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A generalization of Elsayed's solution to the difference equation $x_{n+1} = \frac{x_{n-5}}{-1+x_{n-2}x_{n-5}}$

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Abstract

In this paper, we obtain solutions to difference equations of the form

$$x_{n+1} = \frac{x_{n-5}}{a_n + b_n x_{n-2} x_{n-5}},$$

where (a_n) and (b_n) are sequences of real numbers. Consequently, a result of Elsayed is generalized. To achieve this, we use Lie symmetry analysis.

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1. Introduction

The method of Lie symmetries for differential equations has been applied to difference equations in recent times and tremendous progress has been made (see [8, 10, 11]). The idea behind the method is to find the group of transformations that map the set of solutions to the difference equation under study onto itself. For lower order difference equations, the method is applicable with less computation, but for higher order equations, the computations are quite complex, however rewarding.

Elsayed [3] obtained the solution of

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2}x_{n-5}}. \tag{1.1}$$

Similar work on related difference equations has been done (see [1, 2, 4–7, 9, 12]). In this paper we derive solutions of the following difference equations via the invariant of their group of transformations:

$$x_{n+1} = \frac{x_{n-5}}{a_n + b_n x_{n-2}x_{n-5}} \tag{1.2}$$

for some arbitrary sequences of real numbers (a_n) and (b_n).

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The solution to our difference equation provides a generalization of the solution of Elsayed [3] to equation (1.1) in a more general setting.

Without loss of generality, we instead study the difference equation

$$u_{n+6} = \frac{u_n}{A_n + B_n u_{n+3} u_n}. \tag{1.3}$$

1.1. Preliminaries

Definition 1.1 ([8])**.** A parameterized set of point transformations,

$$\Gamma_\varepsilon : x \mapsto \hat{x}(x; \varepsilon),$$

where $x = x_i$, $i = 1, \ldots, p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

1. Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$;
2. $\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0;
3. each $\hat{x_i}$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore

$$\hat{x_i}(x : \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2), i = 1, \ldots, p.$$

Given a sixth order ordinary difference equation

$$u_{n+6} = \omega(n, u_n, u_{n+1}, \ldots, u_{n+5}) \tag{1.4}$$

such that the derivative of ω with respect to u_n is nonzero, we consider the point transformations of the form

$$\Gamma_\varepsilon : (n, u_n) \mapsto (n, u_n + \varepsilon Q(n, u_n)),$$

where ε is the parameter and Q the characteristic. We refer to

$$X = Q(n, u_n) \frac{\partial}{\partial u_n} + Q(n+1, u_{n+1}) \frac{\partial}{\partial u_{n+1}} + Q(n+2, u_{n+2}) \frac{\partial}{\partial u_{n+2}} + \cdots + Q(n+5, u_{n+5}) \frac{\partial}{\partial u_{n+5}}$$

as the corresponding symmetry generator. The linearized symmetry condition is given by

$$\mathcal{S}^{(6)} Q(n, u_n) - X\omega = 0 \tag{1.5}$$

whenever (1.4) holds. The map $S : n \mapsto n+1$ is called the shift operator. In this paper, we follow the notation given in [8].

2. Symmetries

Consider the equation (1.3), i.e.,

$$u_{n+6} = \frac{u_n}{A_n + B_n u_{n+3} u_n},$$

Imposing the symmetry condition (1.5) to the equation above, we obtain

$$Q(n+6, u_{n+6}) + \frac{B_n {u_n}^2}{(B_n u_n u_{n+3} + A_n)^2} Q(n+3, u_{n+3}) - \frac{A_n}{(B_n u_n u_{n+3} + A_n)^2} Q(n, u_n) = 0. \tag{2.1}$$

The characteristic Q takes different arguments, and solving for it becomes difficult. So we first eliminate u_{n+6} by implicit differentiability with respect to u_n. This yields

$$\frac{A_n}{(B_n u_n u_{n+3}+A_n)^2}Q'(n+3,u_{n+3})-\frac{A_n}{(B_n u_n u_{n+3}+A_n)^2}Q'(n,u_n)+\frac{2A_n}{u_n(B_n u_n u_{n+3}+A_n)^2}Q(n,u_n)$$

and after simplification, we get

$$Q'(n+3,u_{n+3})-Q'(n,u_n)+\frac{2}{u_n}Q(n,u_n)=0,$$

where $'$ denotes the derivative with respect to the continuous variable. We now eliminate u_{n+3} by differentiating the above equation with respect to u_n and get the following equation:

$$Q''(n,u_n)-\frac{2}{u_n}Q'(n,u_n)+\frac{2}{{u_n}^2}Q(n,u_n)=0. \tag{2.2}$$

Thus, the solution of (2.2) is

$$Q=\alpha_n u_n+\beta_n {u_n}^2, \tag{2.3}$$

where α_n and β_n are functions of n. We substitute (2.3) in (2.1) and with simplification, we are led to

$$\begin{aligned}B_n\beta_{n+3}{u_{n+3}}^2{u_n}^2+B_n\alpha_{n+3}u_{n+3}{u_n}^2+B_n\alpha_{n+6}u_{n+3}{u_n}^2-A_n\beta_n{u_n}^2\\-A_n\alpha_n u_n+A_n\alpha_{n+6}u_n+\beta_{n+6}{u_n}^2=0.\end{aligned}$$

To solve for α_n and β_n, we use the method of separation by powers of shifts of u_n and we obtain the following reduced system:

$$\alpha_n+\alpha_{n+3}=0, \tag{2.4}$$
$$\beta_n=0. \tag{2.5}$$

The solutions of (2.4) are $(-1)^n$, $\exp(i\pi n/3)$, and $\exp(-i\pi n/3)$. The implication is that we have three non trivial generators

$$\begin{aligned}X_1 =&(-1)^n u_n\partial u_n-(-1)^n u_{n+1}\partial_{u_{n+1}}+(-1)^n u_{n+2}\partial u_{n+2}\\&-(-1)^n u_{n+3}\partial_{u_{n+3}}+(-1)^n u_{n+4}\partial u_{n+4}-(-1)^n u_{n+5}\partial_{u_{n+5}},\\X_2 =&\alpha^n u_n\partial u_n+\alpha^{n+1}u_{n+1}\partial_{u_{n+1}}+\alpha^{n+2}u_{n+2}\partial u_{n+2}\\&+\alpha^{n+3}u_{n+3}\partial_{u_{n+3}}+\alpha^{n+4}u_{n+4}\partial u_{n+4}+\alpha^{n+5}u_{n+5}\partial_{u_{n+5}},\\X_3 =&\bar{\alpha}^n u_n\partial u_n+\bar{\alpha}^{n+1}u_{n+1}\partial_{u_{n+1}}+\bar{\alpha}^{n+2}u_{n+2}\partial u_{n+2}\\&+\bar{\alpha}^{n+3}u_{n+3}\partial_{u_{n+3}}+\bar{\alpha}^{n+4}u_{n+4}\partial u_{n+4}+\bar{\alpha}^{n+5}u_{n+5}\partial_{u_{n+5}},\end{aligned}$$

where $\alpha=\exp(i\pi/3)$.

3. Reduction

We use the known choice of canonical coordinate [8]

$$s_n=\int\frac{du_n}{Q(n,u_n)}.$$

Using X_2 (we could have chosen X_1 or X_3), we have

$$s_n=\int\frac{du_n}{Q_2(n,u_n)}=\frac{1}{\alpha^n}\ln|u_n|.$$

Using (2.5), it can be shown that

$$X_2(\alpha^{n+3}s_{n+3}+\alpha^n s_n)=0$$

and so

$$r_n=\alpha^{n+3}s_{n+3}+\alpha^n s_n$$

is an invariant of X_2. We introduce

$$|V_n|=\exp\{-r_n\},$$

i.e., $V_n=\pm 1/u_n u_{n+3}$. For the rest of the work, we use $V_n=1/u_n u_{n+3}$. One can show that

$$V_{n+3}=A_n V_n+B_n \tag{3.1}$$

and it can be verified that the closed form solution of (3.1) is

$$V_{3n+i}=V_i\left(\prod_{k_1=0}^{n-1}A_{3k_1+i}\right)+\sum_{l=0}^{n-1}\left(B_{3l+i}\prod_{k_2=l+1}^{n-1}A_{3k_2+i}\right), i=0,1,2. \tag{3.2}$$

4. Exact solutions

Working backward to express the result in terms of the original variable u_n, we get

$$\begin{aligned}
|u_n| &= \exp\Bigg((-1)^n c_1+\bar{\alpha}^n c_2+\alpha^n c_3+(-1)^n\sum_{k_1=0}^{n-1}\frac{1}{3}(-1)^{-k_1}\ln|V_{k_1}|+\bar{\alpha}^n\sum_{k_2=0}^{n-1}\frac{1}{3}\alpha^{k_2}\ln|V_{k_2}|\\
&\quad+\alpha^n\sum_{k_3=1}^{n-1}\frac{1}{3}\bar{\alpha}^{k_3}\ln|V_{k_3}|\Bigg)\\
&=\exp\left((-1)^n c_1+\bar{\alpha}^n c_2+\alpha^n c_3+\frac{1}{3}\sum_{k=0}^{n-1}\left((-1)^{n-k}+\bar{\alpha}^n\alpha^k+\alpha^n\bar{\alpha}^k\right)\ln|V_k|\right)\\
&=\exp\left((-1)^n c_1+\bar{\alpha}^n c_2+\alpha^n c_3+\frac{1}{3}\sum_{k=0}^{n-1}[(-1)^{n-k}+2\mathrm{Re}(\alpha^n\bar{\alpha}^k)]\ln|V_k|\right)\\
&=\exp\left((-1)^n c_1+\bar{\alpha}^n c_2+\alpha^n c_3+\frac{1}{3}\sum_{k=0}^{n-1}\left[(-1)^{n-k}+2\cos\left(\frac{(n-k)\pi}{3}\right)\right]\ln|V_k|\right).
\end{aligned}$$

Setting $n:=6n+j$ and $H_j=(-1)^j c_1+\bar{\alpha}^j c_2+\alpha^j c_3$, we obtain

$$|u_{6n+j}|=\exp\left(H_j+\frac{1}{3}\sum_{k=0}^{6n+j-1}\left[(-1)^{j-k}+2\cos\left(\frac{(j-k)\pi}{3}\right)\right]\ln|V_k|\right). \tag{4.1}$$

For $j=0$, (4.1) becomes

$$|u_{6n}|=\exp(H_0)\exp(\ln|V_0|-\ln|V_3|+\ln|V_6|-\ln|V_9|+\cdots+\ln|V_{6n-6}|-\ln|V_{6n-3}|).$$

However, using (4.1) with $j=0, n=0$, $|u_0|=\exp(H_0)$. It can be shown that we do not need the absolute values, thus

$$u_{6n}=u_0\prod_{s=0}^{n-1}\frac{V_{6s}}{V_{6s+3}}.$$

Using (3.2), where $n := 2s$ and $i = 0$ for V_{6s}, and $n := 2s+1$ and $i = 0$ for V_{6s+3}, we have

$$\begin{aligned} V_{6s} &= V_0 \prod_{k_1=0}^{2s-1} A_{3k_1} + \sum_{l=0}^{2s-1} B_{3l} \prod_{k_2=l+1}^{2s-1} A_{3k_2} \\ &= V_0 \left(\prod_{k_1=0}^{2s-1} A_{3k_1} + \frac{1}{V_0} \sum_{l=0}^{2s-1} B_{3l} \prod_{k_2=l+1}^{2s-1} A_{3k_2} \right) \\ &= \frac{1}{u_0 u_3} \left(\prod_{k_1=0}^{2s-1} A_{3k_1} + u_0 u_3 \sum_{l=0}^{2s-1} B_{3l} \prod_{k_2=l+1}^{2s-1} A_{3k_2} \right) \end{aligned}$$

and

$$\begin{aligned} V_{6s+3} &= V_0 \prod_{k_1=0}^{2s} A_{3k_1} + \sum_{l=0}^{2s} B_{3l} \prod_{k_2=l+1}^{2s} A_{3k_2} \\ &= V_0 \left(\prod_{k_1=0}^{2s} A_{3k_1} + \frac{1}{V_0} \sum_{l=0}^{2s} B_{3l} \prod_{k_2=l+1}^{2s} A_{3k_2} \right) \\ &= \frac{1}{u_0 u_3} \left(\prod_{k_1=0}^{2s} A_{3k_1} + u_0 u_3 \sum_{l=0}^{2s} B_{3l} \prod_{k_2=l+1}^{2s-1} A_{3k_2} \right). \end{aligned}$$

Thus

$$u_{6n} = u_0 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} A_{3k_1} + u_0 u_3 \sum_{l=0}^{2s-1} B_{3l} \prod_{k_2=l+1}^{2s-1} A_{3k_2}}{\prod_{k_1=0}^{2s} A_{3k_1} + u_0 u_3 \sum_{l=0}^{2s} B_{3l} \prod_{k_2=l+1}^{2s} A_{3k_2}},$$

which implies that

$$x_{6n-6} = x_{-6} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} a_{3k_1} + x_{-6} x_{-3} \sum_{l=0}^{2s-1} b_{3l} \prod_{k_2=l+1}^{2s-1} a_{3k_2}}{\prod_{k_1=0}^{2s} a_{3k_1} + x_{-6} x_{-3} \sum_{l=0}^{2s} b_{3l} \prod_{k_2=l+1}^{2s} a_{3k_2}}. \tag{4.2}$$

For $j = 1$, (4.1) becomes

$$|u_{6n+1}| = \exp(H_1) \exp(\ln|V_1| - \ln|V_4| + \ln|V_7| - \ln|V_{10}| + \cdots + \ln|V_{6n-5}| - \ln|V_{6n-2}|).$$

However, using (4.1) with $j = 1, n = 0$, $|u_1| = \exp(H_1)$. Thus

$$u_{6n+1} = u_1 \prod_{s=0}^{n-1} \frac{V_{6s+1}}{V_{6s+4}}.$$

Using (3.2) with $i = 1$ and $n := 2s$, we have

$$\begin{aligned} V_{6s+1} &= V_1 \prod_{k_1=0}^{2s-1} A_{3k_1+1} + \sum_{l=0}^{2s-1} B_{3l+1} \prod_{k_2=l+1}^{2s-1} A_{3k_2+1} \\ &= V_1 \prod_{k_1=0}^{2s-1} A_{3k_1+1} + \frac{1}{V_1} \sum_{l=0}^{2s-1} B_{3l+1} \prod_{k_2=l+1}^{2s-1} A_{3k_2+1} \end{aligned}$$

$$= \frac{1}{u_1u_4}\left(\prod_{k_1=0}^{2s-1} A_{3k_1+1} + u_1u_4\sum_{l=0}^{2s-1} B_{3l+1}\prod_{k_2=l+1}^{2s-1} A_{3k_2+1}\right)$$

and similarly,

$$V_{6s+4} = \frac{1}{u_1u_5}\left(\prod_{k_1=0}^{2s} A_{3k_1+1} + u_1u_5\sum_{l=0}^{2s} B_{3l+1}\prod_{k_2=l+1}^{2s} A_{3k_2+1}\right).$$

Now

$$u_{6n+1} = u_1\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{2s-1} A_{3k_1+1} + u_1u_4\sum_{l=0}^{2s-1} B_{3l+1}\prod_{k_2=l+1}^{2s-1} A_{3k_2+1}}{\prod_{k_1=0}^{2s} A_{3k_1+1} + u_1u_5\sum_{l=0}^{2s} B_{3l+1}\prod_{k_2=l+1}^{2s} A_{3k_2+1}},$$

which implies that

$$x_{6n-5} = x_{-5}\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{2s-1} a_{3k_1+1} + x_{-5}x_{-2}\sum_{l=0}^{2s-1} b_{3l+1}\prod_{k_2=l+1}^{2s-1} a_{3k_2+1}}{\prod_{k_1=0}^{2s} a_{3k_1+1} + x_{-5}x_{-2}\sum_{l=0}^{2s} b_{3l+1}\prod_{k_2=l+1}^{2s} a_{3k_2+1}}. \tag{4.3}$$

For $j = 2$, we find that (4.1) becomes

$$|u_{6n+2}| = \exp(H_2)\exp(\ln|V_2| - \ln|V_5| + \ln|V_8| - \cdots + \ln|V_{6n-4}| - \ln|V_{6n-1}|).$$

We set $n = 0$ and $j = 2$ in (4.1) so that $|u_2| = \exp(H_2)$. So we have

$$u_{6n+2} = u_2\prod_{s=0}^{n-1}\frac{V_{6s+2}}{V_{6s+5}}.$$

The expressions for V_{6s+2} and V_{6s+5} are obtained from (3.2), by setting $n = 2s, i = 2$ and $n = 2s+1, i = 0$, respectively. They are as follows:

$$\begin{aligned}V_{6s+2} &= V_2\prod_{k_1=0}^{2s-1} A_{3k_1+2} + \sum_{l=0}^{2s-1} B_{3l+2}\prod_{k_2=l+1}^{2s-1} A_{3k_2+2}\\ &= V_2\left(\prod_{k_1=0}^{2s-1} A_{3k_1+2} + \frac{1}{V_2}\sum_{l=0}^{2s-1} B_{3l+2}\prod_{k_2=l+1}^{2s-1} A_{3k_2+2}\right)\\ &= \frac{1}{u_2u_5}\left(\prod_{k_1=0}^{2s-1} A_{3k_1+2} + u_2u_5\sum_{l=0}^{2s-1} B_{3l+2}\prod_{k_2=l+1}^{2s-1} A_{3k_2+2}\right)\end{aligned}$$

and

$$\begin{aligned}V_{6s+5} &= V_2\prod_{k_1=0}^{2s} A_{3k_1+2} + \sum_{l=0}^{2s-1} B_{3l+2}\prod_{k_2=l+1}^{2s} A_{3k_2+2}\\ &= V_2\left(\prod_{k_1=0}^{2s} A_{3k_1+2} + \frac{1}{V_2}\sum_{l=0}^{2s} B_{3l+2}\prod_{k_2=l+1}^{2s} A_{3k_2+2}\right)\end{aligned}$$

$$= \frac{1}{u_2u_5}\left(\prod_{k_1=0}^{2s} A_{3k_1+2} + u_2u_5\sum_{l=0}^{2s} B_{3l+2}\prod_{k_2=l+1}^{2s} A_{3k_2+2}\right).$$

Hence

$$u_{6n+2} = u_2\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s-1} A_{3k_1+2} + u_2u_5\sum\limits_{l=0}^{2s-1} B_{3l+2}\prod\limits_{k_2=l+1}^{2s-1} A_{3k_2+2}}{\prod\limits_{k_1=0}^{2s} A_{3k_1+2} + u_2u_5\sum\limits_{l=0}^{2s} B_{3l+2}\prod\limits_{k_2=l+1}^{2s} A_{3k_2+2}},$$

which gives

$$x_{6n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s-1} a_{3k_1+2} + x_{-4}x_{-1}\sum\limits_{l=0}^{2s-1} b_{3l+2}\prod\limits_{k_2=l+1}^{2s-1} a_{3k_2+2}}{\prod\limits_{k_1=0}^{2s} a_{3k_1+2} + x_{-4}x_{-1}\sum\limits_{l=0}^{2s} b_{3l+2}\prod\limits_{k_2=l+1}^{2s} a_{3k_2+2}}. \tag{4.4}$$

For $j = 3$, (4.1) becomes

$$|u_{6n+3}| = \exp(H_3)\exp(-\ln|V_0| + \ln|V_3| - \ln|V_6| + \cdots + \ln|V_{6n-3}| - \ln|V_{6n}|).$$

Setting $n = 0$ and $j = 3$ in (4.1), we find that $|u_3| = \exp(H_3)\frac{1}{V_0}$, hence

$$u_{6n+3} = u_3\prod_{s=1}^{n-1}\frac{V_{6s+3}}{V_{6s+6}}.$$

Following a similar approach as was done in the earlier cases ($j = 0, 1, 2$), the reader can verify that

$$u_{6n+3} = u_3\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s} A_{3k_1} + u_0u_3\sum\limits_{l=0}^{2s} B_{3l}\prod\limits_{k_2=l+1}^{2s} A_{3k_2}}{\prod\limits_{k_1=0}^{2s+1} A_{3k_1} + u_0u_3\sum\limits_{l=0}^{2s+1} B_{3l}\prod\limits_{k_2=l+1}^{2s+1} A_{3k_2}}.$$

Thus

$$x_{6n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s} a_{3k_1} + x_{-6}x_{-3}\sum\limits_{l=0}^{2s} b_{3l}\prod\limits_{k_2=l+1}^{2s} a_{3k_2}}{\prod\limits_{k_1=0}^{2s+1} a_{3k_1} + x_{-6}x_{-3}\sum\limits_{l=0}^{2s+1} b_{3l}\prod\limits_{k_2=l+1}^{2s+1} a_{3k_2}}. \tag{4.5}$$

For $j = 4$, (4.1) becomes

$$|u_{6n+4}| = \exp(H_4)\exp(-\ln|V_1| + \ln|V_4| - \ln|V_7| + \cdots + \ln|V_{6n-2}| - \ln|V_{6n+1}|).$$

Setting $j = 4$ and $n = 0$ in (4.1) yields $|u_4| = \exp(H_4)\exp(-\ln|V_1|)$ so that $\exp(H_4) = |u_4||V_1|$. Thus

$$u_{6n+4} = u_4\prod_{s=1}^{n-1}\frac{V_{6s+4}}{V_{6s+7}}.$$

As before, similar steps can be carried out and one obtains

$$u_{6n+4} = u_4\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s} A_{3k_1+1} + u_1u_4\sum\limits_{l=0}^{2s} B_{3l+1}\prod\limits_{k_2=l+1}^{2s} A_{3k_2+1}}{\prod\limits_{k_1=0}^{2s+1} A_{3k_1+1} + u_1u_4\sum\limits_{l=0}^{2s+1} B_{3l+1}\prod\limits_{k_2=l+1}^{2s+1} A_{3k_2+1}},$$

which yields

$$x_{6n-2}=x_{-2}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s}a_{3k_1+1}+x_{-5}x_{-2}\sum\limits_{l=0}^{2s}b_{3l+1}\prod\limits_{k_2=l+1}^{2s}a_{3k_2+1}}{\prod\limits_{k_1=0}^{2s+1}a_{3k_1+1}+x_{-5}x_{-2}\sum\limits_{l=0}^{2s+1}b_{3l+1}\prod\limits_{k_2=l+1}^{2s+1}a_{3k_2+1}}. \tag{4.6}$$

For $j=5$, (4.1) becomes

$$|u_{6n+5}|=\exp(H_5)\exp(-\ln|V_2|+\ln|V_5|-\ln|V_8|+\cdots+\ln|V_{6n+1}|-\ln|V_{6n+2}|).$$

But setting $n=0$ and $j=5$ in the same equation (4.1), we get $|u_5|=\exp(H_5)\exp(-\ln|V_2|)$ so that $\exp(H_5)=|u_5||V_2|$. Thus

$$u_{6n+5}=u_5\prod_{s=0}^{n-1}\frac{V_{6s+5}}{V_{6s+8}}.$$

After performing similar substitutions as before, we get

$$u_{6n+5}=u_5\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s}A_{3k_1+2}+u_2u_5\sum\limits_{l=0}^{2s}B_{3l+2}\prod\limits_{k_2=l+1}^{2s}A_{3k_2+2}}{\prod\limits_{k_1=0}^{2s+1}A_{3k_1+2}+u_2u_5\sum\limits_{l=0}^{2s+1}B_{3l+2}\prod\limits_{k_2=l+1}^{2s+1}A_{3k_2+2}},$$

which leads to

$$x_{6n-1}=x_{-1}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{2s}a_{3k_1+2}+x_{-4}x_{-1}\sum\limits_{l=0}^{2s}b_{3l+2}\prod\limits_{k_2=l+1}^{2s}a_{3k_2+2}}{\prod\limits_{k_1=0}^{2s+1}a_{3k_1+2}+x_{-4}x_{-1}\sum\limits_{l=0}^{2s+1}b_{3l+2}\prod\limits_{k_2=l+1}^{2s+1}a_{3k_2+2}}. \tag{4.7}$$

Thus, our solution to the difference equation is given by the equations (4.2)-(4.7) as long as the denominators are non-zero.

5. The case when a_j and b_j are 3-periodic sequences

In this case, we assume that $\{a_j\}_{j\geqslant 0}=\{a_0,a_1,a_2,a_0,a_1,a_2,\ldots\}$ and $\{b_j\}_{j\geqslant 0}=\{b_0,b_1,b_2,b_0,b_1,b_2,\ldots\}$.

5.1. *The case when* $a_0\neq 1, a_1\neq 1, a_2\neq 1$

By substitution, one obtains the solution given by equations

$$x_{6n-6}=x_{-6}\prod_{s=0}^{n-1}\frac{a_0^{2s}+x_{-6}x_{-3}\frac{b_0(1-a_0^{2s})}{1-a_0}}{a_0^{2s+1}+x_{-6}x_{-3}\frac{b_0(1-a_0^{2s+1})}{1-b_0}},\qquad x_{6n-5}=x_{-5}\prod_{s=0}^{n-1}\frac{a_1^{2s}+x_{-5}x_{-2}\frac{b_1(1-a_1^{2s})}{1-a_1}}{a_1^{2s+1}+x_{-5}x_{-2}\frac{b_1(1-a_1^{2s+1})}{1-a_1}},$$

$$x_{6n-4}=x_{-4}\prod_{s=0}^{n-1}\frac{a_2^{2s}+x_{-4}x_{-1}\frac{b_2(1-a_2^{2s})}{1-a_2}}{a_2^{2s+1}+x_{-4}x_{-1}\frac{b_2(1-a_2^{2s+1})}{1-a_2}},\qquad x_{6n-3}=x_{-3}\prod_{s=0}^{n-1}\frac{a_0^{2s+1}+x_{-6}x_{-3}\frac{b_0(1-a_0^{2s+1})}{1-a_0}}{a_0^{2s+2}+x_{-6}x_{-3}\frac{b_0(1-a_0^{2s+2})}{1-a_0}},$$

$$x_{6n-2}=x_{-2}\prod_{s=0}^{n-1}\frac{a_1^{2s+1}+x_{-5}x_{-2}\frac{b_1(1-a_1^{2s+1})}{1-a_1}}{a_1^{2s+2}+x_{-5}x_{-2}\frac{b_1(1-a_1^{2s+2})}{1-a_1}},\qquad x_{6n-1}=x_{-1}\prod_{s=0}^{n-1}\frac{a_2^{2s+1}+x_{-4}x_{-1}\frac{b_2(1-a_2^{2s+1})}{1-a_2}}{a_2^{2s+2}+x_{-4}x_{-1}\frac{b_2(1-a_2^{2s+2})}{1-a_2}},$$

where the initial conditions satisfy

$$\prod_{i=1}^{2}\left(x_{-6}x_{-3}b_j\frac{1-a_j^{2s+i}}{1-a_j}-a_j^{2s+i}\right)\left(x_{-5}x_{-2}b_j\frac{1-a_j^{2s+i}}{1-a_j}-a_j^{2s+i}\right)\left(x_{-4}x_{-1}b_j\frac{1-a_j^{2s+i}}{1-a_j}-a_j^{2s+i}\right)\neq 0$$

for all $j=0,1,2$ and $s=1,2,\ldots,n-1$.

6. The case when a_j and b_j are 1-periodic

The solution equations are obtained by replacing a_1 and a_2, and b_1 and b_2 by a_0 and b_0, respectively in the previous section.

6.1. *The case when* $a_0 \neq 1$

In this case, we have the solution defined by the following equations

$$x_{6n-6} = x_{-6}\prod_{s=0}^{n-1}\frac{a_0^{2s}+x_{-6}x_{-3}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1}+x_{-6}x_{-3}b_0\frac{1-a_0^{2s+1}}{1-a_0}}, \qquad x_{6n-5} = x_{-5}\prod_{s=0}^{n-1}\frac{a_0^{2s}+x_{-5}x_{-2}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1}+x_{-5}x_{-2}b_0\frac{1-a_0^{2s+1}}{1-a_0}},$$

$$x_{6n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{a_0^{2s}+x_{-4}x_{-1}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1}+x_{-4}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}}, \qquad x_{6n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{a_0^{2s+1}+x_{-6}x_{-3}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2}+x_{-6}x_{-3}b_0\frac{1-a_0^{2s+2}}{1-a_0}},$$

$$x_{6n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{a_0^{2s+1}+x_{-5}x_{-2}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_1^{2s+2}+x_{-5}x_{-2}b_0\frac{1-a_0^{2s+2}}{1-a_0}}, \qquad x_{6n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{a_0^{2s+1}+x_{-4}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2}+x_{-4}x_{-1}b_0\frac{1-a_0^{2s+2}}{1-a_0}},$$

where the initial conditions satisfy

$$\prod_{i=1}^{2}\left(x_{-6}x_{-3}b_0\frac{1-a_0^{2s+i}}{1-a_0}-a_0^{2s+i}\right)\left(x_{-5}x_{-2}b_0\frac{1-a_0^{2s+i}}{1-a_0}-a_0^{2s+i}\right)\left(x_{-4}x_{-1}b_0\frac{1-a_0^{2s+i}}{1-a_0}-a_0^{2s+i}\right)\neq 0$$

for all $s = 1, 2, \ldots, n-1$.

6.1.1. $a_0 = -1$ *and* $b_0 = 1$

In this case, we have the solution defined by the following equations and appears in [3, Theorem 1]

$$x_{6n-6} = x_{-6}(-1+x_{-6}x_{-3})^{-n}, \qquad x_{6n-5} = x_{-5}(-1+x_{-5}x_{-2})^{-n},$$
$$x_{6n-4} = x_{-4}(-1+x_{-4}x_{-1})^{-n}, \qquad x_{6n-3} = x_{-3}(-1+x_{-6}x_{-3})^{n},$$
$$x_{6n-2} = x_{-2}(-1+x_{-5}x_{-2})^{n}, \qquad x_{6n-1} = x_{-1}(-1+x_{-4}x_{-1})^{n},$$

where $x_{-6}x_{-3} \neq 1$, $x_{-5}x_{-2} \neq 1$, and $x_{-4}x_{-1} \neq 1$.

6.1.2. $a_0 = -1$ *and* $b_0 = -1$

In this case, we have the solution defined by the following equations

$$x_{6n-6} = x_{-6}(-1-x_{-6}x_{-3})^{-n}, \qquad x_{6n-5} = x_{-5}(-1-x_{-5}x_{-2})^{-n},$$
$$x_{6n-4} = x_{-4}(-1-x_{-4}x_{-1})^{-n}, \qquad x_{6n-3} = x_{-3}(-1-x_{-6}x_{-3})^{n},$$
$$x_{6n-2} = x_{-2}(-1-x_{-5}x_{-2})^{n}, \qquad x_{6n-1} = x_{-1}(-1-x_{-4}x_{-1})^{n},$$

where $x_{-6}x_{-3} \neq -1$, $x_{-5}x_{-2} \neq -1$ and $x_{-4}x_{-1} \neq -1$.

6.2. $a_0 = 1$

By substitution, one obtains the solution given by equations

$$x_{6n-6} = x_{-6}\prod_{s=0}^{n-1}\frac{1+2sb_0x_{-6}x_{-3}}{1+(2s+1)b_0x_{-6}x_{-3}}, \qquad x_{6n-5} = x_{-5}\prod_{s=0}^{n-1}\frac{1+2sb_0x_{-5}x_{-2}}{1+(2s+1)b_0x_{-5}x_{-2}},$$

$$x_{6n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{1+2sb_0x_{-4}x_{-1}}{1+(2s+1)b_0x_{-4}x_{-1}}, \qquad x_{6n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1+(2s+1)b_0x_{-6}x_{-3}}{1+(2s+2)b_0x_{-6}x_{-3}},$$

$$x_{6n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1+(2s+1)b_0x_{-5}x_{-2}}{1+(2s+2)b_0x_{-5}x_{-2}}, \qquad x_{6n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{1+(2s+1)b_0x_{-4}x_{-1}}{1+(2s+2)b_0x_{-4}x_{-1}},$$

where $jb_0x_{-4}x_{-1} \neq -1$, $jb_0x_{-5}x_{-2} \neq -1$, and $jb_0x_{-6}x_{-3} \neq -1$ for all $j = 1, 2, \ldots, 2n$.

6.2.1. $a_0 = 1$ *and* $b_0 = 1$

In this case, we have the solution defined by the following equations

$$x_{6n-6} = x_{-6} \prod_{s=0}^{n-1} \frac{1 + 2s x_{-6} x_{-3}}{1 + (2s+1) x_{-5} x_{-2}}, \qquad x_{6n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{1 + 2s x_{-5} x_{-2}}{1 + (2s+1) x_{-5} x_{-2}},$$
$$x_{6n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{1 + 2s x_{-4} x_{-1}}{1 + (2s+1) x_{-5} x_{-2}}, \qquad x_{6n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{1 + (2s+1) x_{-6} x_{-3}}{1 + (2s+2) x_{-5} x_{-2}},$$
$$x_{6n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{1 + (2s+1) x_{-5} x_{-2}}{1 + (2s+2) x_{-5} x_{-2}}, \qquad x_{6n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{1 + (2s+1) x_{-4} x_{-1}}{1 + (2s+2) x_{-4} x_{-1}},$$

where $jx_{-4}x_{-1} \neq -1$, $jx_{-5}x_{-2} \neq -1$, and $jx_{-6}x_{-3} \neq -1$ for all $j = 1, 2, \ldots, 2n$.

6.2.2. $a_0 = 1$ *and* $b_0 = -1$

We obtain the following solution equations

$$x_{6n-6} = x_{-6} \prod_{s=0}^{n-1} \frac{1 - 2s x_{-6} x_{-3}}{1 - (2s+1) x_{-5} x_{-2}}, \qquad x_{6n-5} = x_{-5} \prod_{s=0}^{n-1} \frac{1 - 2s x_{-5} x_{-2}}{1 - (2s+1) x_{-5} x_{-2}},$$
$$x_{6n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{1 - 2s x_{-4} x_{-1}}{1 - (2s+1) x_{-5} x_{-2}}, \qquad x_{6n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{1 - (2s+1) x_{-6} x_{-3}}{1 - (2s+2) x_{-5} x_{-2}},$$
$$x_{6n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{1 - (2s+1) x_{-5} x_{-2}}{1 - (2s+2) x_{-5} x_{-2}}, \qquad x_{6n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{1 - (2s+1) x_{-4} x_{-1}}{1 - (2s+2) x_{-4} x_{-1}},$$

where $jx_{-4}x_{-1} \neq 1$, $jx_{-5}x_{-2} \neq 1$, and $jx_{-6}x_{-3} \neq 1$ for all $j = 1, 2, \ldots, 2n$.

7. Conclusion

We obtained three non trivial symmetries and exact solutions to difference equations of the form (1.2). In particular, we generalized the result of Elsayed [3]. Our method of solution finding involved Lie symmetry analysis.

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