{R}^3} [V_0(x) - V(x)] u_n \varphi_n dx \\
&\quad - \int_{\mathbb{R}^3} [f_0(x,u_n) - f(x,u)]\varphi_n dx - \int_{\mathbb{R}^3} [K(x)\phi_{u_n} u_n \varphi_n - K_0(x)\tilde{\phi}_{u_n} u_n \varphi_n] dx \\
&= I'(u_n)\varphi_n + o_n(1).
\end{aligned}$$

So, by (3.1), we get $I_0'(\tilde{u}) = 0$.

By Lemma 2.9, similar to above, we have

$$I(u_n) - I_0(u_n) \to 0, \quad I'(u_n)u_n - I_0'(u_n)u_n \to 0.$$

Then

$$I_0(u_n) \to c, \quad I_0'(u_n)u_n \to 0.$$

By (iv) of (H_6), $\forall u \in \mathbb{R}$, we have $4F_0(x,u) \leqslant f_0(x,u)$. So

$$\begin{aligned}
c + o_n(1) &= I_0(u_n) - \frac{1}{4}I_0'(u_n)u_n \\
&= \frac{1}{4}\|u_n\|_0^2 + \int_{\mathbb{R}^3}[\frac{1}{4}f_0(x,u_n)u_n - F_0(x,u_n)]dx \\
&= \frac{1}{4}\|\tilde{u}_n\|_0^2 + \int_{\mathbb{R}^3}[\frac{1}{4}f_0(x,\tilde{u}_n)\tilde{u}_n - F_0(x,\tilde{u}_n)]dx \\
&\geqslant \frac{1}{4}\|\tilde{u}\|_0^2 + \int_{\mathbb{R}^3}[\frac{1}{4}f_0(x,\tilde{u})\tilde{u} - F_0(x,\tilde{u})]dx + o_n(1) \\
&= I_0(\tilde{u}) - \frac{1}{4}I_0'(\tilde{u})\tilde{u} + o_n(1) \\
&= I_0(\tilde{u}) + o_n(1).
\end{aligned}$$

Therefore $I_0(\tilde{u}) \leqslant c$.

We shall verify that $\max_{t\geqslant 0} I_0(t\tilde{u}) = I_0(\tilde{u})$. Let

$$\chi(t) = I_0(t\tilde{u}) = \frac{t^2}{2}\|\tilde{u}\|_0^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K_0(x)\phi_{\tilde{u}}\tilde{u}^2dx - \int_{\mathbb{R}^3} F_0(x,t\tilde{u})dx.$$

So,

$$\begin{aligned}
\chi'(t) &= t\|\tilde{u}\|_0^2 + t^3\int_{\mathbb{R}^3} K_0(x)\phi_{\tilde{u}}\tilde{u}^2dx - \int_{\mathbb{R}^3} f_0(x,t\tilde{u})\tilde{u}dx \\
&= t^3\left(\frac{1}{t^2}\|\tilde{u}\|_0^2 + \int_{\mathbb{R}^3} K_0(x)\phi_{\tilde{u}}\tilde{u}^2dx - \int_{\mathbb{R}^3}\frac{f_0(x,t\tilde{u})\tilde{u}}{t^3}\right)dx = t^3A(t).
\end{aligned}$$

Since $I_0'(\tilde{u}) = 0$, $A(1) = 0$. It follows from part (iv) of (H_6) that A is strictly decreasing in $(0,\infty)$, then $A(t) > 0$ when $t \in (0,1)$ and $A(t) < 0$ when $t \in (1,\infty)$. Therefore

$$\chi'(t) > 0 \text{ when } t \in (0,1) \text{ and } \chi'(t) < 0 \text{ when } t \in (1,\infty).$$

Hence, $\max_{t\geqslant 0} I_0(t\tilde{u}) = I_0(\tilde{u})$.

By the definition of c, (V) and part (iii) of (H_6), we have that

$$c \leqslant \max_{t\geqslant 0} I(t\tilde{u}) \leqslant \max_{t\geqslant 0} I_0(t\tilde{u}) = I_0(\tilde{u}) \leqslant c.$$

We can now invoke Theorem 2.5 to conclude that I possesses a critical point at level $c > 0$. This finishes the proof. □

Proof of Theorem 1.2. It is easy to see that Lemmas 2.2, 2.6, 2.7, and 2.8 are all hold by using the conditions of Theorem 1.1. From Lemma 2.6 and Theorem 2.4, there exists Cerami sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$, i.e.,

$$I_0(u_n) \to c_0 \text{ and } (1 + \|u_n\|_0)I_0'(u_n) \to 0, \text{ as } n \to +\infty.$$

where c_0 is the mountain pass level of I_0.

By Lemmas 2.7, we conclude that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$. Similar to proof of Theorem 1.1, we have $I_0'(u) = 0$.

Following, we only need to consider the case in which $u = 0$. By Lemma 2.8, there is a sequence $(y_n) \subset \mathbb{Z}^3$, $R > 0$, $\beta > 0$ such that $|y_n| \to \infty$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \int_{B_{R(y_n)}} |u_n|^2 \geqslant \beta > 0. \tag{3.3}$$

Let $\tilde{u}_n(x) = u_n(x + y_n)$, then $\|\tilde{u}_n\|_0 = \|u_n\|_0$. Up to a subsequence, we have

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ weakly in } H^1(\mathbb{R}^3), \quad \tilde{u}_n \to \tilde{u} \text{ in } L^2_{loc}(\mathbb{R}^3), \quad \tilde{u}_n(x) \to \tilde{u} \text{ almost everywhere in } \mathbb{R}^3.$$

By (3.3), $\tilde{u} \neq 0$. Similar to proof of Theorem 1.1, we get $I_0'(\tilde{u}) = 0$.

So $m = \inf\{I_0(u) : u \in H^1(\mathbb{R}^3), I'(u) = 0\} > 0$ is well defined. Next, to prove m is achieved. Indeed, let $\{u_n\} \subset H^1(\mathbb{R}^3)$ be a minimizing sequence for m, i.e.,

$$I_0(u_n) \to m, \quad I_0'(u_n) = 0 \text{ and } u_n \neq 0.$$

Obviously, $\{u_n\}$ is a Cerami sequence for I_0. So, from Lemma 2.7, $\{u_n\}$ is bounded. Moreover, from $I_0'(u_n)u_n = 0$ and Lemma 2.2, there exists $\sigma > 0$ such that $\|u_n\|_0 \geqslant \sigma$. Thus, arguing as in the preceding paragraph, we obtain a translated subsequence $\{\tilde{u}_n\}$, which has a non-zero weak limit u_0 such that $I_0'(u_0) = 0$ and $\tilde{u}_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^N$. By Fatou's lemma

$$\begin{aligned} m = \lim_{n\to\infty} I_0(u_n) = \lim_{n\to\infty} I_0(\tilde{u}_n) &= \liminf_{n\to\infty} \frac{\|\tilde{u}_n\|_0}{4} + \liminf_{n\to\infty} \int_{\mathbb{R}^3} \hat{F}_0(x, \tilde{u}_n)dx \\ &\geqslant \frac{\|u_0\|_0}{4} + \int_{\mathbb{R}^3} \hat{F}_0(x, u_0)dx = I_0(u_0). \end{aligned}$$

Consequently, $I_0(u_0) = m$, and therefore $u_0 \neq 0$ is a ground-state solution. □

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