



A topology on lattice-ordered groups



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Abstract

We introduce the concept of the strong-positive cone in a lattice-ordered group (G, \leq, \cdot) and define the continuous lattice-ordered group. We also investigate the C-topology and bi-C-topology given on a lattice-ordered group. The main results obtained in this paper are as follows: (1) (G, \leq, \cdot) is a continuous lattice-ordered group if and only if (G, \leq) is a continuous poset; (2) for the bi-C-topology τ in a continuous lattice-ordered group (G, \leq, \cdot) , (G, \cdot, τ) is a topological group and (G, \leq, τ) is a topological lattice.

Keywords: Lattice-ordered group, continuous, topology, topological group, topological lattice.

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1. Introduction

Lattice-ordered groups (also called l-groups) are an important class of partially ordered algebraic systems. The study of l-groups was initiated by Birkhoff [9, 12] and followed by many other researchers [6, 16, 24]. The monographs [10, 15] gave a systematic account of the basic theory of l-groups. Recently, l-groups have cropped up in many other areas of mathematics, for instance, in the theory of bézout domains [36, 37], ordered fields [7, 15], category theory [11], topology theory [5, 8, 21] and others [22].

In the 1960s, Scott introduced domain theory [31], aiming to provide mathematical models for computer program languages. Later, domain theory has got a great development in the areas of abstract theories of computation [28, 32], the semantics of programming languages [29, 34], logic and lambda calculus [3, 33], and in other branches of mathematics [14, 17]. Now, domain theory is still hot in lambda calculus and programming languages [1, 2], stable domain theory [35, 38], category theory [20, 23], and has many applications in classical mathematics [18, 25, 26, 30].

Based on the characteristics of l-group and domain, we introduce continuous l-group, aiming to establish the application of domain theory in algebraic structure and also broaden the study space of l-group

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theory. As we all know that domain is a continuous dcpo and the approximation is the core of continuity. Therefore, in the course of the study of continuity, scholars pay more attention to the infinite case. And, by Proposition 2.1 given soon afterwards, we know that an l-group G is an infinite set except that $G = \{e\}$. Thus, in the research of l-groups, scholars tend to study in the infinite set. Due to this consideration, we come up with an idea that we need to introduce continuity in l-group. In the field of mathematics, topology theory has always been the focus of our concern. Especially in recent years, topologies on l-group have received much more attention [8, 21, 22]. In this paper, the C-topology and bi-C-topology τ given on an l-group are studied. We also obtain the result that if (G, \leq, \cdot) is a continuous l-group, then (G, \cdot, τ) is a topological group and (G, \leq, τ) is a topological lattice.

The main task of this paper reads as follows. In Section 2, we shall first briefly introduce l-group and related basic facts. In Section 3, we introduce the strong-positive cone C and give the definition of the continuous l-group. And then some properties of the strong-positive cone and continuous l-group are studied. Finally, we investigate the C-topology and bi-C-topology defined on an l-group.

2. Preliminaries

In this section, we shall recall some notions and notations used in this paper. Let X be a set and τ a family of subsets of X . Then τ is called a topology on X if it satisfies: (1) both the empty set and X are elements of τ ; (2) any union of elements of τ is an element of τ ; (3) any intersection of finitely many elements of τ is an element of τ . If τ is a topology on X , then the pair (X, τ) is called a topological space. The members of τ are called open sets in X . A subset of X is said to be closed if its complement is in τ . A basis \mathcal{B} for a topological space (X, τ) is a collection of open sets in τ such that every open set in τ can be written as a union of elements of \mathcal{B} . A function or map from one topological space to another is called continuous if the inverse image of any open set is open.

Let P be a set. An order (a partial order) on P is a binary relation \leq on P such that, for all $x, y, z \in P$, (1) $x \leq x$, (2) $x \leq y$ and $y \leq x$ imply $x = y$, (3) $x \leq y$ and $y \leq z$ imply $x \leq z$, are satisfied. A set P equipped with an order relation \leq is said to be a partially ordered set (poset). A poset P is called a lattice if the infimum and supremum of any pair of elements in P exist. A poset P is called a complete lattice if the infimum and supremum of any subset of P exist. Let D be a non-empty subset of a poset P . Then D is said to be directed if, for every pair of elements $x, y \in D$, there exists $z \in D$ such that $x \leq z$ and $y \leq z$. A subset $A \subseteq P$ is called an upper set if for every $x \in A$, $y \in P$ with $x \leq y$, we have $y \in A$. Dually, a subset $A \subseteq P$ is called a lower set if for every $x \in A$, $y \in P$ with $x \geq y$, we have $y \in A$. A subset $A \subseteq P$ is called a filter if A is a filtered upper set. For two elements x, y in a poset P , $x \vee y$, $x \wedge y$ denote the supremum and infimum of $\{x, y\}$, respectively and for every subset A of P , $\bigvee A$, $\bigwedge A$ denote the supremum and infimum of A , respectively. For $x \in P$, we write $\downarrow x = \{a \in P : a \leq x\}$ and $\uparrow x = \{a \in P : x \leq a\}$.

A lattice-ordered group (G, \leq, \cdot) is a group (G, \cdot) with a binary relation \leq such that (G, \leq) is a lattice and satisfies: $x \leq y$ implies $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for any $x, y, z \in G$. A quantale (S, \leq, \cdot) is a semigroup (S, \cdot) with a binary relation \leq such that (S, \leq) is a complete lattice and satisfies: $x \cdot (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \cdot y_i)$ and $(\bigvee_{i \in I} y_i) \cdot x = \bigvee_{i \in I} (y_i \cdot x)$. In the following paper, e represents the identity element of a group G . For any $x \in G$, $A, B \subseteq G$, we write $x \cdot A = \{x \cdot a : a \in A\}$, $A \cdot x = \{a \cdot x : a \in A\}$, $A \cdot B = \{a \cdot b : a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$. Sometimes we will note an l-group (G, \leq, \cdot) as G .

Proposition 2.1. *Let (G, \leq, \cdot) be an l-group. Then $G = \{e\}$ or $|G|$ is infinite, where $|G|$ denotes the number of elements of G .*

Proof. It is obvious that $G = \{e\}$ is an l-group. Assume that $|G| > 1$. Since G is a lattice, there exist $a, b \in G$ such that $a < b$. Denote $c = a^{-1} \cdot b$ and so $c \in G$, $c > e$. Then we can construct an infinite subset $\{x_n : n \in \mathbf{N}\}$ of G . Let $x_1 = c \cdot c$, $x_2 = x_1 \cdot x_1$, \dots , $x_n = x_{n-1} \cdot x_{n-1}$, \dots . Now we will show that for any $n \in \mathbf{N}$, $x_n > e$ and x_n is an increasing sequence. Since $c > e$, we have $x_n = c^{2^n} > c > e$. Because $x_n = x_{n-1} \cdot x_{n-1} > x_{n-1}$, we have $\{x_n\}$ is an increasing sequence. Therefore $\{x_n : n \in \mathbf{N}\}$ is an infinite subset of G . Whence $G = \{e\}$ or $|G|$ is infinite. \square

Definition 2.2 ([15, 24]). The element x of an l -group G is positive if $x \geq e$. The set of all positive elements of G is called the positive cone.

Definition 2.3 ([17, 18]). Let P be a poset. For every $x, y \in P$, we say $x \ll y$ if, for every directed subset D of P for which $\bigvee D$ exists and $y \leq \bigvee D$, there exists $d \in D$ such that $x \leq d$.

P is called a continuous poset, if for each $x \in P$, $\{y \in P : y \ll x\}$ is directed and $x = \bigvee \{y \in P : y \ll x\}$.

Proposition 2.4 ([24]). Let G be an l -group. Then $(x \vee y)^{-1} = x^{-1} \wedge y^{-1}$ and $(x \wedge y)^{-1} = x^{-1} \vee y^{-1}$ hold for any $x, y \in G$.

Proposition 2.5 ([15, 24]). Let G be an l -group. If D is a directed subset of G with $\bigvee D$ exists, then for any $a, b \in G$, $a \cdot D \cdot b = \{a \cdot d \cdot b : d \in D\}$ is directed and $\bigvee (a \cdot D \cdot b) = a \cdot (\bigvee D) \cdot b$.

Proposition 2.6 ([15]). Let G be an l -group.

(1) If $A, B \subseteq G$ with $\bigvee A, \bigvee B$ exist, then $\bigvee (A \cdot B) = \bigvee A \cdot \bigvee B$.

(2) If $A, B \subseteq G$ with $\bigwedge A, \bigwedge B$ exist, then $\bigwedge (A \cdot B) = \bigwedge A \cdot \bigwedge B$.

Proposition 2.7 ([15, 24]). Let G be an l -group. If D is a directed subset of G with $\bigvee D$ exists, then D^{-1} is a filtered set and $\bigwedge (D^{-1}) = (\bigvee D)^{-1}$. Dually, if F is a filtered subset of G with $\bigwedge F$ exists, then F^{-1} is a directed set and $\bigvee (F^{-1}) = (\bigwedge F)^{-1}$.

Proposition 2.8 ([27]). Let X be a set and \mathcal{B} a family of subsets of X . \mathcal{B} is a basis of a topology of X if it satisfies:

(1) $\bigcup \{B : B \in \mathcal{B}\} = X$;

(2) for any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition 2.9 ([4]). $(G, \cdot, \tau(G))$ is called a topological group if it satisfies:

(1) (G, \cdot) is a group and $\tau(G)$ is a topology on G ;

(2) the map $f : G \times G \rightarrow G$ as mapping (x, y) to $x \cdot y$ is continuous with respect to $\tau(G \times G)$ and $\tau(G)$;

(3) the map $g : G \rightarrow G$ as mapping x to x^{-1} is continuous with respect to $\tau(G)$.

Definition 2.10 ([19]). $(L, \leq, \tau(L))$ is called a topological lattice if it satisfies:

(1) (L, \leq) is a lattice and $\tau(L)$ is a topology on L ;

(2) the map $f : L \times L \rightarrow L$ as mapping (x, y) to $x \vee y$ is continuous with respect to $\tau(L \times L)$ and $\tau(L)$;

(3) the map $g : L \times L \rightarrow L$ as mapping (x, y) to $x \wedge y$ is continuous with respect to $\tau(L \times L)$ and $\tau(L)$.

3. Continuous lattice-ordered group

In this section, we introduce the strong-positive cone which is a subset of the positive cone of an l -group. Furthermore, we study many properties of the strong-positive cone and give some good examples. Based on the strong-positive cone, we put forward the definition of continuous l -group and show that the direct product of two continuous l -groups is also a continuous l -group.

Definition 3.1. Let G be an l -group with the positive cone P . An $x \in P$ is called a strong-positive element if for any directed subset D of G with $\bigvee D$ exists and $x = \bigvee D$, $D \cap P \neq \emptyset$ holds. The set of all strong-positive elements in G is called the strong-positive cone.

In domain theory, for elements x, y in a poset L , we say $x \ll y$ if, for every directed subset D of L for which $\bigvee D$ exists and $y \leq \bigvee D$, there exists $d \in D$ such that $x \leq d$. So for an element x in an l -group, $e \ll x$ means that for every directed subset D of G for which $\bigvee D$ exists and $x \leq \bigvee D$, there exists $d \in D$ such that $e \leq d$, that is $D \cap P \neq \emptyset$. Therefore, for an element x in an l -group, $e \ll x$ implies that x is a strong-positive element. However, if x is a strong-positive element in an l -group, then we can't get that $e \ll x$ according to the definitions. In the following paper, you can see that " $e \ll x$ " and " x is a strong-positive element" in an l -group are equivalent because of the particularity of l -groups. This will greatly simplify the definition of " \ll " in l -group and continuous l -groups. For a deeper understanding of the strong-positive cone in an l -group, we give the following remark.

Remark 3.2. Let G be an l -group with the positive cone P and the strong-positive cone C . $x \in P \setminus C$ if and only if $x \in P$ and there is a directed subset D of $G \setminus P$ with $\bigvee D$ exists and $x = \bigvee D$.

Example 3.3.

- (1) Let $(\mathbf{Z}, \leq, +)$ be the additive group with the standard order of integer number \mathbf{Z} . $(\mathbf{Z}, \leq, +)$ is an l -group. The positive cone is $\{x \in \mathbf{Z} : x \geq 0\}$ and the strong-positive cone is $\{x \in \mathbf{Z} : x > 0\}$.
- (2) Let $(\mathbf{R}, \leq, +)$ be the additive group with the standard order of \mathbf{R} . Obviously, $(\mathbf{R}, \leq, +)$ is an l -group. The positive cone is $\{x \in \mathbf{R} : x \geq 0\}$ and the strong-positive cone is $\{x \in \mathbf{R} : x > 0\}$.
- (3) Let $(\mathbf{Q}^+, \leq, \cdot)$ be the multiplicative group of positive rationals with the standard order. Then $(\mathbf{Q}^+, \leq, \cdot)$ is an l -group. The positive cone of $(\mathbf{Q}^+, \leq, \cdot)$ is $\{x \in \mathbf{Q}^+ : x \geq 1\}$ and the strong-positive cone is $\{x \in \mathbf{Q}^+ : x > 1\}$.
- (4) Let $(\mathbf{R}^n, +)$ be the real n -space with the standard additive operator. The partial order is defined on $(\mathbf{R}^n, +)$ by $(x_1, x_2, \dots, x_n) \leq (y_1, y_2, \dots, y_n)$ if $x_i \leq y_i$ for $1 \leq i \leq n$. The positive cone of $(\mathbf{R}^n, \leq, +)$ is $\{(x_1, x_2, \dots, x_n) : x_i \geq 0\}$ and the strong-positive cone is $\{(x_1, x_2, \dots, x_n) : x_i > 0\}$.

It must be noted that, the strong-positive cone in an l -group may be empty, as you can see in the example below.

Example 3.4. Let X be an infinite set, $G = F(X, \mathbf{R})$ be a set of all functions $f : X \rightarrow \mathbf{R}$ with the coordinate order: $f \leq g$ for $f, g \in G$ if $f(x) \leq g(x)$ for all $x \in X$. Then G is an l -group under the partial order \leq and the addition: $(f + g)(x) = f(x) + g(x)$. The positive cone P of G is $\{f \in G : f(x) \geq 0 \text{ for every } x \in X\}$. The strong-positive cone C of G is empty.

Proof. Let $f \in P$. For any finite set $F \subseteq X$, define $f^F \in G$ with $f^F(x) = f(x)$ for $x \in F$ and $f^F(x) = -1$ for $x \in X \setminus F$. Consider $D = \{f^F : F \subseteq_{\text{fin}} X\}$. It is obvious that $D \subseteq G \setminus P$. For any $f^{F_1}, f^{F_2} \in D$, $f^{F_1} \leq f^{F_1 \cup F_2}$ and $f^{F_2} \leq f^{F_1 \cup F_2}$, so D is directed and $\bigvee D = f$. By Remark 3.2, $f \in P \setminus C$. Therefore the strong-positive cone C is empty. □

In the following paper, we study the l -group in which the strong-positive cone is nonempty.

Proposition 3.5. *Let G be an l -group with the positive cone P and the strong-positive cone C . Then,*

- (1) C is an upper set;
- (2) $P \cdot C \subseteq C, C \cdot P \subseteq C$;
- (3) for any $x \in G, x^{-1} \cdot C \cdot x \subseteq C$;
- (4) for any $x \in G, x \cdot C = C \cdot x$ and for any $A \subseteq G, A \cdot C = C \cdot A$.

Proof.

(1) Let $x \in C, y \in G$ with $x \leq y$. It is obvious that $y \in P$. Let D be a directed subset of G with $\bigvee D$ exists and $y = \bigvee D$. By Proposition 2.5, $D' = x \cdot y^{-1} \cdot D$ is directed and $\bigvee D' = x$. Since $x \in C$, we have $D' \cap P \neq \emptyset$ by Definition 3.1. That is, there exists $d \in D$ such that $x \cdot y^{-1} \cdot d \in P$. So $x \cdot y^{-1} \cdot d \geq e$. Since $y \cdot x^{-1} \geq e$, we have $d = (y \cdot x^{-1}) \cdot (x \cdot y^{-1} \cdot d) \geq e$. So $D \cap P \neq \emptyset$ and by Definition 3.1, $y \in C$. Therefore C is an upper set.

(2) Let $x \in P, y \in C$. Then $x \cdot y \geq y$. By (1), $x \cdot y \in C$. Thus $P \cdot C \subseteq C$. Similarly, $C \cdot P \subseteq C$.

(3) Let $y \in C$. It is obvious that $x^{-1} \cdot y \cdot x \in P$. Suppose that D is a directed subset of G with $\bigvee D$ exists and $x^{-1} \cdot y \cdot x = \bigvee D$. By Proposition 2.5, $D' = \{x \cdot d \cdot x^{-1} : d \in D\}$ is directed and $\bigvee D' = y$. So there exists $d \in D$ such that $x \cdot d \cdot x^{-1} \in P$ by Definition 3.1. Hence $d \in P$ and $D \cap P \neq \emptyset$. Thus $x^{-1} \cdot y \cdot x \in C$ by Definition 3.1. Therefore $x^{-1} \cdot C \cdot x \subseteq C$.

(4) Let $x \in G, a \in C$. By (3), $x \cdot a \cdot x^{-1} \in C$. So $x \cdot a = (x \cdot a \cdot x^{-1}) \cdot x \in C \cdot x$. Thus $x \cdot C \subseteq C \cdot x$. Similarly, $C \cdot x \subseteq x \cdot C$. Therefore $x \cdot C = C \cdot x$. Let $A \subseteq G$. Since for any $x \in G, x \cdot C = C \cdot x$, we have $A \cdot C = C \cdot A$. □

Proposition 3.6. *Let G be an l -group with the strong-positive cone C . For any $x, y \in G$, if $x \leq y$, then $x \cdot C^{-1} \subseteq y \cdot C^{-1}$ and $y \cdot C \subseteq x \cdot C$.*

Proof. Let $x, y \in G$ with $x \leq y$. For any $z \in x \cdot C^{-1}$, we have $z^{-1} \cdot x \in C$. Since $x \leq y$, we have $z^{-1} \cdot x \leq z^{-1} \cdot y$. Thus $z^{-1} \cdot y \in C$ by Proposition 3.5 and $z \in y \cdot C^{-1}$. Therefore $x \cdot C^{-1} \subseteq y \cdot C^{-1}$. Since $x \leq y$, we have $y^{-1} \leq x^{-1}$. Hence $y^{-1} \cdot C^{-1} \subseteq x^{-1} \cdot C^{-1}$. Because $x^{-1} \cdot C^{-1} = (C \cdot x)^{-1}$, we have $(C \cdot y)^{-1} \subseteq (C \cdot x)^{-1}$ and $C \cdot y \subseteq C \cdot x$. Hence $y \cdot C \subseteq x \cdot C$ by Proposition 3.5 (4). \square

Proposition 3.7. *Let G be an l -group with the positive cone P and the strong-positive cone C . Then C is a filter.*

Proof. By Proposition 3.5 (3), C is an upper set. Let $x, y \in C$. We will show that $x \wedge y \in C$. Suppose that D is a directed subset of G with $\bigvee D$ exists and $\bigvee D = x \wedge y$. Consider $D_1 = x \cdot (x \wedge y)^{-1} \cdot D$. By Proposition 2.5, D_1 is directed and $\bigvee D_1 = x$. Since $x \in C$, we have $D_1 \cap P \neq \emptyset$. That is, there exists $d_1 \in D$ such that $x \cdot (x \wedge y)^{-1} \cdot d_1 \in P$. Thus $d_1 \geq (x \wedge y) \cdot x^{-1}$. Similarly, there exists $d_2 \in D$ such that $d_2 \geq (x \wedge y) \cdot y^{-1}$. Since D is directed, there exists $d \in D$ such that $d \geq (x \wedge y) \cdot x^{-1}$ and $d \geq (x \wedge y) \cdot y^{-1}$. So $d \cdot (x \wedge y)^{-1} \geq x^{-1}$ and $d \cdot (x \wedge y)^{-1} \geq y^{-1}$. Hence $d \cdot (x \wedge y)^{-1} \geq x^{-1} \vee y^{-1}$. By Proposition 2.4, $x^{-1} \vee y^{-1} = (x \wedge y)^{-1}$. So $d \cdot (x \wedge y)^{-1} \geq (x \wedge y)^{-1}$ and $d \geq e$. Thus $D \cap P \neq \emptyset$ and $x \wedge y \in C$. Therefore C is a filter. \square

Definition 3.8. Let G be an l -group with the strong-positive cone C . G is called a continuous l -group if, $\bigwedge C$ exists and $e = \bigwedge C$.

Comparing with the definition of continuous in domain theory, this definition is more simple and one can judge whether an l -group is continuous more quickly. Of course, by Theorem 3.14 given soon afterward, you can see these two definitions in an l -group are equivalent.

Example 3.9.

- (1) Every totally ordered group is a continuous l -group.
- (2) Let G be an l -group with the strong-positive cone C . If $e \in C$, then G is a continuous l -group.
- (3) Consider the Example 3.3 (4). In $(\mathbf{R}^n, \leq, +)$, the strong-positive cone is $\{(x_1, x_2, \dots, x_n) : x_i > 0\}$. Since $\bigwedge \{(x_1, x_2, \dots, x_n) : x_i > 0 \text{ for } 1 \leq i \leq n\} = (0, 0, \dots, 0)$, $(\mathbf{R}^n, \leq, +)$ is a continuous l -group.
- (4) Consider the Example 3.4. In $(G, \leq, +)$, the strong-positive cone is empty. Since there does not exist the largest element in $(G, \leq, +)$, we have $\bigwedge \emptyset$ does not exist. Thus $(G, \leq, +)$ is not a continuous l -group.

Proposition 3.10. *Let G be an l -group with the strong-positive cone C . If there exists a set $F \subseteq C$ such that $\bigwedge F = e$, then G is a continuous l -group.*

Proof. It is obvious that e is a lower bound of C . Suppose that x is another lower bound of C . Since $F \subseteq C$, we have e, x are lower bounds of F . $\bigwedge F = e$, so $x \leq e$. Thus e is the largest lower bound of C . Hence G is a continuous l -group by Definition 3.8. \square

Proposition 3.11. *Let G be a continuous l -group with the strong-positive cone C . Then for any $x \in G$, $x \cdot C^{-1}$ is directed and $\bigvee (x \cdot C^{-1}) = x$.*

Proof. By Proposition 3.7, we have C is filtered. Since G is a continuous l -group, $\bigwedge C = e$. So C^{-1} is directed and $\bigvee C^{-1} = e$ by Proposition 2.7. Hence for any $x \in G$, the set $x \cdot C^{-1}$ is directed and $\bigvee (x \cdot C^{-1}) = x$ by Proposition 2.5. \square

Proposition 3.12. *Let G be a continuous l -group with the strong-positive cone C . For any $x \in C$, if D is a directed subset of G with $\bigvee D$ exists and $x = \bigvee D$, then $D \cap C \neq \emptyset$.*

Proof. Assume that $x \in C$ and D is a directed subset of G with $\bigvee D$ exists and $x = \bigvee D$. By Proposition 3.11, we have C^{-1} is directed and $\bigvee C^{-1} = e$. So $D \cdot C^{-1}$ is directed and $\bigvee (D \cdot C^{-1}) = x$ by Proposition 2.6. Hence $(D \cdot C^{-1}) \cap P \neq \emptyset$ by Definition 3.1. That is, there exists $d \in D, a \in C$ such that $d \cdot a^{-1} \geq e$. Thus $d \geq a$. Since C is an upper set by Proposition 3.5 (1), we have $d \in C$ and $D \cap C \neq \emptyset$. \square

Proposition 3.13. *Let G be a continuous l-group with the strong-positive cone C . Then $C \cdot C = \{x_1 \cdot x_2 : x_1 \in C, x_2 \in C\} = C$.*

Proof. Since $C \subseteq P$, we have $C \cdot C \subseteq P \cdot C \subseteq C$ by Proposition 3.5 (2). Thus $C \cdot C \subseteq C$. For any $x \in C$, $x \cdot C^{-1}$ is directed and $\bigvee(x \cdot C^{-1}) = x$ by Proposition 3.11. Hence $(x \cdot C^{-1}) \cap C \neq \emptyset$ by Proposition 3.12. That is, there exists $a \in C$ such that $x \cdot a^{-1} \in C$. Since $x = (x \cdot a^{-1}) \cdot a \in C \cdot C$, we have $C \subseteq C \cdot C$. Therefore $C \cdot C = C$. \square

Theorem 3.14. *Let (G, \leq, \cdot) be an l-group. Then (G, \leq, \cdot) is a continuous l-group if and only if (G, \leq) is a continuous poset.*

Proof. Let P be the positive cone of G , C the strong-positive cone of G . We will show that for any $x \in G$, $\{y \in G : y \ll x\} = x \cdot C^{-1}$. Let $y \in G$ with $y \ll x$. Assume that D is a directed subset of G with $\bigvee D$ exists and $y^{-1} \cdot x = \bigvee D$. By Proposition 2.5, $y \cdot D$ is directed and $x = \bigvee(y \cdot D)$. Hence there exists $d \in D$ such that $y \leq y \cdot d$ by Definition 2.3. So $d \in P$ and $D \cap P \neq \emptyset$. Thus $y^{-1} \cdot x \in C$ by Definition 3.1. Because $y = x \cdot (y^{-1} \cdot x)^{-1}$, we have $y \in x \cdot C^{-1}$. Hence $\{y \in G : y \ll x\} \subseteq x \cdot C^{-1}$. Let $a \in C$. We will show that $x \cdot a^{-1} \ll x$. Assume that D is a directed subset of G with $\bigvee D$ exists and $x \leq \bigvee D$. By Proposition 2.5, $a \cdot (\bigvee D)^{-1} \cdot D$ is directed and $a = \bigvee(a \cdot (\bigvee D)^{-1} \cdot D)$. So $(a \cdot (\bigvee D)^{-1} \cdot D) \cap P \neq \emptyset$ by Definition 3.1. That is, there exists $d \in D$ such that $a \cdot (\bigvee D)^{-1} \cdot d \geq e$. Hence $d \geq (\bigvee D) \cdot a^{-1} \geq x \cdot a^{-1}$. We have $x \cdot a^{-1} \ll x$ by Definition 2.3. Thus $x \cdot C^{-1} \subseteq \{y \in G : y \ll x\}$. Therefore $\{y \in G : y \ll x\} = x \cdot C^{-1}$.

“ \Rightarrow ” : Assume that (G, \leq, \cdot) is a continuous l-group. Then for any $x \in G$, $x \cdot C^{-1}$ is a directed set and $\bigvee(x \cdot C^{-1}) = x$ by Proposition 3.11. Since $\{y \in G : y \ll x\} = x \cdot C^{-1}$, we have $\{y \in G : y \ll x\}$ is directed and $x = \bigvee\{y \in G : y \ll x\}$. Hence (G, \leq) is a continuous poset by Definition 2.3.

“ \Leftarrow ” : Since (G, \leq) is a continuous poset and for any $x \in G$, $\{y \in G : y \ll x\} = x \cdot C^{-1}$, we have $x \cdot C^{-1}$ is directed and $x = \bigvee(x \cdot C^{-1})$. Hence $e = \bigvee C^{-1}$ by Proposition 2.5 and $e = \bigwedge C$ by Proposition 2.7. Therefore (G, \leq, \cdot) is a continuous l-group. \square

Proposition 3.15. *Let (G_1, \leq_1, \cdot) and $(G_2, \leq_2, *)$ be l-groups with the positive cones P_1 and P_2 , respectively. Then the direct product $(G_1 \times G_2, \leq, \star)$ with $(a_1, b_1) \leq (a_2, b_2)$ defined by $a_1 \leq_1 a_2, b_1 \leq_2 b_2$ and $(a_1, b_1) \star (a_2, b_2) = (a_1 \cdot b_1, a_2 * b_2)$ is an l-group and the positive cone of $G_1 \times G_2$ is $P_1 \times P_2$. If C_1, C_2 are the strong-positive cones of G_1, G_2 , respectively, then in $G_1 \times G_2$ the strong-positive cone C is $C_1 \times C_2$.*

Proof. It is obvious that $G_1 \times G_2$ is an l-group and the positive cone is $P_1 \times P_2$. Now we will show that the strong-positive cone $C = C_1 \times C_2$. Let $(x_1, x_2) \in C$. Suppose that D_1 is a directed subset of G_1 with $x_1 = \bigvee D_1$. Then $D_1 \times \{x_2\}$ is a directed subset of $G_1 \times G_2$. Since $(x_1, x_2) \in C$, there exists $d_1 \in D_1$ such that $(d_1, x_2) \in P_1 \times P_2$ which implies $d_1 \in P_1$. Thus $x_1 \in C_1$. Similarly, $x_2 \in C_2$. Therefore $C \subseteq C_1 \times C_2$. Let $(x_1, x_2) \in C_1 \times C_2$. Suppose that $D_1 \times D_2$ is a directed subset of $G_1 \times G_2$ with $(x_1, x_2) = \bigvee(D_1 \times D_2)$. So D_1 is a directed subset of G_1 with $x_1 = \bigvee D_1$ and D_2 is a directed subset of G_2 with $x_2 = \bigvee D_2$. Since $(x_1, x_2) \in C_1 \times C_2$, $D_1 \cap P_1 \neq \emptyset$ and $D_2 \cap P_2 \neq \emptyset$. So $(D_1 \times D_2) \cap (P_1 \times P_2) \neq \emptyset$. Thus $(x_1, x_2) \in C$ and $C_1 \times C_2 \subseteq C$. Therefore $C = C_1 \times C_2$. \square

Theorem 3.16. *Let (G_1, \leq_1, \cdot) and $(G_2, \leq_2, *)$ be continuous l-groups. Then the direct product $G_1 \times G_2$ is a continuous l-group.*

Proof. This result can be proved by Proposition 3.15. \square

4. A topology on lattice-ordered groups

In this section, the C -topology and bi- C -topology τ given on an l-group are studied. We investigate some properties of these topologies and obtain that for a continuous l-group (G, \leq, \cdot) , (G, \cdot, τ) is a topological group and (G, \leq, τ) is a topological lattice.

Proposition 4.1. *Let G be an l -group with the strong-positive cone C . The sets of the form $x \cdot C$, $x \in G$, constitute a basis of a topology on G (we call this topology the C -topology and denote it by $\beta(G)$).*

Proof. Let $x \in G$ and $y \in x \cdot C^{-1}$. Then there exists $a \in C$ such that $y = x \cdot a^{-1}$. Thus $x = y \cdot a \in y \cdot C \subseteq \bigcup\{x \cdot C : x \in G\}$. Therefore $G \subseteq \bigcup\{x \cdot C : x \in G\} \subseteq G$ and $G = \bigcup\{x \cdot C : x \in G\}$. Let $x, y \in G$. Since G is an l -group, $x \vee y$ exists. By Proposition 3.6, $(x \vee y) \cdot C \subseteq x \cdot C$ and $(x \vee y) \cdot C \subseteq y \cdot C$. Hence $\{x \cdot C : x \in G\}$ constitutes a basis of a topology on G by Proposition 2.8. \square

Example 4.2.

- (1) Consider Example 3.3 (1). In $(\mathbf{Z}, \leq, +)$, the basis of the C -topology is $\{[x, +\infty) : x \in \mathbf{Z}\}$.
- (2) Consider Example 3.3 (4). In $(\mathbf{R}^n, \leq, +)$, the basis of the C -topology is $\{(\uparrow(x_1, x_2, \dots, x_n)) \setminus (x_1, x_2, \dots, x_n) : x_{i(1 \leq i \leq n)} \in \mathbf{R}\}$.

Proposition 4.3. *Let G be an l -group with the strong-positive cone C .*

- (1) $U \in \beta(G)$ if and only if $U = U \cdot C$.
- (2) If $U \in \beta(G)$, then $x \cdot U \in \beta(G)$.
- (3) If A is a closed set in $\beta(G)$, then for every $x \in G$, $x \cdot A$ is also a closed set.

Proof.

(1) " \Rightarrow ": Let $U \in \beta(G)$. Then $U = \bigcup_{x \in A \subseteq G} x \cdot C = A \cdot C$ by Proposition 4.1. So $U \cdot C = A \cdot C \cdot C$. By Proposition 3.13, $C \cdot C = C$. Hence $U \cdot C = A \cdot C$. Therefore $U = U \cdot C$.

" \Leftarrow ": Since $U = U \cdot C = \bigcup_{u \in U} u \cdot C$, we have $U \in \beta(G)$ by Proposition 4.1.

- (2) By (1), $U = U \cdot C$. Thus $x \cdot U = x \cdot U \cdot C = (x \cdot U) \cdot C$. Hence $x \cdot U$ is also an open set by (1).
- (3) We will show that $x \cdot (G \setminus A) = G \setminus (x \cdot A)$. Let $a \in G \setminus A$ which implies $a \notin A$. So $x \cdot a \notin x \cdot A$, that is $x \cdot a \in G \setminus (x \cdot A)$. Thus $x \cdot (G \setminus A) \subseteq G \setminus (x \cdot A)$. Let $a \in G \setminus (x \cdot A)$ which implies $a \notin x \cdot A$. So $x^{-1} \cdot a \notin A$, that is $x^{-1} \cdot a \in G \setminus A$. Thus $a = x \cdot (x^{-1} \cdot a) \in x \cdot (G \setminus A)$ and $G \setminus (x \cdot A) \subseteq x \cdot (G \setminus A)$. Therefore $x \cdot (G \setminus A) = G \setminus (x \cdot A)$. Since A is a closed set, we have $G \setminus A$ is an open set. Thus $x \cdot (G \setminus A)$ is also an open set by (2). Hence $G \setminus (x \cdot A)$ is an open set and $x \cdot A$ is a closed set. \square

Definition 4.4. A quantale G is called completely distributive if for any family $\{x_{j,k} : j \in J, k \in K(j)\}$ in G the identity

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = \bigvee_{f \in M} \bigwedge_{j \in J} x_{j,f(j)}$$

holds, where M is the set of choice functions defined on J with values $f(j) \in K(j)$.

Proposition 4.5 ([13]). *Let P be a non-empty poset. P is a complete lattice if and only if P has a least element and $\bigvee S$ exists in P for any non-empty subset S of P .*

Proposition 4.6. *Let (G, \leq, \cdot) be an l -group with the strong-positive cone C . $(\beta(G), \subseteq, \cdot)$ is a completely distributive quantale, where $U_1 \cdot U_2 = \{u_1 \cdot u_2 : u_1 \in U_1, u_2 \in U_2\}$ for any $U_1, U_2 \in \beta(G)$.*

Proof. By Proposition 4.5, we have $(\beta(G), \subseteq)$ is a complete lattice in which for $U_{i(i \in I)} \in \beta(G)$

$$\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i, \quad \bigwedge_{i \in I} U_i = \bigcup\{V \in \beta(G) : V \subseteq \bigcap_{i \in I} U_i\}.$$

For any $U_1, U_2 \in \beta(G)$, $U_1 \cdot U_2 = U_1 \cdot C \cdot U_2 \cdot C$. By Proposition 3.5 (4), $C \cdot U_2 = U_2 \cdot C$. So $U_1 \cdot U_2 = U_1 \cdot U_2 \cdot C \cdot C$. By Proposition 3.13, $C \cdot C = C$. So $U_1 \cdot U_2 = U_1 \cdot U_2 \cdot C$ and by Proposition 4.3 (1), $U_1 \cdot U_2 \in \beta(G)$. Thus $(\beta(G), \cdot)$ is a semigroup. Let $A, B_{i(i \in I)} \in \beta(G)$. We have $A \cdot (\bigvee_{i \in I} B_i) = A \cdot (\bigcup_{i \in I} B_i) =$

$A \cdot (\bigcup_{i \in I} B_i) \cdot C = \bigcup_{i \in I} (A \cdot B_i \cdot C) = \bigcup_{i \in I} (A \cdot B_i) = \bigvee_{i \in I} (A \cdot B_i)$. Thus $A \cdot (\bigvee_{i \in I} B_i) = \bigvee_{i \in I} (A \cdot B_i)$. Similarly, $(\bigvee_{i \in I} B_i) \cdot A = \bigvee_{i \in I} (B_i \cdot A)$. Therefore $(\beta(G), \subseteq, \cdot)$ is a quantale.

Then we will show that G is completely distributive. For any family $\{U_{j,k} : j \in J, k \in K(j)\}$ in G , the inequality $\bigwedge_{j \in J} \bigvee_{k \in K(j)} U_{j,k} \geq \bigvee_{f \in M} \bigwedge_{j \in J} U_{j,f(j)}$ holds in any complete lattice. So we only need to show that $\bigwedge_{j \in J} \bigvee_{k \in K(j)} U_{j,k} \leq \bigvee_{f \in M} \bigwedge_{j \in J} U_{j,f(j)}$, i.e., $\bigwedge_{j \in J} \bigvee_{k \in K(j)} U_{j,k} \subseteq \bigvee_{f \in M} \bigwedge_{j \in J} U_{j,f(j)}$. For any $x \in \bigwedge_{j \in J} \bigvee_{k \in K(j)} U_{j,k}$, there exists $y \in \bigwedge_{j \in J} \bigvee_{k \in K(j)} U_{j,k}$ such that $x \in y \cdot C$ by Proposition 4.3 (1). Since $\bigwedge_{j \in J} \bigvee_{k \in K(j)} U_{j,k} = \bigcup \{V \in \beta(G) : V \subseteq \bigcap_{j \in J} (\bigvee_{k \in K(j)} U_{j,k})\}$, we have $y \in \bigvee_{k \in K(j)} U_{j,k}$ for any $j \in J$. Because $\bigvee_{k \in K(j)} U_{j,k} = \bigcup_{k \in K(j)} U_{j,k}$, we conclude that for any $j \in J$ there exists $k_j \in K(j)$ such that $y \in U_{j,k_j}$. We denote this k_j by $f(j)$. By these choices we have found a function $f \in M$ such that $y \in \bigcap_{j \in J} U_{j,f(j)}$. Since $U_{j,f(j)} \in \beta(G)$, we have $y \cdot C \subseteq U_{j,f(j)}$ for any $j \in J$ by Proposition 4.3 (1). Thus $y \cdot C \subseteq \bigcap_{j \in J} U_{j,f(j)}$. Because $y \cdot C \in \beta(G)$, we have $y \cdot C \subseteq \bigwedge_{j \in J} U_{j,f(j)} = \bigcup \{V \in \beta(G) : V \subseteq \bigcap_{j \in J} U_{j,f(j)}\}$. Thus $y \cdot C \subseteq \bigvee_{f \in M} \bigwedge_{j \in J} U_{j,f(j)}$ and $x \in \bigvee_{f \in M} \bigwedge_{j \in J} U_{j,f(j)}$. Hence $\bigwedge_{j \in J} \bigvee_{k \in K(j)} U_{j,k} \subseteq \bigvee_{f \in M} \bigwedge_{j \in J} U_{j,f(j)}$. Therefore $\bigwedge_{j \in J} \bigvee_{k \in K(j)} U_{j,k} = \bigvee_{f \in M} \bigwedge_{j \in J} U_{j,f(j)}$.

From the above discussion, we have $(\beta(G), \subseteq, \cdot)$ is a completely distributive quantale. □

Definition 4.7 ([18]). Let P be a poset. We say that $U \subseteq P$ is Scott-open (open for the Scott topology of P) if it satisfies the following two conditions:

- (1) U is an upper set;
- (2) $\bigvee D \in U$ implies $D \cap U \neq \emptyset$, for any directed subset D of P with $\bigvee D$ exists.

Lemma 4.8 ([18]). Let P be a poset. For any $x \in P$, $\downarrow x$ is closed in the Scott topology of P .

Theorem 4.9. Let G be an l -group with the strong-positive cone C . Then the following conditions are equivalent:

- (1) $\downarrow e$ is closed in the C -topology;
- (2) G is a continuous l -group;
- (3) the C -topology $\beta(G)$ is equivalent to the Scott topology of (G, \leq) .

Proof.

(1) \Rightarrow (2) Since $C \subseteq P$, e is a lower bound of C . Assume that $x \in G$ is another lower bound of C . Then we will show that $x \leq e$. Suppose $x \not\leq e$. By (1), $G \setminus \downarrow e$ is open which contains x . So there exists $y \in G \setminus \downarrow e$ such that $x \in y \cdot C$ by Proposition 4.3 (1). Thus $y^{-1} \cdot x \in C$. Since x is a lower bound of C , we have $x \leq y^{-1} \cdot x$. Hence $y \leq e$ which is in contradiction with $y \in G \setminus \downarrow e$. So $x \leq e$ and e is the largest lower bound of C . Therefore G is a continuous l -group.

(2) \Rightarrow (3) Let $U \in \beta(G)$ and $x \in U$, $y \in G$ with $x \leq y$. Then there exist $u \in U$, $a \in C$ such that $x = u \cdot a$ by Proposition 4.3 (1). Since $x = u \cdot a \leq y$, we have $a \leq u^{-1} \cdot y$. So $u^{-1} \cdot y \in C$ by Proposition 3.5 (1). Thus $y = u \cdot (u^{-1} \cdot y) \in u \cdot C \subseteq U$. Hence $y \in U$ and U is an upper set. Assume that D is a directed subset of G with $\bigvee D$ exists and $\bigvee D \in U$. Then there exists $u \in U$ such that $\bigvee D \in u \cdot C$ by Proposition 4.3 (1). So $u^{-1} \cdot (\bigvee D) \in C$. By Proposition 2.5, $u^{-1} \cdot D$ is directed and $\bigvee (u^{-1} \cdot D) = u^{-1} \cdot (\bigvee D) \in C$. Hence $(u^{-1} \cdot D) \cap C \neq \emptyset$ by Proposition 3.12. That is, there exists $d \in D$ such that $u^{-1} \cdot d \in C$. Thus $d \in u \cdot C \subseteq U$ and $D \cap U \neq \emptyset$. Therefore U is Scott open.

Conversely, let U be Scott open. Suppose that $x \in U$, $a \in C$. Then $x \cdot a \geq x$ and $x \cdot a \in U$ because U is an upper set. Thus $U \cdot C \subseteq U$. By Proposition 3.11, for any $x \in G$, $x \cdot C^{-1}$ is directed and $\bigvee (x \cdot C^{-1}) = x$. Hence $(x \cdot C^{-1}) \cap U \neq \emptyset$ by Proposition 4.7. That is, there exists $c \in C$ such that $x \cdot c^{-1} \in U$. So $x = (x \cdot c^{-1}) \cdot c \in U \cdot c \subseteq U \cdot C$. Thus $U \subseteq U \cdot C$. Hence $U = U \cdot C$ and $U \in \beta(G)$ by Proposition 4.3 (1).

From the above discussion, we conclude that if (G, \leq, \cdot) is a continuous l -group, the C -topology $\beta(G)$ is equivalent to the Scott topology of (G, \leq) .

(3) \Rightarrow (1) This result can be proved by Lemma 4.8. □

Lemma 4.10 ([27]). *Let (X, τ) be a topological space and $A \subseteq X$. $A \in \tau$ if and only if for any $x \in A$, there exists $U \in \tau$ such that $x \in U \subseteq A$.*

Proposition 4.11. *Let G_1, G_2 be continuous l-groups with the strong-positive cones C_1, C_2 , respectively. A function $f : G_1 \rightarrow G_2$ is continuous with respect to the C-topologies of G_1 and G_2 if and only if for any $x \in G_1$, $f(x) = \bigvee \{f(w) : w \in x \cdot C_1^{-1}\}$.*

Proof.

“ \Rightarrow ” : Firstly, we will show that $f(x)$ is an upper bound of $\{f(w) : w \in x \cdot C_1^{-1}\}$. Assume that there exists $w_0 \in x \cdot C_1^{-1}$ such that $f(w_0) \not\leq f(x)$. By Proposition 4.9 and Proposition 4.3 (3), $f(x) \cdot \downarrow e = \downarrow f(x)$ is closed and so, $V = G_2 \setminus \downarrow f(x)$ is an open set and $f(w_0) \in V, f(x) \notin V$. Since f is continuous, $U = f^{-1}(V)$ is an open set and $w_0 \in U, x \notin U$. Hence $w_0 \cdot C_1 \subseteq U$ by Proposition 4.3 (1). We have $x \in w_0 \cdot C_1 \subseteq U$. This contradicts $x \notin U$. Thus for any $w \in x \cdot C_1^{-1}, f(w) \leq f(x)$.

Then we will show that $f(x)$ is the least upper bound of $\{f(w) : w \in x \cdot C_1^{-1}\}$. Suppose that $t \in G_2$ is another upper bound of $\{f(w) : w \in x \cdot C_1^{-1}\}$. We only need to show that $f(x) \leq t$. Assume that $f(x) \not\leq t$. Then $G_2 \setminus \downarrow t$ is an open set and $f(x) \in G_2 \setminus \downarrow t$. Since f is continuous, $f^{-1}(G_2 \setminus \downarrow t)$ is an open set and $x \in f^{-1}(G_2 \setminus \downarrow t)$. Hence there exists $d \in f^{-1}(G_2 \setminus \downarrow t)$ such that $x \in d \cdot C_1$ by Proposition 4.3 (1). We have $f(d) \in G_2 \setminus \downarrow t$ and $d \in x \cdot C_1^{-1}$. Thus $f(d) \not\leq t$. This contradicts that t is an upper bound of $\{f(w) : w \in x \cdot C_1^{-1}\}$. Therefore $f(x) = \bigvee \{f(w) : w \in x \cdot C_1^{-1}\}$.

“ \Leftarrow ” : Let $x, y \in G_1$ with $x \leq y$. By Proposition 3.6, we have $\{f(w) : w \in x \cdot C_1^{-1}\} \subseteq \{f(w) : w \in y \cdot C_1^{-1}\}$. Thus $f(x) = \bigvee \{f(w) : w \in x \cdot C_1^{-1}\} \leq \bigvee \{f(w) : w \in y \cdot C_1^{-1}\} = f(y)$ and so, f is monotone.

Let $U \in \beta(G_2)$ and $x \in f^{-1}(U)$. By Proposition 3.11, $x \cdot C_1^{-1}$ is directed in G_1 . Since f is monotone, we have $\{f(w) : w \in x \cdot C_1^{-1}\}$ is directed in G_2 . Hence $y^{-1} \cdot \{f(w) : w \in x \cdot C_1^{-1}\}$ is directed and $\bigvee (y^{-1} \cdot \{f(w) : w \in x \cdot C_1^{-1}\}) = y^{-1} \cdot f(x)$ by Proposition 2.5. Because $f(x) \in U$, there exists $y \in U$ such that $f(x) \in y \cdot C_2 \subseteq U$ by Proposition 4.3 (1). So $y^{-1} \cdot f(x) \in C_2$. Thus $(y^{-1} \cdot \{f(w) : w \in x \cdot C_1^{-1}\}) \cap C_2 \neq \emptyset$ by Proposition 3.12. This means that there exists $w \in x \cdot C_1^{-1}$ such that $y^{-1} \cdot f(w) \in C_2$. Hence $x \in w \cdot C_1$ and $f(w) \in y \cdot C_2 \subseteq U$. Now we will show that $w \cdot C_1 \subseteq f^{-1}(U)$. Let $z \in w \cdot C_1$, i.e., $w \in z \cdot C_1^{-1}$. Then $f(w) \leq f(z)$ for $f(z) = \bigvee \{f(w) : w \in z \cdot C_1^{-1}\}$. By Proposition 3.5, we have $y \cdot C_2$ is an upper set. Thus $f(z) \in y \cdot C_2 \subseteq U$ and $z \in f^{-1}(U)$. Hence $w \cdot C_1 \subseteq f^{-1}(U)$. Since $x \in w \cdot C_1$ and $w \cdot C_1 \in \beta(G_1)$, we have $f^{-1}(U) \in \beta(G_1)$ by Proposition 4.10. Therefore f is continuous. □

Proposition 4.12. *Let G be an l-group with the strong-positive cone C . The sets of the form $x \cdot C^{-1}, x \in G$, constitute a basis of a topology on G (we call this topology the C^{-1} -topology).*

Proof. Let $x \in G$ and $y \in x \cdot C$. Then there exists $a \in C$ such that $y = x \cdot a$. Thus $x = y \cdot a^{-1} \in y \cdot C^{-1} \subseteq \bigcup_{u \in G} (u \cdot C^{-1})$. Therefore $G \subseteq \bigcup_{x \in G} (x \cdot C^{-1}) \subseteq G$ and $G = \bigcup_{x \in G} (x \cdot C^{-1})$. Let $x, y \in G$. Since G is an l-group, we conclude that $x \wedge y$ exists and $(x \wedge y) \cdot C^{-1} \subseteq x \cdot C^{-1}$ and $(x \wedge y) \cdot C^{-1} \subseteq y \cdot C^{-1}$ by Proposition 3.6. Hence the family $\{x \cdot C^{-1} : x \in G\}$ constitutes a basis of a topology on G by Proposition 2.8. □

If $G = (G, \leq, \cdot)$ is an l-group with the positive cone P , the strong-positive cone C and $G^{op} = (G, \geq, \cdot)$ denotes the dual of G , then in G^{op} the positive cone is P^{-1} and the strong-positive cone is C^{-1} . So the C^{-1} -topology of G is equivalent to the C -topology of G^{op} . Thus we denote the C^{-1} -topology of G by $\beta(G^{op})$. Moreover, if G is a continuous l-group, then G^{op} is also a continuous l-group. So we have the following properties:

Proposition 4.13. *Let G be an l-group with the strong-positive cone C . U is an open set in the C^{-1} -topology if and only if $U = U \cdot C^{-1}$.*

Proposition 4.14. *Let G_1, G_2 be continuous l-groups with the strong-positive cones C_1, C_2 , respectively. A function $f : G_1 \rightarrow G_2$ is continuous with respect to the C^{-1} -topologies of G_1 and G_2 if and only if for any $x \in G_1$, $f(x) = \bigwedge \{f(w) : w \in x \cdot C_1\}$.*

Definition 4.15. Let G be an l -group with the strong-positive cone C . Then the common refinement of the C -topology and the C^{-1} -topology is called the bi- C -topology.

Obviously, the bi- C -topology has a basis $\{U \cap V : U \in \beta(G), V \in \beta(G^{op})\}$, where $\beta(G)$ denotes the C -topology of G and $\beta(G^{op})$ denotes the C^{-1} -topology of G .

Corollary 4.16. Let G_1, G_2 be continuous l -groups with the strong-positive cones C_1, C_2 , respectively. If a function $f : G_1 \rightarrow G_2$ satisfies: for any $x \in G_1$, $f(x) = \bigvee\{f(w) : w \in x \cdot C_1^{-1}\} = \bigwedge\{f(w) : w \in x \cdot C_1\}$, then f is continuous with respect to the bi- C -topologies of G_1 and G_2 .

Proof. This result can be proved by Proposition 4.11 and Proposition 4.14. □

Remark 4.17. In an l -group G , a basis of the bi- C -topology is $\{U \cap V : U \in \beta(G), V \in \beta(G^{op})\}$, where $\beta(G)$ denotes the C -topology of G . So a basis of the bi- C -topology of G^{op} is $\{U \cap V : U \in \beta(G^{op}), V \in \beta(G)\}$, where $\beta(G)$ denotes the C -topology of G . Therefore, open sets in the bi- C -topology of G and open sets in the bi- C -topology of G^{op} are exactly the same.

Thus we have the following corollary.

Corollary 4.18. Let G_1, G_2 be continuous l -groups with the strong-positive cones C_1, C_2 , respectively. If a function $f : G_1 \rightarrow G_2$ satisfies: for any $x \in G_1$, $f(x) = \bigwedge\{f(w) : w \in x \cdot C_1^{-1}\} = \bigvee\{f(w) : w \in x \cdot C_1\}$, then f is continuous with respect to the bi- C -topologies of G_1 and G_2 .

Proposition 4.19. Let G be a continuous l -group with the strong-positive cone C . Then the mapping

$$f : G \times G \rightarrow G, \quad (x, y) \mapsto x \cdot y$$

is continuous with respect to the C -topologies and the bi- C -topologies of $G \times G$ and G .

Proof. Let $(x, y) \in G \times G$. Because $\bigvee(x \cdot C^{-1}) = x$ and $\bigvee(y \cdot C^{-1}) = y$ by Proposition 3.11, we have $\bigvee(x \cdot C^{-1} \cdot y \cdot C^{-1}) = \bigvee(x \cdot C^{-1}) \cdot \bigvee(y \cdot C^{-1}) = x \cdot y$ by Proposition 2.6. Since $\{f(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot (C \times C)^{-1}\} = \{w_1 \cdot w_2 : w_1 \in x \cdot C^{-1}, w_2 \in y \cdot C^{-1}\} = x \cdot C^{-1} \cdot y \cdot C^{-1}$, we conclude that $\bigvee\{f(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot (C \times C)^{-1}\} = x \cdot y = f(x, y)$. Therefore f is a continuous map with respect to the C -topologies of $G \times G$ and G by Proposition 4.11.

Because $\bigwedge(x \cdot C) = x$ and $\bigwedge(y \cdot C) = y$ by Proposition 3.11, we have $\bigwedge(x \cdot C \cdot y \cdot C) = \bigwedge(x \cdot C) \cdot \bigwedge(y \cdot C) = x \cdot y$ by Proposition 2.6. Since $\{f(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot C \times C\} = \{w_1 \cdot w_2 : w_1 \in x \cdot C, w_2 \in y \cdot C\} = x \cdot C \cdot y \cdot C$, we have $\bigwedge\{f(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot C \times C\} = x \cdot y = f(x, y)$. Therefore f is a continuous map with respect to the bi- C -topologies of $G \times G$ and G by Corollary 4.16. □

Proposition 4.20. Let G be a continuous l -group with the strong-positive cone C . Then the mapping

$$f : G \rightarrow G, \quad x \mapsto x^{-1}$$

is continuous with respect to the bi- C -topology.

Proof. Let $x \in G$. Then $\bigwedge\{f(w) : w \in x \cdot C^{-1}\} = \bigwedge\{w^{-1} : w \in x \cdot C^{-1}\} = \bigwedge(x^{-1} \cdot C) = x^{-1} = f(x)$. Similarly, $\bigvee\{f(w) : w \in x \cdot C\} = f(x)$. Hence f is continuous with respect to the bi- C -topology by Corollary 4.18. □

Lemma 4.21 ([13]). Let L be a lattice and $A, B \subseteq L$. If $\bigvee A, \bigvee B$ exist in L , then $\bigvee\{a \vee b : a \in A, b \in B\} = (\bigvee A) \vee (\bigvee B)$. Dually, if $\bigwedge A, \bigwedge B$ exist in L , then $\bigwedge\{a \wedge b : a \in A, b \in B\} = (\bigwedge A) \wedge (\bigwedge B)$.

Lemma 4.22 ([18]). Let P be a continuous poset and D_1, D_2 are directed subsets of P . If $\bigvee D_1, \bigvee D_2$ exist in P , then $\bigvee\{d_1 \wedge d_2 : d_1 \in D_1, d_2 \in D_2\} = (\bigvee D_1) \wedge (\bigvee D_2)$. Dually, let F_1, F_2 be filtered subsets of P . If $\bigwedge F_1, \bigwedge F_2$ exist in P , then $\bigwedge\{f_1 \vee f_2 : f_1 \in F_1, f_2 \in F_2\} = (\bigwedge F_1) \vee (\bigwedge F_2)$.

Theorem 4.23. Let G be a continuous l -group with the bi- C -topology τ . Then (G, \cdot, τ) is a topological group and (G, \leq, τ) is a topological lattice.

Proof. We have (G, \cdot, τ) is a topological group by Proposition 4.19 and Proposition 4.20.

Let $(x, y) \in G \times G$. We will show that $f : G \times G \rightarrow G$ by $(x, y) \mapsto x \vee y$ and $g : G \times G \rightarrow G$ by $(x, y) \mapsto x \wedge y$ are continuous with respect to the bi-C-topologies of $G \times G$ and G .

Since $\bigvee(x \cdot C^{-1}) = x$ and $\bigvee(y \cdot C^{-1}) = y$ by Proposition 3.11 and $\bigvee\{f(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot (C \times C)^{-1}\} = \bigvee\{w_1 \vee w_2 : w_1 \in x \cdot C^{-1}, w_2 \in y \cdot C^{-1}\}$, we conclude that $\bigvee\{f(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot (C \times C)^{-1}\} = x \vee y = f(x, y)$ by Lemma 4.21. By the continuity of G , $x \cdot C$, $y \cdot C$ are filtered sets and $\bigwedge(x \cdot C) = x$, $\bigwedge(y \cdot C) = y$. Because $\bigwedge\{f(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot C \times C\} = \bigwedge\{w_1 \vee w_2 : w_1 \in x \cdot C, w_2 \in y \cdot C\}$, we have $\bigwedge\{f(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot C \times C\} = x \vee y = f(x, y)$ by Theorem 3.14 and Lemma 4.22. Hence f is continuous with respect to the bi-C-topologies of $G \times G$ and G by Corollary 4.16.

By Proposition 3.11, $x \cdot C^{-1}$, $y \cdot C^{-1}$ are directed sets and $\bigvee(x \cdot C^{-1}) = x$, $\bigvee(y \cdot C^{-1}) = y$. Because $\bigvee\{g(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot (C \times C)^{-1}\} = \bigvee\{w_1 \wedge w_2 : w_1 \in x \cdot C^{-1}, w_2 \in y \cdot C^{-1}\}$, we have $\bigvee\{g(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot (C \times C)^{-1}\} = x \wedge y = g(x, y)$ by Theorem 3.14 and Lemma 4.22. Since $\bigwedge(x \cdot C) = x$ and $\bigwedge(y \cdot C) = y$ by the continuity of G and $\bigwedge\{g(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot C \times C\} = \bigwedge\{w_1 \wedge w_2 : w_1 \in x \cdot C, w_2 \in y \cdot C\}$, we conclude that $\bigwedge\{g(w_1, w_2) : (w_1, w_2) \in (x, y) \cdot C \times C\} = x \wedge y = g(x, y)$ by Lemma 4.21. Hence g is continuous with respect to the bi-C-topologies of $G \times G$ and G by Corollary 4.16. Therefore (G, \leq, τ) is a topological lattice. \square

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