Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 723–733
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

On spectral gap for multicolored disordered lattice gas of exclusion processes

Ali Bey Touati, Laila Benaon, Halim Zeghdoudi*

LaPS laboratory, Badji-Mokhtar University, Box 12, Annaba, 23000, Algeria.

Communicated by M. Bohner

Abstract

We consider a system of multicolored disordered lattice gas in a volume Λ of $\mathbb{Z}^d$ driven by a disordered Markov generator similar to that of Faggionato and Martinelli [A. Faggionato, F. Martinelli, Probab. Theory Related Fields, **127** (2003), 535–608]. The aim of our work is to give a new and elementary computation of the spectral gap of multicolored disordered lattice gas which is an important step towards obtaining the hydrodynamic limit.

Keywords: A simple exclusion, Markov generator, spectral gap.

2010 MSC: 60K35, 60F10, 60J25.

1. Introduction

In this work, which is inspired on work by Faggionato and Martinelli [8], we are interested in the movement of free electrons in a crystal in the presence of impurities. Contrary to the case of non-presence of impurities where the crystal creates a periodic electromagnetic field which influences the transport of electrons, here one of the modelings of impurities, consists with regards the field a randomness and the electrons as particles evolving/moving on a network made up of sites, subjected to a principle of exclusion where the random walks describing the jumps between the sites, locally depend on the random chemical potential field (the disorder), and for the decomplexification of the model we neglect the interactions between the electrons.

More precisely, this phenomenon can be described as follows: a particle sitting on a site x of the cubic lattice $\mathbb{Z}^d$ waits an exponential time and then attempts to jump to a neighbor site y. If the site y is occupied then the jump is canceled, otherwise it is achieved with a rate c^{α}_{xy} depending only on the values (α_x, α_y) of some external quenched disorder field $\{\alpha_x\}_{x\in\mathbb{Z}^d}$ that, for simplicity, is assumed to be a collection of i.i.d. bounded random variables. Given an external chemical potential λ, the Hamiltonian of the system is defined as $H(\eta) = -\sum_x(\alpha_x + \lambda)\eta_x$, where η_x is the particle occupation number at site x. In

*Corresponding author
Email addresses: alibeytouati@gmail.com (Ali Bey Touati), lbenaoun23@gmail.com (Laila Benaon), halim.zeghdoudi@univ-annaba.dz (Halim Zeghdoudi)

doi: 10.22436/jnsa.011.05.12

Received: 2017-02-22 Revised: 2017-11-06 Accepted: 2018-03-08

these last years, several investigations are motivated to determine the speed of convergence to equilibrium of conservative stochastic dynamics. To attack this question it is necessary to estimate the spectral gap of the corresponding Markov generator or by proving a Poincaré inequality. In this direction the important achievements are the diffusive estimates established for Kawasaki dynamics in high temperature by Lu and Yau [9] and by Cancrini and Martinelli [3]. Also, Boudou et al. [2] developed a general technique based on a Bochner-type identity, to estimate spectral gap of class of Markov generator. Moreover, Caputo [4] prove Poincaré inequalities in product spaces with one or more conservation laws. In the preceding works they relate to the monocolor case.

Previous works of Dermoune and Heinrich [5, 6] were a first step to get the hydrodynamic limit of this colored disordered simple exclusion process similar to [8].

As observed in [1, 11, 12] their approach can be generalized for multicolored disordered lattice gas of exclusion processes. More precisely, we propose the explicit form for the canonical measures and spectral gap who does not depend on disorder α, although this last one plays an important role in the study of hydrodynamic limit.

The rest of this paper is organized as follows: In Section 2 we introduce the preliminaries and the notations whose we deal on dynamics in a volume Λ, Markov generator, grand canonical, canonical measures and spectral gap. Section 3 is devoted for state the main results and its proofs. Finally, we make a conclusion.

2. Preliminaries and notations

In the multicolored case similar to that of Dermoune and Martinez [7], the finite set of colors is $I = \{0, 1, \ldots, n\}$ and in $I_0 = I \cup \{0\}$, the value 0 expresses the absence of color (or that a site is empty). The set of configurations is I_0^{Λ}. In this approach we study the projection on the monocolour system and we derive an estimate of the closeness between grand canonical and canonical Gibbs measures.

Consider a finite subset Λ of the d-dimensional lattice $\mathbb{Z}^d$. To each site x, we assign a disorder, that is a random variable α_x. A configuration means an application $\eta = \{\eta_x, x \in \Lambda\} \in \Omega_\Lambda := \{0, 1, .., n\}^{\Lambda}$, such that

$$\eta_x = \begin{cases} 0, & \text{if there is no particle at } x, \\ i, & \text{if there is a particle of color "}i\text{" at } x, \text{ with } i = \overline{1, n}. \end{cases}$$

As in [8], we assume that the α_x's are i.i.d, and bounded by some constant B. The corresponding product measure (Resp. expectation) on $\Omega_D = [-B, B]^{\Lambda}$ will be denoted by $\mathbb{P}$(Resp. $\mathbb{E}$). We set for simplicity

$$\eta_x^i = \mathbf{1}_{\{\eta_x = i\}},$$

where $\eta_x^i \times \eta_x^j = 0$ for $i \neq j$.

Dynamics in the volume Λ. For a (particles) configuration $\{\eta_x, x \in \Lambda\}$ which is simply denoted by η, if $\{x, y\}$ is a pair of sites, we denoted by $\eta^{x,y}$ the configuration derived from η by permuting η_x with η_y. Namely, $\eta_x^{x,y} = \eta_y$, $\eta_y^{x,y} = \eta_x$ and the rest is unchanged. The dynamics of the particles are given by a Markov process $\{\eta(t), t \in \mathbb{R}^+\}$ and can then be described as follows: a particle at x waits an exponential time and attempts to jump to a neighbor site y. If this site is occupied then the jump is aborted, otherwise it is realized with a probabilistic rate

$$c_{x,y}^{\alpha}(\eta) = f_e(\alpha_x, \frac{\eta_x}{i}, \alpha_y, \frac{\eta_y}{i}),$$

where f_e is a bounded function on $(\mathbb{R} \times \{0, 1\})^2$ satisfying the following conditions:

1. $f_e(a, s, a', s') = f_e(a', s', a, s)$ (symmetry condition);
2. $ss' \neq 0 \Longrightarrow f_e(a, s, a', s') = 0$ (exclusion condition);
3. $ss' = 0 \Longrightarrow f_e(a, s, a', s') \geqslant \delta > 0$ (uniform bound condition);
4. $f_e(a, s, a', s') = f_e(a, s', a', s) \exp(-(s' - s)(a' - a)$ (balance condition) (see [8]).

Markov generator. These conditions allow us to define a disordered Markov generator $\mathbf{L}$. Disordered means depending on the random collection $\alpha = \{\alpha_x, x \in \Lambda\}$, which is also called disorder. The mentioned generator $\mathbf{L} := \mathbf{L}^{\alpha}_{\Lambda}$ is given for a bounded function f on Ω_Λ by

$$\mathbf{L}f(\eta) = \sum_{x,y\in\Lambda} c^{\alpha}_{x,y}(\eta)\,[f(\eta^{x,y}) - f(\eta)]\,.$$

Grand canonical and canonical measures. We consider the product Gibbs measure $\mu = \mu^{\alpha,\lambda_1,\ldots,\lambda_n}_{\Lambda}$ on Ω_Λ defined by

$$\mu(\eta^i_x) = \frac{\exp(\alpha_x + \lambda_i)}{1 + \exp(\alpha_x + \lambda)} \quad \left(\text{for } x \in \Lambda, \text{ and } i = \overline{1,n}\right). \tag{2.1}$$

In the following theorem, we construct Gibbs's measures such that the dynamics is time reversible for several external chemical potentials.

Theorem 2.1. *Set $\eta^i_x = 1_{\{\eta_x = i\}}$. Let $\mu^{\alpha,\lambda_1,\ldots,\lambda_n}_{\Lambda}$ be a disordered probability measure on Ω_Λ such that, for almost all α, under $\mu^{\alpha,\lambda_1,\ldots,\lambda_n}_{\Lambda}$, the random variables $\eta \mapsto \eta_x$, $x \in \Lambda$, are independent. Then, for almost all α, the uniform bound condition 3 and the detailed balance condition 4 holds if and only if there exists $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that*

(I) $\sum_{i=1}^{n} e^{\lambda_i} = e^{\lambda}$;
(II) $\mu^{\alpha}(\eta^i_x) = e^{\alpha_x+\lambda_i}/(1 + e^{\alpha_x+\lambda_1} + \cdots + e^{\alpha_x+\lambda_n})(x \in \Lambda)$.

Proof. See [12]. □

We define the Hamiltonian $H^{\alpha,\lambda_1,\ldots,\lambda_n}_{\Lambda}$ of the system as well as the empirical and annealed chemical potential $\lambda_1, \ldots, \lambda_n$ in the volume Λ as follows:

$$H^{\alpha,\lambda_1,\ldots,\lambda_n}_{\Lambda}(\eta) = -\sum_{x\in\Lambda}[\alpha_x\eta_x + \sum_{i=1}^{n}\lambda_i\eta^i_x],$$

where λ is such that $e^{\lambda} = \sum_{i=1}^{n} e^{\lambda_i}$.

Then the corresponding grand canonical Gibbs measure on Ω_Λ coincides with $\mu^{\alpha,\lambda_1,\ldots,\lambda_n}_{\Lambda}$, namely

$$\mu^{\alpha,\lambda_1,\ldots,\lambda_n}_{\Lambda}(\{\eta\}) = Z_1 \exp(-H^{\alpha,\lambda_1,\ldots,\lambda_n}_{\Lambda}(\eta)),$$

where Z_1 is a normalizing constant. For each $(m_1, \ldots, m_n) \in [0,1]^n$ such that $\sum_{i=1}^{n} m_i \leqslant 1$, we define also the canonical measures $\nu^{\alpha}_{\Lambda,m_1,\ldots,m_n}$ as follows. Let $N^i_\Lambda(\eta)$ be the number of particles of color "i" in the volume Λ and $m \in \left\{0, \frac{1}{|\Lambda|}, \ldots, 1\right\}$. Then

$$\nu^{\alpha}_{\Lambda,m_1,\ldots,m_n}(\{\eta\}) = \mu^{\alpha,\lambda_1,\ldots,\lambda_n}_{\Lambda}(\{\eta\}|\, N^1_\Lambda = |\Lambda|\, m_1, \ldots, N^n_\Lambda = |\Lambda|\, m_n).$$

Spectral gap. Note that $I - \mathcal{P}$ is a non-negative, bounded self-adjoint operator on $L^2(\nu)$. Any constant is an eigenfunction with eigenvalue 0 and the spectral gap $\lambda = \lambda(|\Lambda|)$ is defined as

$$\lambda(|\Lambda|) = \inf_{f\in L^2(\nu),\nu(f)=0} \frac{\nu(f(I-\mathcal{P})f)}{\nu(f^2)},$$

where $\nu(f)$ stands for the expectation $\int f d\nu$. Note that this linear operator $\mathcal{P}$ on $L^2(\nu)$ preserves positivity, and is of norm less or equal to one, and satisfies $\mathcal{P}I = I$. These properties ensure that $\mathcal{P} - I$ is a Markov generator. Moreover, note that $\mathcal{P} - I$ has reversible (and thus invariant) measure ν since $\nu(f(\mathcal{P} - I)g) = \nu(g(\mathcal{P} - I)f$ for all f, $g \in L^{\dot{2}}(\nu)$.

In the continuation, we assume that the random variables $\eta \mapsto \eta_x$, $x \in \Lambda$ are independent with respect to μ (w.r.t μ) and we will use the notation

$$\xi_x(\eta) = \mathbf{1}_{[\eta(x)\neq 0]}$$

for $x \in \Lambda$ so that $\xi(\eta) \in \{0,1\}^{\Lambda}$ denotes the configuration of occupied sites associated to η.

Goal. Previous works [5, 6] were first steps to get the hydrodynamic limit of this colored disordered simple exclusion process similar to [8]. The aim of the present paper is the calculation of the spectral gap for multicolored disordered lattice gas of exclusion processes which plays an important role in the study of hydrodynamic limit with $c^{\alpha}_{x,y}(\eta) = 1$. Set the Dirichlet form defined by

$$\mathfrak{D}(f) = \frac{1}{|\Lambda|} \sum_{x,y\in\Lambda} \nu(|f(\eta^{x,y}) - f(\eta)|^2)$$

and

$$\mathcal{P}f = \frac{1}{|\Lambda|} \sum_{y\in\Lambda} \nu(f|\eta_y) \ \left(\text{for every } f \in L^2(\nu)\right).$$

Let be $s_1 < s_2 < \cdots < s_{3|\Lambda|}$ eigenvalues of $3|\Lambda|$ by $3|\Lambda|$ blocks symmetric matrix $\mathbf{A}$, so that $s_1 = s_1(|\Lambda|)$ is the smallest eigenvalue of matrix $\mathbf{A} = \begin{bmatrix} \mathbf{Y} & \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Y} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Y} \end{bmatrix}$, where $\mathbf{Y} = (y_{ij})$ is a $|\Lambda|$ by $|\Lambda|$ with $y_{ii} = 1 + \frac{1}{|\Lambda|(|\Lambda|-1)}, y_{ij} = \frac{1}{|\Lambda|(|\Lambda|-1)}$, $\forall i \neq j$ and $\mathbf{Z} = (z_{ij})$ is a $|\Lambda|$ by $|\Lambda|$ with $z_{ii} = \frac{|\Lambda|-1}{|\Lambda|}, z_{ij} = 0, \forall i \neq j$.

3. Main result

We are now able to state our main result.

Theorem 3.1. *For $|\Lambda| \geqslant n$. The spectral gap is equal to smallest eigenvalue of matrix A.*

Note that by definitions of ν and μ, and by the fact that η^i_x and η^j_y are independent w.r.t. μ for all $i, j \in \{1, \ldots, n\}$, we have:

$$\nu\left(\eta^i_x\eta^j_y\right) = \frac{\mu(\eta^i_x)\mu(\eta^j_y)}{\mu(S)} \times \mu\left(N^i_{\Lambda\setminus\{x,y\}} = N^i - \delta^i_i - \delta^i_j, N^j_{\Lambda\setminus\{x,y\}} = N^j - \delta^j_i - \delta^j_j\right), \tag{3.1}$$

$$\nu\left(\eta^i_x\eta^i_y\right) = \frac{\mu(\eta^i_x)\mu(\eta^i_y)}{\mu(S)} \times \mu\left(N^i_{\Lambda\setminus\{x,y\}} = N^i - 2\delta^i_i, N^j_{\Lambda\setminus\{x,y\}} = N^j - 2\delta^j_j\right), \tag{3.2}$$

and

$$\nu\left(\eta^i_x\right) = \frac{\mu(\eta^i_x)}{\mu(S)} \times \mu\left(N^i_{\Lambda\setminus\{x\}} = N^i - \delta^i_i, N^j_{\Lambda\setminus\{x\}} = N^j - \delta^j_i\right), \tag{3.3}$$

where δ^j_i is the Kronecker symbol. We put

$$\theta^{i,j}_{x,y} = \mu\left(N^i_{\Lambda\setminus\{x,y\}} = N^i - \delta^i_i - \delta^i_j, N^-_{\Lambda\setminus\{x,y\}} = N^j - \delta^j_i - \delta^j_j\right),$$
$$\theta^{i,i}_{x,y} = \mu\left(N^i_{\Lambda\setminus\{x,y\}} = N^i - 2\delta^i_i, N^j_{\Lambda\setminus\{x,y\}} = N^j - 2\delta^j_j\right),$$

and

$$\theta^i_x = \mu\left(N^i_{\Lambda\setminus\{x\}} = N^i - \delta^i_i, N^j_{\Lambda\setminus\{x\}} = N^j - \delta^j_i\right).$$

Moreover, as it is shown in Caputo [4], it is sufficient to prove the Theorem 3.1 for $f = \sum_{x\in\Lambda}\sum_{i=1}^n f^i_x\overline{\eta^i_x}$, where $\overline{\eta^i_x} = \eta^i_x - \nu(\eta^i_x)$, with the row $(f^i_x, x \in \Lambda, i = \overline{1,n})$ of real numbers.

The following theorem proposes the explicit form of the canonical measures for a multicolored disordered lattice gas.

Theorem 3.2. *We have for $|\Lambda| \geqslant n$*

1) $\nu\left(\eta^i_x\eta^j_y\right) = \frac{N^iN^j}{|\Lambda|(|\Lambda|-1)} e^{\alpha_x(1-\xi_x(\eta))+\alpha_y(1-\xi_y(\eta))+\sum\limits_{\substack{k=1\\k\neq i,j}}^{n}\lambda_k N^k}$;

2) $\nu\left(\eta_x^i\eta_y^i\right) = \frac{N^i(N^i-1)}{|\Lambda|(|\Lambda|-1)} e^{\alpha_x(1-\xi_x(\eta))+\alpha_y(1-\xi_y(\eta))+\sum\limits_{\substack{k=1\\k\neq i,j}}^{n}\lambda_k N^k}$;

3) $\nu\left(\eta_x^i\right) = \frac{N^i}{|\Lambda|} e^{\alpha_x(1-\xi_x(\eta))+\sum\limits_{\substack{k=1\\k\neq i}}^{n}\lambda_k N^k}$.

Proof of Theorem 3.2.

Computation of $\mu(S)$, $\theta_{x,y}^{i,j}$, $\theta_{x,y}^{i,i}$, and θ_x^i. We have

$$\mu(S) = \sum_{\substack{V^1\cup\cdots\cup V^n\cup V^0=\Lambda\\ |V^1|=N^1,\ldots,|V^n|=N^n,|V^0|=|\Lambda|-N}} \prod_{z\in V^1}\mu\left(\eta_z^1\right)\times\cdots\times\prod_{z\in V^-}\mu\left(\eta_z^n\right)\times\prod_{z\in V^0}(1-\mu(\eta_z)).$$

The sum is taken on $\left\{V^1,\ldots,V^n,V^0\right\}$ such that $V^1,\ldots,V^n,V^0\subset\Lambda$, where V^i is the set of sites occupied by particles of color "i" with $i=\overline{1,n}$, V^0 is the set of empty sites, and $N=\sum\limits_{i=1}^{n}N^i$. The formula (2.1) implies that:

$$\begin{aligned}\mu(S) &= \sum_{V^+\cup V^-\cup V^0=\Lambda} e^{\sum\limits_{i=1}^{n}\lambda_i N^i}\prod_{z\in V^1}\frac{e^{\alpha_z}}{(1+e^{\alpha_z+\lambda})}\times\cdots\times\prod_{z\in V^n}\frac{e^{\alpha_z}}{(1+e^{\alpha_z+\lambda})}\times\prod_{z\in V^0}\frac{1}{(1+e^{\alpha_z+\lambda})}\\ &= \frac{e^{\sum\limits_{i=1}^{n}\lambda_i N^i}}{\prod\limits_{z\in\Lambda}(1+e^{\alpha_z+\lambda})}\sum_{V^1\cup\cdots\cup V^n\cup V^0=\Lambda}\ \prod_{z\in V^1\cup\cdots\cup V^n} e^{\alpha_z}.\end{aligned}$$

Note that the number of $\left\{V^1,\ldots,V^n,V^0\right\}$ is the number of permutations with repetition, therefore according to the fact that

$$\sum_{z\in V^1\cup\cdots\cup V^n}\alpha_z = \sum_{z\in\Lambda}\alpha_z\xi_z = u,$$

where

$$\xi_z(\eta) = \mathbf{1}_{[\eta(x)\neq 0]},$$

we have

$$\mu(S) = \frac{e^{\sum\limits_{k=1}^{n}\lambda_k N^k}}{\prod\limits_{z\in\Lambda}(1+e^{\alpha_z+\lambda})}\frac{|\Lambda|!}{N^1!\times\cdots\times N^n!\,(|\Lambda|-N)!}e^{u}. \tag{3.4}$$

Using the following notations

$$\sum_{z\in\Lambda\setminus\{x,y\}}\alpha_z\xi_z(\eta) = u_{x,y} \text{ and } \sum_{z\in\Lambda\setminus\{x\}}\alpha_z\xi_z(\eta) = u_x$$

we find with the same manner that

$$\theta_{x,y}^{i,j} = \frac{e^{\lambda^i(N^i-1)+\lambda^j(N^j-1)}}{\prod\limits_{z\in\Lambda\setminus\{x,y\}}(1+e^{\alpha_z+\lambda})}\frac{(|\Lambda|-2)!}{N^1!\times\cdots\times(N^i-1)!(N^j-1)!\times\cdots\times N^n!\,(|\Lambda|-N)!}e^{u_{x,y}}, \text{ with } i\neq j,$$

$$\theta_{x,y}^{i,i} = \frac{e^{\lambda_i(N^i-2)+\lambda_j N^j}}{\prod\limits_{z\in\Lambda\setminus\{x,y\}}(1+e^{\alpha_z+\lambda})}\frac{(|\Lambda|-2)!}{N^1!\times\cdots\times(N^i-2)!(N^j)!\times\cdots\times N^n!\,(|\Lambda|-N)!}e^{u_{x,y}},$$

and

$$\theta_x^i = \frac{e^{\lambda_i(N^i-1)+\lambda_j N^j}}{\prod\limits_{z\in\Lambda\setminus\{x\}}(1+e^{\alpha_z+\lambda})}\frac{(|\Lambda|-1)!}{N^1!\times\cdots\times(N^i-1)!N^j!\times\cdots\times N^n!\,(|\Lambda|-N)!}e^{u_x}.$$

Then we can write

$$\mu(\eta_x^i)\mu(\eta_y^j)\theta_{x,y}^{i,j} = \frac{e^{\lambda_i N^i+\lambda_j N^j}}{\prod\limits_{z\in\Lambda}(1+e^{\alpha_z+\lambda})}\frac{(|\Lambda|-2)!}{N^1!\times\cdots\times(N^i-1)!(N^j-1)!\times\cdots\times N^n!\,(|\Lambda|-N)!}e^{u_{x,y}+\alpha_x+\alpha_y}, \quad (3.5)$$

$$\mu(\eta_x^i)\mu(\eta_y^i)\theta_{x,y}^{i,i} = \frac{e^{\lambda_i N^i+\lambda_j N^j}}{\prod\limits_{z\in\Lambda}(1+e^{\alpha_z+\lambda})}\frac{(|\Lambda|-2)!}{N^1!\times\cdots\times(N^i-2)!(N^j)!\times\cdots\times N^n!\,(|\Lambda|-N)!}e^{u_{x,y}+\alpha_x+\alpha_y}, \quad (3.6)$$

and

$$\mu(\eta_x^i)\theta_x^i = \frac{e^{\lambda_i N^i+\lambda_j N^j}}{\prod\limits_{z\in\Lambda}(1+e^{\alpha_z+\lambda})}\frac{(|\Lambda|-1)!}{N^1!\times\cdots\times(N^i-1)!N^j!\times\cdots\times N^n!\,(|\Lambda|-N)!}e^{u_x+\alpha_x}. \quad (3.7)$$

We replace (3.4), (3.5), (3.6), (3.7) in (3.1), (3.2), (3.3) respectively, we conclude that

$$\nu\left(\eta_x^i\eta_y^j\right) = \frac{N^iN^j}{|\Lambda|\,(|\Lambda|-1)}e^{\alpha_x(1-\xi_x(\eta))+\alpha_y(1-\xi_y(\eta))+\sum\limits_{\substack{k=1\\k\neq i,j}}^{n}\lambda_k N^k},$$

$$\nu\left(\eta_x^i\eta_y^i\right) = \frac{N^i(N^i-1)}{|\Lambda|\,(|\Lambda|-1)}e^{\alpha_x(1-\xi_x(\eta))+\alpha_y(1-\xi_y(\eta))+\sum\limits_{\substack{k=1\\k\neq i,j}}^{n}\lambda_k N^k},$$

and

$$\nu\left(\eta_x^i\right) = \frac{N^i}{|\Lambda|}e^{\alpha_x(1-\xi_x(\eta))+\sum\limits_{\substack{k=1\\k\neq i,j}}^{n}\lambda_k N^k},$$

which achieves the proof. □

Proof of Theorem 3.1.

Step1. We only take three colors. Now, we take the matrix representations $\nu(f^2)$ and $\nu(f\mathcal{P}f)$ used in [6]. Let for $x,y\in\Lambda$ and $i,j\in\{1,2,3\}$

$$(C_{xy}^{ij}) = \nu(\eta_x^i;\eta_y^j),\ R_{xy}^{ij} = \frac{C_{xy}^{ij}}{\sqrt{C_{xx}^{ij}C_{yy}^{ij}}},$$

where $C=(C_{xy}^{ij})$ is a $3\,|\Lambda|$ by$3\,|\Lambda|$ covariance matrix and $R=(R_{xy}^{ij})$ the corresponding correlation matrix. Let $(f_x^i, x\in\Lambda, i=\overline{1,3})\in\mathbb{R}^{3|\Lambda|}$, we will identify $f=\sum_{x\in\Lambda}f_x^1\overline{\eta_x^1}+f_x^2\overline{\eta_x^2}+f_x^3\overline{\eta_x^3}$, where $\overline{\eta_x^i}=\eta_x^i-\nu(\eta_x^i)$, hence we can write

$$\nu(f^2) = fCf^T, \quad (3.8)$$

$$\nu(f\mathcal{P}f) = \frac{1}{|\Lambda|}fCDCf^T, \quad (3.9)$$

where $D=\left(D_{xy}^{ij}\right)$ is the $3\,|\Lambda|$ by$3\,|\Lambda|$ symmetric matrix defined by

$$D_{xx}^{ij} = \frac{(2\delta_i^j-1)C_{xx}^{ij}}{\det(C_{xx})},\ D_{xy}^{ij}=0 \text{ for } x\neq y,$$

where δ_i^j denotes the Kronecker symbol. Now, we shall find a non-negative matrix Q and a row g such that

$$\nu(f^2) = gQg^T, \ \nu(f\mathcal{P}f) = \frac{1}{|\Lambda|} gQ^2 g^T$$

if we set furthermore

$$\begin{gathered} h = Q^{\frac{1}{2}} g \quad \text{and} \quad \Gamma = I - Q, \\ \nu(f^2) = hh^T, \nu(f(I-\mathcal{P})f = h\left[\frac{(|\Lambda|-1)\,I}{|\Lambda|} + \frac{\Gamma}{|\Lambda|}\right] h^T. \end{gathered} \tag{3.10}$$

We seek for $3\,|\Lambda|$ by $3\,|\Lambda|$ upper triangular matrix U such that $D = U^T U$, where

$$D = \begin{bmatrix} D^{11} & D^{12} & D^{13} \\ D^{13} & D^{22} & D^{23} \\ D^{13} & D^{23} & D^{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} U^{11} & 0 & 0 \\ U^{12} & U^{22} & 0 \\ U^{13} & U^{23} & U^{33} \end{bmatrix}$$

so that U must satisfy the following block identities

$$\begin{aligned} &U^{21} = U^{31} = U^{32} = 0, \\ &D^{11} = U^{11}U^{11} + U^{12}U^{12} + U^{13}U^{13}, \\ &D^{12} = U^{12}U^{22} + U^{13}U^{23}, \\ &D^{13} = U^{13}U^{33}, \\ &D^{22} = U^{22}U^{22} + U^{23}U^{23}, \\ &D^{23} = U^{33}U^{23}, \\ &D^{33} = U^{33}U^{33}. \end{aligned}$$

This implies that $U_{xy}^{ij} = 0$, if $x \neq y$ and

$$U_{xx}^{33} = \sqrt{D_{xx}^{33}}, \quad U_{xx}^{23} = \frac{D_{xx}^{23}}{\sqrt{D_{xx}^{33}}}, \quad U_{xx}^{13} = \frac{D_{xx}^{13}}{\sqrt{D_{xx}^{33}}}, \quad U_{xx}^{22} = \sqrt{D_{xx}^{22} - \frac{|D_{xx}^{23}|^2}{D_{xx}^{33}}}, \tag{3.11}$$

$$U_{xx}^{12} = \frac{D_{xx}^{12} - \frac{D_{xx}^{13} D_{xx}^{23}}{D_{xx}^{33}}}{\sqrt{D_{xx}^{22} - \frac{|D_{xx}^{23}|^2}{D_{xx}^{33}}}}, \quad U_{xx}^{11} = \sqrt{D_{xx}^{11} - \frac{D_{xx}^{12} - \frac{D_{xx}^{13} D_{xx}^{23}}{D_{xx}^{33}}}{D_{xx}^{22} - \frac{|D_{xx}^{23}|^2}{D_{xx}^{33}}} - \frac{|D_{xx}^{13}|^2}{D_{xx}^{33}}}. \tag{3.12}$$

Since D has a dominating diagonal. Set $g = fU^{-1}$ and $Q = UCU^T$, so that (3.8) and (3.9) yield (3.10) and thus (3.11), (3.12).

Using Theorem 3.2 for $i,j \in \{1,2,3\}$, we find

$$\inf_{f\in L^2(\nu), \nu(f)=0} \frac{\nu(f(I-\mathcal{P})f}{\nu(f^2)} = \inf_{h\neq 0} \frac{hAh^T}{hh^T}.$$

There exists an orthogonal matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{S}\mathbf{P}$ where $\mathbf{S} = \mathrm{Diag}(s_1, s_2, \ldots, s_{3|\Lambda|})$ is a diagonal matrix whose diagonal elements are eigenvalues of $\mathbf{A}$. If $\mathbf{P}$ is partitioned as $\mathbf{P} = (e_1, e_2, \ldots, e_{3|\Lambda|})$ where e_i is an eigenvector of $\mathbf{A}$, then $\mathbf{A}$ can be written as

$$\mathbf{A} = \sum_{i=1}^{3|\Lambda|} e_i e_i^T.$$

If $h \neq 0$ we have

$$h = \sum_{i=1}^{3|\Lambda|} h_i e_i,$$

we deduce

$$\frac{h\mathbf{A}h^T}{hh^T} = \frac{\sum_{i=1}^{3|\Lambda|} s_i h_i^2}{\sum_{i=1}^{3|\Lambda|} h_i^2} \geqslant s_1 \frac{\sum_{i=1}^{3|\Lambda|} h_i^2}{\sum_{i=1}^{3|\Lambda|} h_i^2} = s_1.$$

Now, if we choose $h = e_1$, therefore

$$\inf_{h\neq 0} \frac{h\mathbf{A}h^T}{hh^T} = s_1,$$

which achieves the proof.

Step 2. Now, we seek the exact value of the smallest eigenvalue of matrix $\mathbf{A}$. Let $\mathbf{A} = \begin{bmatrix} Y & Z & Z \\ Z & Y & Z \\ Z & Z & Y \end{bmatrix}$. For simplicity put $|\Lambda| = m$, $\alpha = \dfrac{1}{m(m-1)}$, and $\beta = \dfrac{m-1}{m}$, so Y is a $m \times m$ matrix with entries $y_{ii} = 1 + \alpha$, $y_{ij} = \alpha$ if $i \neq j$ and Z is a $m \times m$ matrix with entries $z_{ii} = \beta$, $z_{ij} = 0$ if $i \neq j$.

Let

$$\mathcal{P}(\lambda) = \det(\mathbf{A} - \lambda I_{3m}) = \begin{vmatrix} Y-\lambda I_m & Z & Z \\ Z & Y-\lambda I_m & Z \\ Z & Z & Y-\lambda I_m \end{vmatrix},$$

where I_{3m} (I_m) denotes the identity matrix of size $3m \times 3m$ $(m \times m)$, be the characteristic polynomial of $\mathbf{A}$. According to the fact that $Y - \lambda I_m$ and Z commute, $\mathcal{P}(\lambda)$ take the form

$$\begin{aligned}
\mathcal{P}(\lambda) &= \det\Big((Y-\lambda I_m)^3 - (Y-\lambda I_m)Z^2 - Z^2(Y-\lambda I_m) + Z^3 + Z^3 - Z^2(Y-\lambda I_m)\Big) \\
&= \det\Big((Y-\lambda I_m)^3 - 3Z^2(Y-\lambda I_m) + 3Z^3 - Z^3\Big) \\
&= \det\Big((Y-\lambda I_m)^3 - Z^3 - 3Z^2(Y-\lambda I_m - Z)\Big) \\
&= \det\Big((Y-\lambda I_m - Z)\Big((Y-\lambda I_m)^2 + Z(Y-\lambda I_m) - 2Z^2\Big)\Big),
\end{aligned}$$

since

$$(Y-\lambda I_m)^2 = (Y-\lambda I_m - Z + Z)^2 = (Y-\lambda I_m - Z)^2 + 2Z(Y-\lambda I_m - Z) + Z^2,$$

then

$$\begin{aligned}
(Y-\lambda I_m)^2 + Z(Y-\lambda I_m) - 2Z^2 &= (Y-\lambda I_m - Z)^2 + 2Z(Y-\lambda I_m - Z) + Z^2 + Z(Y-\lambda I_m - Z) + Z^2 - 2Z^2 \\
&= (Y-\lambda I_m - Z)(Y-\lambda I_m + 2Z)
\end{aligned}$$

and

$$\mathbf{P}(\lambda) = \det(Y-\lambda I_m - Z)^2(Y-\lambda I_m + 2Z) = (\det(Y-\lambda I_m - Z))^2 \det(Y-\lambda I_m + 2Z).$$

Now to compute $\det(Y-\lambda I_m - Z) = P_m$, we have

$$P_m = \begin{vmatrix} 1+\alpha-\beta-\lambda & \alpha & \alpha & . & . & \alpha \\ \alpha & 1+\alpha-\beta-\lambda & \alpha & . & . & \alpha \\ \alpha & \alpha & . & . & . & \alpha \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \alpha & . & . & . & . & 1+\alpha-\beta-\lambda \end{vmatrix}$$

and by using the linear transformation on the rows, we obtain:

$$P_m = \begin{vmatrix} 1-\beta-\lambda & -(1-\beta-\lambda) & 0 & 0 & . & 0 \\ \alpha & 1+\alpha-\beta-\lambda & \alpha & . & . & \alpha \\ \alpha & \alpha & . & . & . & \alpha \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \alpha & . & . & . & . & 1+\alpha-\beta-\lambda \end{vmatrix}$$

$$= (1-\beta-\lambda) \begin{vmatrix} 1+\alpha-\beta-\lambda & \alpha & \alpha & . & . & \alpha \\ \alpha & 1+\alpha-\beta-\lambda & \alpha & . & . & \alpha \\ \alpha & \alpha & . & . & . & \alpha \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \alpha & . & . & . & . & 1+\alpha-\beta-\lambda \end{vmatrix}$$

$$+ (1-\beta-\lambda) \begin{vmatrix} \alpha & \alpha & \alpha & . & . & \alpha \\ \alpha & 1+\alpha-\beta-\lambda & \alpha & . & . & \alpha \\ \alpha & \alpha & . & . & . & \alpha \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \alpha & . & . & . & . & 1+\alpha-\beta-\lambda \end{vmatrix}.$$

We define P_m recursively. After a linear transformation on the rows, the second determinant Q_{m-1} on the right-hand side becomes

$$Q_{m-1} = \begin{vmatrix} 0 & -(1-\beta-\lambda) & 0 & 0 & . & 0 \\ \alpha & 1+\alpha-\beta-\lambda & \alpha & . & . & \alpha \\ \alpha & \alpha & . & . & . & \alpha \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \alpha & . & . & . & . & 1+\alpha-\beta-\lambda \end{vmatrix},$$

which implies that

$$Q_{m-1} = (1-\beta-\lambda)\, Q_{m-2} \quad \text{and} \quad P_m = (1-\beta-\lambda)\, P_{m-1} + (1-\beta-\lambda)\, Q_{m-1},$$

hence

$$Q_{m-1} = (1-\beta-\lambda)^{m-3}\, Q_2 \quad \text{where} \quad Q_2 = \begin{vmatrix} \alpha & \alpha \\ \alpha & 1+\alpha-\beta-\lambda \end{vmatrix} = \alpha\,(1-\beta-\lambda),$$

so that

$$Q_{m-1} = \alpha\,(1-\beta-\lambda)^{m-2},$$

and

$$P_m = (1-\beta-\lambda)\, P_{m-1} + \alpha\,(1-\beta-\lambda)^{m-1}, \tag{3.13}$$

then

$$P_{m-1} = (1-\beta-\lambda)\, P_{m-2} + \alpha\,(1-\beta-\lambda)^{m-2},$$

which we substitute in (3.13) so

$$P_m = (1-\beta-\lambda)\left((1-\beta-\lambda)\, P_{m-2} + \alpha\,(1-\beta-\lambda)^{m-2}\right) + \alpha\,(1-\beta-\lambda)^{m-1}$$

$$\begin{aligned} P_m &= (1-\beta-\lambda)\left((1-\beta-\lambda)\, P_{m-2} + \alpha\,(1-\beta-\lambda)^{m-2}\right) + \alpha\,(1-\beta-\lambda)^{m-1} \\ &= (1-\beta-\lambda)^2\, P_{m-2} + 2\alpha\,(1-\beta-\lambda)^{m-1}. \end{aligned}$$

We repeat this procedure for $P_{m-2}, P_{m-3}, \ldots, P_3$, to obtain

$$P_m = (1-\beta-\lambda)^{m-2} P_2 + (m-2)\,\alpha\,(1-\beta-\lambda)^{m-1},$$

since

$$P_2 = \begin{vmatrix} 1+\alpha-\beta-\lambda & \alpha \\ \alpha & 1+\alpha-\beta-\lambda \end{vmatrix} = (1-\beta-\lambda)(1+2\alpha-\beta-\lambda),$$

then,

$$\begin{aligned} P_m = \det(Y-\lambda I_m - Z) &= (1-\beta-\lambda)^{m-1}(1+2\alpha-\beta-\lambda) + (m-2)\,\alpha\,(1-\beta-\lambda)^{m-1} \\ &= (1-\beta-\lambda)^{m-1}(1+m\alpha-\beta-\lambda). \end{aligned}$$

With the same manner, we obtain that

$$\det(Y-\lambda I_m + 2Z) = (1+2\beta-\lambda)^{m-1}(1+m\alpha+2\beta-\lambda).$$

Finally

$$\mathbf{P}(\lambda) = (1-\beta-\lambda)^{2m-2}(1+m\alpha-\beta-\lambda)^2(1+2\beta-\lambda)^{m-1}(1+m\alpha+2\beta-\lambda).$$

We substitute the real values of α and β, then

$$\mathbf{P}(\lambda) = \left(\frac{1}{m}-\lambda\right)^{2m-2}\left(\frac{1}{m}+\frac{1}{m-1}-\lambda\right)^2\left(1+2\frac{m-1}{m}-\lambda\right)^{m-1}\left(1+\frac{1}{m-1}+2\frac{m-1}{m}-\lambda\right).$$

We conclude that the smallest eigenvalue of $\mathbf{A}$ is $s_1 = \dfrac{1}{m}$. □

4. Conclusion

We conclude that the spectral gap does not depend on the disorder α, color and then, finding the exact value of the spectral gap is the smallest eigenvalue (s_1) of $\mathbf{A}$, which $s_1 = \dfrac{1}{|\Lambda|}$.

For future studies, we can consider more general cases such as follows.

- Calculation of the spectral gap of a generalized exclusion process for a multicolored disordered system.
- Calculation of the spectral gap such that the matrix A is random.
- Calculation of the spectral gap for lattice gas evolving in a bounded cylinder of length 2N+1 and interacting via a Neuman Kac interaction of range N, in contact with particles reservoirs at different densities (see [10]).

Acknowledgment

The authors are grateful for the comments and suggestions by the referee and the Editor. Their comments and suggestions greatly improved the article. Also, this work was financed by European Mathematical Society.

References

[1] B. T. Ali, H. Zeghdoudi, H. Boutabia, *Computation of spectral gap for a colored disordered lattice gas*, Afr. Stat., **6** (2011), 371–380. 1

[2] A.-S. Boudou, P. Caputo, P. Dai Pra, G. Posta, *Spectral gap estimates for interacting particle systems via a Bochner-type identity*, J. Funct. Anal., **232** (2006), 222–258. 1

[3] N. Cancrini, F. Martinelli, *On the spectral gap of Kawasaki dynamics under a mixing condition revisited.Probabilistic techniques in equilibrium and nonequilibrium statistical physics*, J. Math. Phys., **41** (2000), 1391–1423. 1

[4] P. Caputo, *Spectral gap inequalities in product spaces with conservation laws. Stochastic analysis on large scale interacting systems*, Adv. Stud. Pure Math., Math. Soc. Japan, Tokyo, (2004), 53–88. 1, 3

[5] A. Dermoune, P. Heinrich, *A small step towards the hydrodynamic limit of a colored disordered lattice gas*, C. R. Math. Acad. Sci. Paris, **339** (2004), 507–511. 1, 2

[6] A. Dermoune, P. Heinrich, *Spectral gap inequality for a colored disordered lattice gas*, Sminaire de Probabilits XLI, Lecture Notes in Math., Springer, **2008** (2008), 18 pages. 1, 2, 3

[7] A. Dermoune, S. Martinez, *Around multicolour disordered lattice gas*, J. Stat. Phys., **123** (2006), 181–192. 2

[8] A. Faggionato, F. Martinelli, *Hydrodynamic limit of a disordered lattice gas*, Probab. Theory Related Fields, **127** (2003), 535–608. 1, 2, 2, 2

[9] S. L. Lu, H.-T. Yau, *Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics*, Comm. Math. Phys., **156** (1993), 399–433. 1

[10] M. Mourragui, *Large deviations of the empirical current for the boundary driven Kawasaki process with long range interaction*, ALEA Lat. Am. J. Probab. Math. Stat., **11** (2014), 643–678. 4

[11] H. Zeghdoudi, H. Boutabia, *Computation for the Canonical Measures of a Colored Disordered Lattice Gas and Spectral Gap*, J. Math. Phys., **2009** (2009), 8 pages. 1

[12] H. Zeghdoudi, H. Boutabia, *Gibbs's Measures of a Multi-Colored Disordered Lattice Gas*, Afr. J. Math. Phys., **10** (2011), 49–54. 1, 2