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On m-skew complex symmetric operators

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Abstract

In this paper, the definition of m-skew complex symmetric operators is introduced. Firstly, we prove that $\Delta_m^-(T)$ is complex symmetric with the conjugation C and give some properties of $\Delta_m^-(T)$. Secondly, let T be m-skew complex symmetric with conjugation C, if n is odd, then T^n is m-skew complex symmetric with conjugation C; if n is even, with the assumption $T^*CTC = CTCT^*$, then T^n is m-complex symmetric with conjugation C. Finally, we give some properties of m-skew complex symmetric operators.

Keywords: m-skew complex symmetric operator, conjugation, spectral.

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1. Introduction

Throughout this paper, we denote by $\mathcal{H}$ a complex separable Hilbert space endowed with the inner product $\langle .,. \rangle$ and by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_{su}(T)$, $\sigma_{comp}(T)$, $\sigma_r(T)$, $\sigma_c(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, and $\sigma_{re}(T)$ for the spectrum, the point spectrum, the approximate point spectrum, the surjective spectrum, the compression spectrum, the residual spectrum, the continuous spectrum, the essential spectrum, the left essential spectrum, and the right essential spectrum of T, respectively.

An operator C on $\mathcal{H}$ is called a conjugation, if C is conjugate-linear, $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be skew complex symmetric if there exists a conjugation C on $\mathcal{H}$ such that $CTC = -T^*$. An operator T is said to be complex symmetric if $CTC = T^*$ for some conjugation C on $\mathcal{H}$. This terminology is due to the fact that T is a complex symmetric operator if and only if it is unitary equivalent of a symmetric matrix with complex entries, regarded as an operator acting on an l^2-space of the appropriate dimension ([7]). All normal operators, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators and some Volterra integration operators are included in the class of complex symmetric operators ([8, 14]). A lot of authors have studied the complex symmetric operators, however, less attention has been paid to skew complex symmetric operators. There are several

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motivations for such operators. We remark that $T \in \mathcal{L}(\mathcal{H})$ is skew symmetric if and only if T admits a skew symmetric matrix representation with respect to some orthonormal basis of $\mathcal{H}$. In particular, skew symmetric matrices have many applications in pure mathematics, applied mathematics and even in engineering disciplines. Real skew symmetric matrices play an important role in function theory, the solution to linear quadratic optional control problems, robust control problems, model reduction, crack following in anisotropic materials and so on. In view of these applications, it is natural to study skew symmetric operators on the Hilbert space $\mathcal{H}$. Recently there has been growing interest in skew symmetric, which is closely related to the study of complex symmetric operators. Muneo Chō, Eungil Ko and Ji Eun Lee [2–4] have studied m-complex symmetric operators.

In [10], Helton initiated the study of operators $T \in \mathcal{L}(\mathcal{H})$ which satisfy the form

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}T^{m-j} = 0.$$

In [2], Chō et al. defined m-complex symmetric operators as follows: An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an m-complex symmetric operator if there exists some conjugation C such that

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C = 0$$

for some positive integer m. In this case, we say that T is m-complex symmetric with conjugation C. Let $\Delta_m(T) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C$, then T is an m-complex symmetric operator with conjugation C, if and only if $\Delta_m(T) = 0$. Note that

$$T^*\Delta_m(T) - \Delta_m(T)(CTC) = \Delta_{m+1}(T).$$

In contrast to the m-complex symmetric operators, we define m-skew complex symmetric operators as follows: an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an m-skew complex symmetric operator if there exists some conjugation C such that

$$\sum_{j=0}^{m}\binom{m}{j}T^{*j}CT^{m-j}C = 0$$

for some positive integer m. In this case, we say that T is m-skew complex symmetric with conjugation C. Let $\Delta_m^-(T) := \sum_{j=0}^{m}\binom{m}{j}T^{*j}CT^{m-j}C$, then T is an m-skew complex symmetric operator with conjugation C if and only if $\Delta_m^-(T) = 0$. For an m-skew complex symmetric operator with conjugation C, we have

$$T^*\Delta_m^-(T) + \Delta_m^-(T)(CTC) = \Delta_{m+1}^-(T). \tag{1.1}$$

In fact,

$$\begin{aligned} T^*\Delta_m^-(T) + \Delta_m^-(T)(CTC) &= T^*\sum_{j=0}^{m}\binom{m}{j}T^{*j}CT^{m-j}C + \sum_{j=0}^{m}\binom{m}{j}T^{*j}CT^{m-j}C(CTC) \\ &= T^*\left\{\binom{m}{0}T^{*0}CT^{m}C + \binom{m}{1}T^{*}CT^{m-1}C + \cdots + \binom{m}{m}T^{*m}CT^{0}C\right\} \\ &\quad + \left\{\binom{m}{0}T^{*0}CT^{m}C + \binom{m}{1}T^{*}CT^{m-1}C + \cdots + \binom{m}{m}T^{*m}CT^{0}C\right\}CTC \\ &= \left\{\left[\binom{m}{0} + \binom{m}{1}\right]T^{*}CT^{m}C + \cdots + \left[\binom{m}{m-1} + \binom{m}{m}\right]T^{*m}CT^{1}C\right\} \\ &\quad + \left\{\binom{m}{0}T^{*0}CT^{m+1}C + \binom{m}{m}T^{*m+1}CT^{0}C\right\} \end{aligned}$$

$$\begin{aligned}
&= \binom{m+1}{0} T^{*0} C T^{m+1} C + \binom{m+1}{1} T^{*} C T^{m} C + \binom{m+1}{2} T^{*2} C T^{m} C + \cdots \\
&\quad + \binom{m+1}{m} T^{*m} C T^{1} C + \binom{m+1}{m+1} T^{*m+1} C T^{0} C \\
&= \sum_{j=0}^{m+1} \binom{m+1}{j} T^{*j} C T^{m+1-j} C = \Delta^{-}_{m+1}(T).
\end{aligned}$$

Hence, if T is m-skew complex symmetric with conjugation C, then T is n-skew complex symmetric with conjugation C for all $n \geqslant m$. It is obvious that 1-skew complex symmetric is skew complex symmetric.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the single-valued extension property, if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T-\lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and a vector $x \in \mathcal{H}$, the local resolvent set $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) \equiv x$ on G. The local spectrum of T at x is given by $\sigma_T(x) = \mathbb{C}\backslash\rho_T(x)$. We define the local spectral subspace of $T \in \mathcal{L}(\mathcal{H})$ by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have Dunford's property $(\mathcal{C})$, if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have Bishop's property (β), if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$ -valued analytic functions on G such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, we get that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be decomposable, if for every open cover $\{U, V\}$ of $\mathbb{C}$, there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{H} = \mathcal{X} + \mathcal{Y}$, $\sigma(T \mid x) \subset \overline{U}$ and $\sigma(T \mid y) \subset \overline{V}$. It is well-known that

$$\text{Decomposable} \Rightarrow \text{Bishop's property}(\beta); \text{Decomposable} \Rightarrow \text{Dunford's property}(\mathcal{C}) \Rightarrow \text{SVEP}.$$

In this paper, the definition of m-skew complex symmetric operators is introduced. Firstly, we prove that $\Delta^{-}_m(T)$ is complex symmetric with the conjugation C and give some properties of $\Delta^{-}_m(T)$. Secondly, let T be m-skew complex symmetric with conjugation C, if n is odd, then T^n is m-skew complex symmetric with conjugation C; if n is even, with the assumption $T^*CTC = CTCT^*$, then T^n is m-complex symmetric with conjugation C. Finally, we give some properties of m-skew complex symmetric operators.

2. Some Properties of $\Delta^{-}_m(T)$

Let T be an operator on $\mathcal{H}$ and C be a conjugation on $\mathcal{H}$. In [3], the following statements hold.

(i) if m is even, then $\Delta_m(T)$ is complex symmetric with the conjugation C;

(ii) if m is odd, then $\Delta_m(T)$ is skew complex symmetric with the conjugation C.

For $\Delta^{-}_m(T)$, we have the following theorem.

Theorem 2.1. *Let T be an operator on $\mathcal{H}$ and C be a conjugation on $\mathcal{H}$. Then $\Delta^{-}_m(T)$ is complex symmetric with the conjugation C.*

Proof. Since $\binom{m}{j} = \binom{m}{m-j}$, we obtain

$$\begin{aligned}
C(\Delta^{-}_m(T))^*C = C\left(\sum_{j=0}^{m}\binom{m}{j} T^{*j} C T^{m-j} C\right)^* C = C\left(\sum_{j=0}^{m}\binom{m}{j} C T^{*m-j} C T^{j}\right) C = \sum_{j=0}^{m}\binom{m}{j} C C T^{*m-j} C T^{j} C \\
= \sum_{j=0}^{m}\binom{m}{j} T^{*m-j} C T^{j} C = \sum_{j=0}^{m}\binom{m}{m-j} T^{*m-j} C T^{j} C \\
= \sum_{i=0}^{m}\binom{m}{i} T^{*i} C T^{m-i} C = \Delta^{-}_m(T).
\end{aligned}$$

Hence, $\Delta^{-}_m(T)$ is complex symmetric with conjugation C. □

Corollary 2.2. *Let T be an operator on $\mathcal{H}$ and C be a conjugation on $\mathcal{H}$. If $\Delta_m^-(T)$ is p-hyponormal, then it is normal.*

Proof. By Theorem 2.1, then $\Delta_m^-(T)$ is a complex symmetric operator with the conjugation C. Since $\Delta_m^-(T)$ is p-hyponormal, then $\Delta_m^-(T)$ is normal by [15, Lemma 3.1]. □

Corollary 2.3. *Let T be an operator on $\mathcal{H}$ and C be a conjugation on $\mathcal{H}$, then $\sigma(\Delta_m^-(T)) = \sigma_a(\Delta_m^-(T))$.*

Proof. By Theorem 2.1, then $\Delta_m^-(T)$ is a complex symmetric operator with the conjugation C. Thus $\sigma(\Delta_m^-(T)) = \sigma_a(\Delta_m^-(T))$ by [11, Lemma 3.22]. □

By (1.1), we know if T is m-skew complex symmetric with the conjugation C, then T is n-skew complex symmetric with the conjugation C for all $n \geqslant m$. In the following corollary, we state the conditions that $(m+1)$-skew complex symmetric operators become m-skew complex symmetric operators. Let us recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said a normaloid operator if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T. A vector $x \in \mathcal{L}(\mathcal{H})$ is said isotropic if $\langle x, Cx\rangle = 0$ ([6]).

Corollary 2.4. *Let $T \in \mathcal{L}(H)$ be an operator and C be a conjugation on $\mathcal{H}$. Suppose $\Delta_{m+1}^-(T) = 0$, $\Delta_m^-(T)$ is normaloid and an eigenvector corresponding to every eigenvalue in $\sigma_p(\Delta_m^-(T))$ is not isotropic. For every $\mu \in \sigma_a(\Delta_m^-(T))$, there exists a sequence $\{x_n\}$ of unit vectors and $\lambda \in \sigma(\Delta_1^-(T))$, such that $|\lambda|^m = |\mu|$ and*

$$\lim_{n\to\infty} \|(\Delta_m^-(T) - \mu)x_n\| = \lim_{n\to\infty} \|(\Delta_1^-(T) - \lambda)x_n\| = 0,$$

then $\Delta_m^-(T) = 0$.

Proof. By Theorem 2.1, then $\Delta_m^-(T)$ is complex symmetric with conjugation C. Since $\Delta_m^-(T)$ is normaloid, then there exists $\mu \in \sigma(\Delta_m^-(T))$ such that $|\mu| = \|\Delta_m^-(T)\|$. For every $\mu \in \sigma_a(\Delta_m^-(T))$, there exist a sequence $\{x_n\}$ of unit vectors and $\lambda \in \sigma(\Delta_1^-(T))$, such that $|\lambda|^m = |\mu|$ and

$$\lim_{n\to\infty} \|(\Delta_m^-(T) - \mu)x_n\| = \lim_{n\to\infty} \|(\Delta_1^-(T) - \lambda)x_n\| = 0,$$

then $\Delta_m^-(T)x_n = \mu x_n, \Delta_1^-(T)x_n = \lambda x_n$. By [12, Lemma 2.5], then $\Delta_m^-(T)^* x_n = \overline{\mu} x_n$. Moreover, since $\Delta_{m+1}^-(T) = 0$, by (1.1), we have

$$\begin{aligned} 0 = \langle \Delta_{m+1}^-(T)x_n, Cx_n\rangle &= \langle T^*\Delta_m^-(T) + \Delta_m^-(T)CTC, Cx_n\rangle \\ &= \langle \Delta_m^-(T)x_n, TCx_n\rangle + \langle CTCx_n, \Delta_m^-(T)^*Cx_n\rangle \\ &= \langle \mu x_n, TCx_n\rangle + \langle CTCx_n, \overline{\mu}Cx_n\rangle \\ &= \mu\langle (T^* + CTC)x_n, Cx_n\rangle \\ &= \mu\langle \Delta_1^-(T)x_n, Cx_n\rangle \\ &= \mu\lambda\langle x_n, Cx_n\rangle. \end{aligned}$$

Since $\langle x, Cx\rangle \neq 0$ by the hypothesis, so $\langle x_n, Cx_n\rangle \neq 0$. Hence $\Delta_m^-(T) = 0$. □

3. m-skew complex symmetric operators

In this section, we give some properties of m-skew complex symmetric operators. In [2, Theorem 4.5], if $T \in \mathcal{L}(\mathcal{H})$ is an m-complex symmetric operator with conjugation C, then T^n (for some $n \in N$) is also m-complex symmetric with conjugation C. For m-skew complex symmetric operators, we have the following result.

Theorem 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be m-skew complex symmetric with conjugation C. If n is odd, then T^n is m-skew complex symmetric with conjugation C.*

Proof. Since T is an m-skew complex symmetric operator with conjugation C, then

$$\Delta_m^-(T)=\sum_{j=0}^{m}\binom{m}{j}T^{*j}CT^{m-j}C=(T^*+CTC)^m=0.$$

If n is odd, we have

$$(a^n+b^n)^m=\omega_0 a^{m(n-1)}(a+b)^m+\omega_1 a^{m(n-1)-1}(a+b)^m(-b)+\cdots+\omega_{m(n-1)}(a+b)^{m(n-1)},$$

where ω_i are constants for $i=0,1,2,\dots,m(n-1)$.

For m-skew complex symmetric operators with conjugation C,

$$\Delta_m^-(T^n)=\sum_{i=0}^{m(n-1)}\omega_i T^{*m(n-1)-i}\Delta_m^-(T)(-CTC)^i. \tag{3.1}$$

By (3.1), if $\Delta_m^-(T)=0$, then $\Delta_m^-(T^n)=0$. Hence, if n is odd, T^n is m-skew complex symmetric with conjugation C. □

However, when n is even, this result is not true. Let C be a conjugation given by $C(z_1,z_2,z_3)=(\overline{z_3},\overline{z_2},\overline{z_1})$, if

$$T=\begin{pmatrix}1&1&0\\1&0&-1\\0&-1&-1\end{pmatrix}$$

on $\mathbb{C}^3$. By [1], we know that T is a skew complex symmetric operator with conjugation C. It is easy to obtain that $\Delta_1^-(T^2)\neq 0$, so T^2 is not skew complex symmetric.

We all know that if T is skew complex symmetric with conjugation C, then T^{2n} is complex symmetric with conjugation C. For m-skew complex symmetric operators, with the assumption $T^*CTC=CTCT^*$, we have the following result.

Theorem 3.2. *Let $T\in\mathcal{L}(\mathcal{H})$ be an m-skew complex symmetric operator with conjugation C. If $T^*CTC=CTCT^*$, then T^{2n} is an m-complex symmetric operator with conjugation C.*

Proof. Since T is an m-skew complex symmetric operator with conjugation C, then

$$\Delta_m^-(T)=\sum_{j=0}^{m}\binom{m}{j}T^{*j}CT^{m-j}C=(T^*+CTC)^m=0 \quad\text{and}\quad T^*CTC=CTCT^*.$$

(1) If n is odd,

$$\Delta_m(T^{2n})=\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(T^{2n})^{*j}C(T^{2n})^{m-j}C=(T^{2n*}-CT^{2n}C)^m=\Big[(T^{n*}-CT^nC)(T^{n*}+CT^nC)\Big]^m.$$

Since n is odd, by Theorem 3.1, then $(T^{n*}+CT^nC)^m=0$, so $\Delta_m(T^{2n})=0$.

(2) If n is even,

$$\begin{aligned}\Delta_m(T^{2n})&=\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(T^{2n})^{*j}C(T^{2n})^{m-j}C\\&=(T^{2n*}-CT^{2n}C)^m=\Big[(T^*-CTC)(T^*+CTC)\cdots(T^{n*}+CT^nC)\Big]^m\\&=(T^*-CTC)^m\Delta_m^-(T)(T^{2*}+CT^2C)^m\cdots(T^{n*}+CT^nC)^m=0,\end{aligned}$$

so $\Delta_m(T^{2n})=0$. Hence T^{2n} is an m-complex symmetric operator with conjugation C. □

Corollary 3.3. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be an* m*-skew complex symmetric operator with conjugation* C*. If* n *is odd and* $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$*, then* $\lim_{n\to\infty} \|T^{*mn}Cx\|^{\frac{1}{n}} = 0$.

Proof. Since n is odd and T is an m-skew complex symmetric operator with conjugation C, then by Theorem 3.1, we obtain that T^n is an m-skew complex symmetric operator with conjugation C. So $\Delta_m^-(T^n) = 0$, we obtain that

$$\begin{aligned}\Delta_m^-(T^n) &= \sum_{j=0}^{m} \binom{m}{j} (T^n)^{*j} C (T^n)^{m-j} C \\ &= \binom{m}{0} (T^n)^{*0} C (T^n)^m C \\ &\quad + \binom{m}{1} (T^n)^{*} C (T^n)^{m-1} C + \cdots + \binom{m}{m-1} (T^n)^{*m-1} C T^n C + \binom{m}{m} (T^n)^{*m} C (T^n)^0 C.\end{aligned}$$

So

$$\binom{m}{0} (T^n)^{*0} C (T^n)^m C + \binom{m}{1} (T^n)^{*} C (T^n)^{m-1} C + \cdots + \binom{m}{m-1} (T^n)^{*m-1} C T^n C = -\binom{m}{m} (T^n)^{*m} C (T^n)^0 C.$$

This ensures that

$$\left[\sum_{j=0}^{m-1} \binom{m-1}{j} C (T^{n*})^j C (T^n)^{m-j-1}\right] T^n x = -C T^{*mn} Cx.$$

Moreover, we have

$$\begin{aligned}\|T^{*mn}Cx\| = \|-CT^{*mn}Cx\| &= \|\left[\sum_{j=0}^{m-1} \binom{m-1}{j} C (T^{n*})^j C (T^n)^{m-j-1}\right] T^n x\| \\ &\leqslant \|\sum_{j=0}^{m-1} \binom{m-1}{j} C (T^{n*})^j C (T^n)^{m-j-1}\| \|T^n x\| \leqslant 2^m \|T\|^{n(m-1)} \| \|T^n x\|.\end{aligned}$$

Since $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$, hence $\lim_{n\to\infty} \|T^{*mn}Cx\|^{\frac{1}{n}} = 0$. □

Theorem 3.4. *If* $\{T_k\}$ *is a sequence of* m*-skew complex symmetric operators with conjugation* C *such that* $\lim_{k\to\infty} \|T_k - T\| = 0$*, then* T *is* m*-skew complex symmetric with conjugation* C.

Proof. Let $\{T_k\}$ be a sequence of m-skew complex symmetric operators with conjugation C such that $\lim_{k\to\infty} \|T_k - T\| = 0$. Since C is conjugation, then $\|C\| = 1$. So

$$\begin{aligned}\|\Delta_m^-(T_k) - \Delta_m^-(T)\| &= \|\sum_{j=0}^{m} \binom{m}{j} T_k^{*j} C T_k^{m-j} C - \sum_{j=0}^{m} \binom{m}{j} T^{*j} C T^{m-j} C\| \\ &= \|\sum_{j=0}^{m} \binom{m}{j} T_k^{*j} C T_k^{m-j} C - \sum_{j=0}^{m} \binom{m}{j} T_k^{*j} C T^{m-j} C \\ &\quad + \sum_{j=0}^{m} \binom{m}{j} T_k^{*j} C T^{m-j} C - \sum_{j=0}^{m} \binom{m}{j} T^{*j} C T^{m-j} C\| \\ &\leqslant \sum_{j=0}^{m} \binom{m}{j} \|T_k^{*j}\| \|C T_k^{m-j} C - C T^{m-j} C\| + \sum_{j=0}^{m} \binom{m}{j} \|T_k^{*j} - T^{*j}\| \|CTC\|^{m-j} \qquad (3.2) \\ &\leqslant \sum_{j=0}^{m} \|T_k^*\| \|CT_kC - CTC\| \| \sum_{i=0}^{m-j-1} (CT_kC)^{m-j-1-i} (CTC)^i \|\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{m}\|T_k^*-T^*\|\|CT^{m-j}C\|\|\sum_{i=0}^{j-1}(T_k^*)^{j-i-1}-(T^*)^i\| \\
&\leqslant \sum_{j=0}^{m}\|T_k^*\|\sum_{i=0}^{m-j-1}\|(CT_kC)^{m-j-1-i}(CTC)^i\|\|T_k-T\| \\
&+\sum_{j=0}^{m}\|CT^{m-j}C\|\sum_{i=0}^{j-1}\|(T_k^*)^{j-i-1}-(T^*)^i\|\|T_k-T\|.
\end{aligned}
$$

Since $\lim_{k\to\infty}\|T_k-T\|=0$, it is obvious that the right side of (3.2) tends to zero. Moreover, since T_k is m-skew complex symmetric, then $\Delta_m^-(T_k)=0$. Hence T is also m-skew complex symmetric. □

Theorem 3.5. *Let $T\in\mathcal{L}(\mathcal{H})$ be invertible and C be a conjugation on $\mathcal{H}$, then the following assertions hold.*

(i) *If $T^{*j}CT^{m-j}C=CT^{m-j}CT^{*j}$ for $j=0,1,\dots,m$, then T is m-skew complex symmetric with conjugation C if and only if $CT^{*-1}C$ is m-skew complex symmetric with conjugation C.*

(ii) *T is m-skew complex symmetric with conjugation C if and only if T^{-1} is m-skew complex symmetric with conjugation C.*

Proof.

(i) Suppose that $T^{*j}CT^{m-j}C=CT^{m-j}CT^{*j}$ for $j=0,1,\dots,m$. If T is an m-skew complex symmetric operator with conjugation C, then we have

$$
\begin{aligned}
0=\Delta_m^-(T)=CT^{-m}C\Big[\sum_{j=0}^{m}\tbinom{m}{j}T^{*j}CT^{m-j}C\Big]T^{*-m}
&=\sum_{j=0}^{m}\tbinom{m}{j}CT^{-m}CT^{*j}CT^{m-j}CT^{*-m}\\
&=\sum_{j=0}^{m}\tbinom{m}{j}(CT^{-1}C)^j(T^{*-1})^{m-j}\\
&=\sum_{j=0}^{m}\tbinom{m}{j}(CT^{*-1}C)^{*j}C(CT^{*-1}C)^{m-j}C.
\end{aligned}
$$

Thus $CT^{*-1}C$ is m-skew complex symmetric with conjugation C. We can obtain the converse implication in a similar way.

(ii) Since T is an m-skew complex symmetric operator, then

$$
\Delta_m^-(T)=\sum_{j=0}^{m}\tbinom{m}{j}T^{*j}CT^{m-j}C=C\Big[\sum_{j=0}^{m}\tbinom{m}{j}CT^{*j}CT^{m-j}\Big]C=0,
$$

thus $\sum_{j=0}^{m}\binom{m}{j}CT^{*j}CT^{m-j}=0$. Since

$$
\Delta_m^-(CTC)=\sum_{j=0}^{m}\tbinom{m}{j}CT^{*j}CC[CT^{m-j}C]C=\sum_{j=0}^{m}\tbinom{m}{j}CT^{*j}CT^{m-j}=0,
$$

then $\Delta_m^-(CTC)=0$. It ensures that T is m-skew complex symmetric with conjugation C if and only if CTC is m-skew complex symmetric with conjugation C.

Let T^{-1} be an m-skew complex symmetric operator. Then $CT^{-1}C$ is an m-skew complex symmetric operator. Since

$$\Delta_m^-(T^{-1}) = \sum_{j=0}^{m} \binom{m}{j} (T^{-1})^{*j} C (T^{-1})^{m-j} C = \sum_{j=0}^{m} \binom{m}{j} (T^*)^{-j} C (T^{j-m}) C = 0,$$

then

$$(T^*)^m \Big[\sum_{j=0}^{m} \binom{m}{j} (T^*)^{-j} C (T^{j-m}) C \Big] C T^m C = 0, \qquad \sum_{j=0}^{m} \binom{m}{j} (T^*)^{m-j} C T^j C = 0.$$

Let $m - j = i$, then

$$\sum_{i=0}^{m} \binom{m}{m-i} (T^*)^i C T^{m-i} C = \sum_{i=0}^{m} \binom{m}{i} (T^*)^{m-i} C T^i C = 0.$$

Hence $\Delta_m^-(T) = 0$. The reverse implication is similar. □

4. Some spectral properties of m-skew complex symmetric operators

In this section, we will give some spectral properties of m-skew complex symmetric operators.

Theorem 4.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be m-skew complex symmetric with conjugation C. If λ is an eigenvalue of T, then $-\overline{\lambda}$ is an eigenvalue of T^*.*

Proof. Let $\{x_n\}$ be a sequence of unit vectors such that $\lim_{n\to\infty}(T-\lambda)x_n = 0$. Since $Tx = \lambda x$, we can easily obtain $(CTC)Cx = \overline{\lambda}Cx$. So $\lim_{n\to\infty}(CTC - \overline{\lambda})Cx_n = 0$. Since T is m-skew complex symmetric with conjugation C, it ensures that

$$\begin{aligned} 0 = \lim_{n\to\infty} \Delta_m^-(T) C x_n &= \lim_{n\to\infty} \Big(\sum_{j=0}^{m} \binom{m}{j} T^{*j} C T^{m-j} C \Big) C x_n \\ &= \lim_{n\to\infty} \Big(\sum_{j=0}^{m} \binom{m}{j} T^{*j} \overline{\lambda}^{m-j} \Big) C x_n = \lim_{n\to\infty} (T^* + \overline{\lambda})^m C x_n. \end{aligned}$$

Hence $-\overline{\lambda}$ is an eigenvalue of T^*. □

Theorem 4.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-skew complex symmetric operator with conjugation C. Then $\Delta_m^-(T)^* = \Delta_m^-(T^*) = 0$.*

Proof. Using the mathematical induction, let $m = 2$, we have

$$\begin{aligned} \Delta_2^-(T)^* = \Big[\sum_{j=0}^{2} \binom{2}{j} T^{*j} C T^{2-j} C \Big]^* &= [(T^* + CTC)^2]^* \\ &= [(T^* + CTC)(T^* + CTC)]^* \\ &= [(T^*)^2 + T^*CTC + CTCT^* + CT^2C]^* \\ &= T^2 + CT^*CT + TCT^*C + C(T^*)^2C = (T + CT^*C)^2 = \Delta_2^-(T^*). \end{aligned}$$

We assume the result is true when $k = m-1$, then

$$\Delta_{m-1}^-(T)^* = [(T^* + CTC)^{m-1}]^* = (T + CT^*C)^{m-1} = \Delta_{m-1}^-(T^*).$$

When $k = m$, we have

$$\begin{aligned} \Delta_m^-(T)^* &= [(T^* + CTC)^m]^* \\ &= [(T^* + CTC)^{m-1}(T^* + CTC)]^* \\ &= (T^* + CTC)^*[(T^* + CTC)^{m-1}]^* = (CT^*C + T)(T + CT^*C)^{m-1} = (CT^*C + T)^m = \Delta_m^-(T^*). \end{aligned}$$

Hence $\Delta_m^-(T)^* = \Delta_m^-(T^*) = 0$. □

Theorem 4.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-skew complex symmetric operator with conjugation C. Then the following relations are true.*

(i) $\sigma(T)^* = -\sigma(T^*)$, $\sigma_a(T)^* = -\sigma_a(T^*)$, $\sigma_{su}(T)^* = -\sigma_{su}(T^*)$.

(ii) $\sigma_p(T)^* = -\sigma_p(T^*)$, $\sigma_{comp}(T)^* = -\sigma_{comp}(T^*)$, $\sigma_p(T) = -\sigma_{comp}(T)$, $\sigma_p(T^*) = -\sigma_{comp}(T^*)$.

(iii) $\sigma_e(T)^* = -\sigma_e(T^*)$, $\sigma_{le}(T)^* = -\sigma_{le}(T^*)$, $\sigma_{re}(T)^* = -\sigma_{re}(T^*)$.

Proof.

(i) Let $\lambda \in \sigma_a(T)$ and $\{x_n\}$ be a sequence of unit vectors such that $\lim_{n\to\infty}(T-\lambda)x_n = 0$. We know that $\lim_{n\to\infty}(CTC-\overline{\lambda})Cx_n = 0$. Since T is m-skew complex symmetric with conjugation C, so $\Delta_m^-(T) = 0$. It ensures that

$$\lim_{n\to\infty} \Delta_m^-(T)Cx_n = \lim_{n\to\infty}\left(\sum_{j=0}^{m}\binom{m}{j}T^{*j}CT^{m-j}C\right)Cx_n = \lim_{n\to\infty}\left(\sum_{j=0}^{m}\binom{m}{j}T^{*j}\overline{\lambda}^{m-j}\right)Cx_n = \lim_{n\to\infty}(T^*+\overline{\lambda})^m Cx_n.$$

If $\lim_{n\to\infty}\frac{(T^*+\overline{\lambda})^{m-1}Cx_n}{\|(T^*+\overline{\lambda})^{m-1}Cx_n\|} \neq 0$, then $\overline{\lambda} \in -\sigma_a(T^*)$. Otherwise, $\lim_{n\to\infty}(T^*+\overline{\lambda})^{m-1}Cx_n = 0$.

If $\lim_{n\to\infty}\frac{(T^*+\overline{\lambda})^{m-2}Cx_n}{\|(T^*+\overline{\lambda})^{m-2}Cx_n\|} \neq 0$, then $\overline{\lambda} \in -\sigma_a(T^*)$. Otherwise, $\lim_{n\to\infty}(T^*+\overline{\lambda})^{m-2}Cx_n = 0$. By induction, we have $\lim_{n\to\infty}(T^*+\overline{\lambda})Cx_n = 0$, so $\overline{\lambda} \in -\sigma_a(T^*)$. Hence $\sigma_a(T)^* \subseteq -\sigma_a(T^*)$.

For the converse, let $\lambda \in \sigma_a(T^*)$ and $\{x_n\}$ be a sequence of unit vectors such that $\lim_{n\to\infty}(T^*-\lambda)x_n = 0$. We know that $\lim_{n\to\infty}(CT^*C-\overline{\lambda})Cx_n = 0$. Since T is m-skew complex symmetric with conjugation C, so $\Delta_m^-(T) = 0$, therefore $[\Delta_m^-(T)]^* = 0$. By Theorem 4.2, we obtain $[\Delta_m^-(T^*)] = 0$. It ensures that

$$\lim_{n\to\infty} \Delta_m^-(T^*)Cx_n = \lim_{n\to\infty}\left(\sum_{j=0}^{m}\binom{m}{j}T^{j}CT^{*m-j}C\right)Cx_n = \lim_{n\to\infty}\left(\sum_{j=0}^{m}\binom{m}{j}T^{j}\overline{\lambda}^{m-j}\right)Cx_n = \lim_{n\to\infty}(T+\overline{\lambda})^m Cx_n.$$

If $\lim_{n\to\infty}\frac{(T+\overline{\lambda})^{m-1}Cx_n}{\|(T+\overline{\lambda})^{m-1}Cx_n\|} \neq 0$, then $\overline{\lambda} \in -\sigma_a(T)$. Otherwise, $\lim_{n\to\infty}(T+\overline{\lambda})^{m-1}Cx_n = 0$.

If $\lim_{n\to\infty}\frac{(T+\overline{\lambda})^{m-2}Cx_n}{\|(T+\overline{\lambda})^{m-2}Cx_n\|} \neq 0$, then $\overline{\lambda} \in -\sigma_a(T)$. Otherwise, $\lim_{n\to\infty}(T+\overline{\lambda})^{m-2}Cx_n = 0$. By induction, we have $\lim_{n\to\infty}(T+\overline{\lambda})Cx_n = 0$, then $\overline{\lambda} \in -\sigma_a(T)$. So, $\sigma_a(T^*) \subseteq -\sigma_a(T)^*$. Hence $\sigma_a(T)^* = -\sigma_a(T^*)$.

For any $T \in \mathcal{L}(\mathcal{H})$, we have $\sigma(T) = \sigma_a(T) \cup \sigma_a(T^*)^*$ in [9]. Since $\sigma_a(T)^* = -\sigma_a(T^*)$, then $\sigma(T) = \sigma_a(T) \cup -\sigma_a(T)$, we obtain $\sigma(T)^* = \sigma_a(T)^* \cup -\sigma_a(T)^* = -\sigma_a(T^*) \cup \sigma_a(T^*) = -\sigma(T^*)$. Hence $\sigma(T)^* = -\sigma(T^*)$.

For any $T \in \mathcal{L}(\mathcal{H})$, $\sigma_{su}(T)^* = \sigma_a(T^*)$. So $\sigma_{su}(T^*)^* = \sigma_a(T)$, we have $\sigma_{su}(T^*) = \sigma_a(T)^* = -\sigma_a(T^*) = -\sigma_{su}(T)^*$. Hence $\sigma_{su}(T)^* = -\sigma_{su}(T^*)$.

(ii) Since $\sigma_a(T)^* = -\sigma_a(T^*)$, we can easily obtain $\sigma_p(T)^* = -\sigma_p(T^*)$. For $T \in \mathcal{L}(\mathcal{H})$, $\sigma_{comp}(T)^* = \sigma_p(T^*)$, so $\sigma_{comp}(T^*)^* = \sigma_p(T^{**}) = \sigma_p(T)$. Then $\sigma_{comp}(T^*) = \sigma_p(T)^* = -\sigma_p(T^*) = -\sigma_{comp}(T)^*$. Hence, $\sigma_{comp}(T)^* = -\sigma_{comp}(T^*)$. For $T \in \mathcal{L}(\mathcal{H})$, $\sigma_{comp}(T)^* = \sigma_p(T^*)$, so we have $\sigma_{comp}(T^*)^* = \sigma_p(T^{**}) = \sigma_p(T)$. Since $\sigma_{comp}(T)^* = -\sigma_{comp}(T^*)$, so $\sigma_{comp}(T^*)^* = -\sigma_{comp}(T)$. Hence $\sigma_p(T) = -\sigma_{comp}(T)$, $\sigma_p(T^*) = -\sigma_{comp}(T^*)$.

(iii) For $T \in \mathcal{L}(\mathcal{H})$, we have $\sigma_e(T) \subseteq \sigma(T)$. Since $\sigma(T)^* = -\sigma(T^*)$ and T is m-skew complex symmetric with conjugation C, we obtain $\sigma_e(T)^* = -\sigma_e(T^*)$.

For $T \in \mathcal{L}(\mathcal{H})$, $\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T)$ and $\sigma_{le}(T)^* = \sigma_{re}(T^*)$. Since $\sigma_e(T)^* = -\sigma_e(T^*)$, we can obtain

$$\begin{aligned}\sigma_e(T)^* = \sigma_{le}(T)^* \cup \sigma_{re}(T)^* = -\sigma_e(T^*) &= -\sigma_{le}(T^*) \cup -\sigma_{re}(T^*)\\ &= -\sigma_{le}(T^*) \cup -\sigma_{le}(T)^* = \sigma_{le}(T)^* \cup -\sigma_{le}(T^*).\end{aligned}$$

Hence $\sigma_{le}(T)^* = -\sigma_{le}(T^*)$. Using the proof of $\sigma_{le}(T)^* = -\sigma_{le}(T^*)$, we can obtain $\sigma_{re}(T)^* = -\sigma_{re}(T^*)$. □

Corollary 4.4. *Let T be m-skew complex symmetric with conjugation C. The following statements are true.*

(i) $\sigma(T) = \sigma_a(T) \cup [-\sigma_a(T)]$ *and* $\sigma(T^*) = \sigma_{su}(T^*) \cup [-\sigma_{su}(T^*)]$.
(ii) $\sigma_e(T) = \sigma_{re}(T) \cup [-\sigma_a(T)] = \sigma_{le}(T) \cup [-\sigma_{le}(T)]$ *and* $\sigma_e(T^*) = \sigma_{re}(T^*) \cup [-\sigma_a(T^*)] = \sigma_{le}(T^*) \cup [-\sigma_{le}(T^*)]$.

Proof.

(i) For $T \in \mathcal{L}(\mathcal{H})$, $\sigma(T) = \sigma_a(T) \cup \sigma_a(T^*)^*$ by [9]. Since $\sigma_a(T)^* = -\sigma_a(T^*)$ by Theorem 4.3, we obtain $\sigma(T) = \sigma_a(T) \cup [-\sigma_a(T)]$. Using the proof of $\sigma(T) = \sigma_a(T) \cup [-\sigma_a(T)]$, we can obtain $\sigma(T^*) = \sigma_{su}(T^*) \cup [-\sigma_{su}(T^*)]$.

(ii) Since $T \in \mathcal{L}(\mathcal{H})$, $\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T)$ and $\sigma_{le}(T)^* = \sigma_{re}(T^*)$, so $\sigma_{le}(T)^* = -\sigma_{le}(T^*)$ and $\sigma_{re}(T)^* = -\sigma_{re}(T^*)$ by Theorem 4.3, we obtain $\sigma_{le}(T) = -\sigma_{re}(T)$ and $\sigma_{le}(T^*) = -\sigma_{re}(T^*)$. Hence we obtain that $\sigma_e(T) = \sigma_{re}(T) \cup [-\sigma_a(T)] = \sigma_{le}(T) \cup [-\sigma_{le}(T)]$ and $\sigma_e(T^*) = \sigma_{re}(T^*) \cup [-\sigma_a(T^*)] = \sigma_{le}(T^*) \cup [-\sigma_{le}(T^*)]$. □

Corollary 4.5. *Let* T *be* m*-skew complex symmetric with conjugation* C. *The following statements are equivalent.*

(i) $T - \lambda$ *is invertible.*
(ii) $T \pm \lambda$ *is bounded below.*
(iii) $T \pm \lambda$ *is one-to-one and have closed range.*

Proof.

(i) $\Rightarrow$(ii) and (iii). If $T - \lambda$ is invertible, then $\lambda \notin \sigma_a(T) \cup [-\sigma_a(T)]$ from Corollary 4.4, so $\lambda \notin \sigma_a(T)$ and $-\lambda \notin \sigma_a(T)$. By [5], we can imply that $T - \lambda$ and $T + \lambda$ are bounded below. Equivalently, $T \pm \lambda$ is one-to-one and have closed range.

(ii) $\Leftrightarrow$ (iii). It is trivial by [5].

(ii) $\Longrightarrow$(i). If $T \pm \lambda$ is bounded below, then $\pm\lambda \notin \sigma_a(T)$ and $\pm\lambda \notin \sigma_a(T) \cup [-\sigma_a(T)]$. By Corollary 4.4, we have $\pm\lambda \notin \sigma(T)$. The proof is similar to [13, Corollary 3.6]. □

Corollary 4.6. *Let* T *be* m*-skew complex symmetric with conjugation* C. *The following statements are equivalent.*

(i) $\lambda \notin \sigma_e(T)$.
(ii) $\dim \ker (T \pm \lambda) < \infty$ *and* $\text{ran}(T \pm \lambda)$ *are closed.*
(iii) $\dim[\text{ran}(T \pm \lambda)]^\perp < \infty$ *and* $\text{ran}(T \pm \lambda)$ *are closed.*

Proof. We can obtain the corollary by Theorem 4.3 and [5] immediately. □

Theorem 4.7. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be* m*-skew complex symmetric with conjugation* C. *Then* T^* *has the property* (β) *if and only if* T *is decomposable.*

Proof. The proof is similar to [2, Theorem 4.7], so $-T$ has the property (β), it is easy to obtain that T has the property (β). □

Theorem 4.8. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be* m*-skew complex symmetric with conjugation* C. *If* T^* *has the single-valued extension property, then* T *has the single-valued extension property.*

Proof. The proof is similar to [2, Theorem 4.7, Theorem 4.10], so $-T$ has the single-valued extension property, it is easy to obtain that T has the single-valued extension property. □

Recall that an operator $N \in \mathcal{L}(\mathcal{H})$ is said nilpotent of order n, if $N^n = 0$ and $N^{n-1} \neq 0$ for some positive integer n. In the following theorem, we will study some properties of the operator $T + N$.

Theorem 4.9. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be* m*-skew complex symmetric with conjugation* C *and* N *be nilpotent of order* $n > 2$ *with* $NT = TN$. *Then* $T + N$ *is* $(2n + m - 1)$*-skew complex symmetric.*

Proof. Let $R = T + N$ and $k = 2n + m - 1$. Since

$$[(a+b)+(c+d)]^k = [(a+c)+(b+d)]^k = \sum_{i=0}^{k} \binom{k}{i}(a+c)^i(b+d)^{k-i} = \sum_{i=0}^{k}\sum_{j=0}^{k-i} \binom{k}{i}\binom{k-i}{j}(a+c)^i b^j d^{k-i-j}.$$

We have

$$\begin{aligned}
\Delta_k^-(R) = \sum_{i=0}^{k}(R^*)^i C R^{k-i} C = (R^* + CRC)^k
&= [(T^* + N^*) + (CTC + CNC)]^k \\
&= [(T^* + CTC) + (N^* + CNC)]^k \\
&= \sum_{i=0}^{k}\binom{k}{i}(T^* + CTC)^i (N^* + CNC)^{k-i} \\
&= \sum_{i=0}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(T^* + CTC)^i (N^*)^j C N^{k-i-j} C \\
&= \sum_{i=0}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}\Delta_i^-(T)(N^*)^j C N^{k-i-j} C.
\end{aligned}$$

(i) If $j \geqslant n$ or $k - i - j \geqslant n$, then $(N^*)^j = 0$ and $CN^{k-i-j}C = 0$. We can imply that $\Delta_k^-(R) = 0$, by the fact $N^n = 0$.

(ii) If $j < n$ and $k - i - j < n$, then $i > k - n - j \geqslant k - n - (n-1) = m$, thus $\Delta_i^-(T) = 0$ and $\Delta_k^-(R) = 0$. Hence $k = m + 2n - 1$ and $T + N$ is a $(2n + m - 1)$-skew complex symmetric operator with conjugation C. □

Theorem 4.10. *Let $T \in \mathcal{L}(\mathcal{H})$ be m-skew complex symmetric with conjugation C and N be nilpotent of order $n > 2$ with $NT = TN$. Let $R = T + N$, then the following arguments hold.*

(i) *If T^* has the single-valued extension property, then R and R^* have the single-valued extension property.*
(ii) *If T has the Dunford's property $(\mathcal{C})$ and $\sigma_T(x) \subset \sigma_R(N^{n-1}x) \cap \sigma_R(x)$ for all $x \in \mathcal{H}$, then R has the Dunford's property $(\mathcal{C})$.*

Proof.

(i) If T^* has the single-valued extension property, then T has the single-valued extension property by Theorem 4.8. Using the similar proof of [4, Theorem 3.13], we obtain that R has the single-valued extension property. Similarly, we get that R^* has the single-valued extension property. Hence R and R^* have the single-valued extension property.

(ii) The proof is similar to [4, Theorem 3.13]. □

Corollary 4.11. *Let $T \in \mathcal{L}(\mathcal{H})$ be m-skew complex symmetric with conjugation C and N be nilpotent of order $n > 2$ with $NT = TN$. Let $R = T + N$, if T^* has the single-valued extension property, then the following arguments hold.*

(i) $\sigma(R) = \sigma_{su}(R) = \sigma_{ap}(R) = \sigma_{se}(R)$, $\sigma_{es}(R) = \sigma_b(R) = \sigma_\omega(R) = \sigma_e(R)$.
(ii) $H_0(R - \lambda) = H_R(\{\lambda\})$ *and* $H_{R^*}(\{\lambda\}) = H_0(R^* - \lambda)$ *for all* $\lambda \in \mathbb{C}$.

Proof. The proof is similar to [4, Corollary 3.14]. □

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