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$BL^m_{p,\nu}$ estimates for the Riesz transforms associated with Laplace-Bessel operator

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Abstract

In this paper, we introduce higher order Riesz-Bessel transforms which we can express partial derivatives of order α of $I_{m,\nu}f$ for $f \in L_{p,\nu}$. In addition, we establish relationship between Riesz potential with higher order Riesz-Bessel transform related to generalized shift operator. By using this relationship, we make some improvements of integral estimates for $I_{m,\nu}f$ and higher order Riesz-Bessel transform R^m_ν in the Beppo Levi space $BL^m_{p,\nu}$. We prove an estimate for the singular integral operator with convolution type generated by generalized shift operator in the Beppo Levi spaces.

Keywords: Laplace-Bessel operator, Bessel generalized shift operator, Riesz-Bessel transform, fractional integral operator, Beppo Levi spaces.

2010 MSC: 47H10, 45E10, 47B37.

1. Introduction

The study of Riesz transforms on Euclidean spaces has a very long history. These operators provide the most important examples of Calderon-Zygmund singular integral operators. They have a variety of applications especially in the area of partial differential equations. The classical Riesz transforms are associated with the standard Laplace on $\mathbb{R}^n$ and therefore it is natural to study analogues of these transforms in the context of other elliptic partial differential operators. Analogues of Riesz transforms are defined and studied for the Dunkl transform and Laplace-Bessel operator. Especially on the L_p space, several authors have studied different aspects of Riesz transforms [1, 5, 6, 7, 12, 14]. Riesz transform is a singular integral operator with convolution type. Thus, we can see to this operator as the operators related to ordinary shift and these transforms are associated with the Laplace operator. Instead of using the ordinary shift operator to generalized shift operators are related to the Laplace-Bessel operator, some

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author have investigated a different singular integral operators with convolution type as called Riesz-Bessel transforms that is related to generalized shift operator. Recently, Riesz transforms have been studied in the context of Laplace-Bessel operator as well. In most of these papers, the authors have investigated weighted and unweighted L_p mapping properties of the Riesz-Bessel transforms.

The L_p boundedness of Riesz transforms have been studied in several contexts. The idea of considering Riesz transforms on L_p spaces investigated with several papers by authors, see [1, 5, 6, 7, 12, 14]. In [1, 5, 6, 7], the authors looked at the Riesz transforms associated with the Laplace-Bessel operator on $\mathbb{R}^n$. Instead of using the ordinary shift operator to generalized shift operator they used a different Riesz-Bessel operator consisting of functions of the form T^y where T^y is the generalized shift operator and Δ_B are Laplace-Bessel operator. When Riesz transforms are defined using this shift operator, the natural spaces to study their boundedness properties turned out to be the p-norm spaces L_p which are defined in terms of weights.

Moreover, relationship between partial derivatives of the Riesz potential of order one and the Riesz transforms is given as follows [18, 19, 20]. For $f \in L_P$ $(p < n)$ $D_j U_1^f = (1-n)c_n^{-1} R_j f$ $j = 1, 2, \ldots, n$. The relationship between the primitive of order m and the potential of order m is established by Kurokawa [14]. By using this relationship it is made some improvements of integral estimates for Riesz potential. Then, it is investigated the Beppo Levi space and give potential and integral representations of Beppo Levi functions for arbitrary positive integer m and $p > 1$. Thus, Kurokawa is concerned with relationship between partial derivatives of the Riesz potentials of order (m, k) and higher Riesz transforms. Thus, since the Riesz transforms are bounded operators on L_p, it see that $R^\alpha f \in L_p$ for $f \in L_P$.

In this paper, we would be interested in higher order Riesz-Bessel transform which is generated derivatives of order α of Bessel potential operator $I_{m,\nu} f$ generated by generalized translation operator related to Laplace-Bessel differential operator [10]. The purpose of this paper is to prove an $BL^m_{p,\nu}$ estimate for singular integral operator related to Laplace Bessel operator and to give relations between higher order Riesz-Bessel transforms and Riesz potentials in $BL^m_{p,\nu}$ spaces.

The paper is organized as follows. In the next section, we collect some notation and preliminaries. In Section 3, we recall the definition of the higher order Riesz-Bessel transforms and Riesz potentials, generalized translation operator and prove derivatives of the Riesz potentials that have been previously used in the paper [2, 3, 8, 9]. The Riesz potentials for the Laplace-Bessel differential operator will be studied and to obtain an estimate of the higher order Riesz-Bessel transform is discussed in Section 4.

2. Notation and preliminaries

Let $\mathbb{R}^n_+$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, ..., x_n)$, defined by the inequality $x_n > 0$. We write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.

$S_+ = S(\mathbb{R}^n)$ be the space of functions which are the restrictions to $\mathbb{R}^n$ of the test functions of the Schwartz that are even with respect to x_n, decreasing sufficiently rapidly at infinity, together with all derivatives of the form

$$D^\alpha_\nu = D^{\alpha'}_{x'} B^{\alpha_n}_n = D^{\alpha_1}_1 ... D^{\alpha_{n-1}}_{n-1} B^{\alpha_n}_n = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots\cdots \frac{\partial^{\alpha_{n-1}}}{\partial x_{n-1}^{\alpha_{n-1}}} B^{\alpha_n}_n,$$

i.e., for all $\varphi \in S_+$, $\sup_{x \in \mathbb{R}^n+} \left| x^\beta D^\alpha_\nu \varphi \right| < \infty$ where $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ are multi-indexes, and $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$ and $B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\nu}{x_n} \frac{\partial}{\partial x_n}$ is the Bessel differential expansion. For a fixed parameter $\nu > 0$, let $L_{p,\nu} = L_{p,\nu}(\mathbb{R}^n)$ be the space of measurable functions with a finite norm

$$\|f\|_{L_{p,\nu}} \equiv \left(\int_{\mathbb{R}^n_+} |f(x)|^p x_n^\nu \, dx \right)^{1/p}$$

is denoted by $L_{p,\nu} \equiv L_{p,\nu}(\mathbb{R}^n_+)$, $1 \leqslant p < \infty$. The space of the essentially bounded measurable function on $\mathbb{R}^n$ is denoted by $L_{\infty,\nu}(\mathbb{R}^n)$.

The Fourier-Bessel transformation and its inverse on $\mathbb{S}_+$ are defined by

$$F_\nu f(x) = \int_{\mathbb{R}^n} f(y)\, e^{-i(x',y')} j_{\frac{\nu-1}{2}}(x_n y_n)\, y_n^\nu dy,$$
$$F_\nu^{-1} f(x) = C_{n,\nu} F_\nu f(-x', x_n),$$

where $(x',y') = x_1y_1 + \ldots + x_{n-1}y_{n-1}$, j_ν, $\nu > -1/2$, is the normalized Bessel function, and

$$C_{n,\nu} = (2\pi)^{n-1} 2^{\nu-1} \Gamma^2((\nu+1)/2).$$

This transform is associated to the Laplace - Bessel differential operator

$$\Delta_\nu = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\nu}{x_n}\frac{\partial}{\partial x_n}, \quad \nu > 0. \tag{2.1}$$

The expression (2.1) is a hybrid of the Hankel transform in the x_n-variable and the ordinary Fourier transform on $\mathbb{R}^{n-1}$.

I. A. Kipriyanov (for $n = 1$ B. M. Levitan [15, 16]) investigated the generalized convolution (Δ_ν-convolution)

$$(f \otimes g)(x) = \int_{\mathbb{R}^n} f(y)\; T^y g(x)\, y_n^\nu dy,$$

associated with the Laplace-Bessel differential operator, where T^y is the generalized shift operator (Δ_ν-shift) defined by

$$T^y f(x) = C_\nu \int_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\theta + y_n^2}\right) \sin^{\nu-1}\theta d\theta,$$

being $C_\nu = \pi^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)[\Gamma\left(\frac{\nu}{2}\right)]^{-1}$ (see [10, 11, 15, 16, 17]) .

It is well known that

$$F_\nu\left(B_n^{\alpha_n} f\right)(x) = (-x_n^2)^{\alpha_n} F_\nu f(x), \tag{2.2}$$

$$F_\nu\left(D_i^{2\alpha_i} f\right)(x) = (-x_i^2)^{\alpha_i} F_\nu f(x), \quad i = 1, \ldots, n-1, \tag{2.3}$$

$$F_\nu\left(\Delta_\nu f\right)(x) = -|x|^2 F_\nu f(x) \quad \text{and} \quad F_\nu(f \otimes g) = F_\nu f\, F_\nu g, \tag{2.4}$$

$$F_\nu\left(D_{x'}^{2\alpha'} B_n^{\alpha_n} f\right)(x) = (-1)^{|\alpha|} x^{2\alpha} F_\nu f(x). \tag{2.5}$$

In this paper, we consider the space $BL^m_{p,\nu}(\mathbb{R}^n_+)$ of Beppo-Levi functions of higher order m, also known as homogeneous Sobolev spaces, that is the space consisting of distributions on $\mathbb{R}^n_+$ whose partial derivatives of mth order all belong to $L_{p,\nu}(\mathbb{R}^n_+)$. Let $1 < p < \infty$. For a non-negative integer m, we denote by $W^m_{p,\nu}(\mathbb{R}^n_+)$ the Sobolev space, the space of all distributions φ such that $D_{x'}^{\alpha'} B_n^{\alpha_n}\varphi \in L_{p,\nu}(\mathbb{R}^n_+)$ for any α with $|\alpha| \leqslant m$. The norm of φ in $W^m_{p,\nu}(\mathbb{R}^n_+)$ is defined by

$$\|\varphi\|_{W^m_{p,\nu}} = \sum_{|\alpha'|+2\alpha_n \leqslant m} \|D_{x'}^{\alpha'} B_n^{\alpha_n}\varphi\|_{L_{p,\nu}},$$

where $\|.\|$ denotes the $L_{p,\nu}$ norm in $\mathbb{R}^n_+$. It is well known that $W^m_{p,\nu}(\mathbb{R}^n_+)$ is a Banach space if $1 < p < \infty$.

According to Deny and Lions [4], we denote by $BL^m_{p,\nu}(\mathbb{R}^n_+)$, $m \geqslant 1$, the space of distributions whose partial derivatives $D_{x'}^{\alpha'} B_n^{\alpha_n} f$ of mth order ($|\alpha'| + 2\alpha_n = m$) all belong to $L_{p,\nu}(\mathbb{R}^n_+)$, i.e.,

$$BL^m_{p,\nu}(\mathbb{R}^n_+) = \Big\{f \in \mathbb{S}_+(\mathbb{R}^n_+) : \sum_{|\alpha'|+2\alpha_n = m} \|D_{x'}^{\alpha'} B_n^{\alpha_n} f\|_{L_{p,\nu} < \infty}\Big\},$$

where $|\alpha'| = \alpha_1 + \ldots + \alpha_{n-1}$ for a multi index $\alpha = (\alpha_1, \ldots, \alpha_n)$. An element in $BL^m_{p,\nu}(\mathbb{R}^n_+)$, which must be locally integrable is called a Beppo Levi function of order m. The norm $\|f\|_{BL^m_{p,\nu}}$ in $BL^m_{p,\nu}$ is given by

$$\|f\|_{BL^m_{p,\nu}} = \|f\|_{L_{p,\nu}} + \sum_{|\alpha'|+2\alpha_n=m} \|D^{\alpha'}_{x'} B^{\alpha_n}_n f\|_{L_{p,\nu}},$$

and so that $BL^m_{p,\nu}(\mathbb{R}^n_+)$ is a Banach space with norm $\|.\|_{BL^m_{p,\nu}}$.

Let us start with an integral representation of fractional integral operator. Let $0 < \alpha < Q$, $Q = n + \nu$, and $S^n_+ = \{x \in \mathbb{R}^n_+ : |x| = 1\}$. In the following we define the Riesz potential by

$$I_{\alpha,\nu} f(x) = c_{\alpha,\nu} \int_{\mathbb{R}^n} T^y f(x) |y|^{\alpha-Q} y_n^\nu dy,$$

where $c_{\alpha,\nu} = 2^{\alpha-n}\Gamma\left(\frac{n+\nu-m}{2}\right)\left[\Gamma\left(\frac{m}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\right]^{-1}$ (see [8, 9]).

To keep on continue the other representation of Beppo Levi functions. Let $K_{m,\nu}(x)$ be the Riesz kernel for $n + \nu > m$ ($|\alpha| = m$) and $m - n - \nu$ odd

$$K_{m,\nu}(x) = c_{\alpha,\nu}|x|^{m-n-\nu}.$$

We note that $\Delta^\alpha_\nu K_{m,\nu}(x) = 0$ for $x \neq 0$ where Δ^α_ν, α times iteration of Δ_ν. A function φ is said to be polyharmonic of degree α on a domain if $\Delta^\alpha_\nu \varphi(x) = 0$. So the Riesz kernel $K_{m,\nu}$ is polyharmonic of degree m in $\mathbb{R}^n_+$. Further the Fourier-Bessel transform of $K_{m,\nu}$ is given by $F_\nu\left[K_{m,\nu}\right] = c_{\alpha,\nu}|x|^{-m}$.

3. The higher order Riesz-Bessel transforms

In this section, we consider generalized shift operator related to the multidimensional Laplace-Bessel operator. Then we introduce higher order Riesz-Bessel transforms of $f \in L_{p,\nu}$, by which we can express partial derivatives of order m of $I_{m,\nu} f$. This work is a different analogy of M. Ohtsuka and T. Kurokawa and Y. Mizuta (see [12, 18, 19]). In addition, we establish relationship between Riesz potential with Riesz-Bessel transform generated by generalized shift operator associated with Laplace-Bessel operator. By using this relationship, we obtain a estimates for $I_{m,\nu} f$ and higher order Riesz-Bessel transform $R^m_\nu f$ in the Beppo Levi space $BL^m_{p,\nu}$.

First note that, let $\Omega(x) = P_k(x)|x|^{-m}$, $K(x) = \Omega(x)|x|^{-n-\nu}$ and P_k range over the homogeneous harmonic polynomials the latter arise in special case $\alpha = 1$. Then for $\alpha > 1$, we call the higher order Riesz-Bessel transform where we refer to α as the degree of the higher order Riesz-Bessel transform [1, 6, 7]. Since P_k is homogeneous B-polynomial of degree k in $\mathbb{R}^n_+$, we shall say that P_k is elliptic if $P_k(x)$ vanishes only at the origin. For any polynomial P_k we consider also its corresponding differential polynomial. Thus if $P_k(x) = \sum a_\alpha x^\alpha$, we write

$$\begin{aligned} P_k\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{n-1}}, B_n\right) = P_k(\Delta_\nu) &= \sum_{|\alpha'|+2\alpha_n=m} a_\alpha\left(\frac{\partial}{\partial x'} + B_n\right)^\alpha \\ &= \sum_{|\alpha'|+2\alpha_n=m} a_\alpha D^{\alpha'}_{x'} B^{2\alpha_n}_n, \end{aligned}$$

where $\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ and with the monomials $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ [21].

Now, we give some technical lemmas which will be used in the proof of the main theorem.

Lemma 3.1. *Let r, m be real numbers and $u(x) \in C^\infty(\mathbb{R}^n_+ \setminus \{0\})$ be homogeneous of degree r. Then for $x \neq 0$*

$$\Delta_\nu(|x|^m u(x)) = |x|^m \Delta_\nu u(x) + m(m + 2r + n - 2)|x|^{m-2} u(x).$$

Lemma 3.2. *Suppose $\mathcal{P}$ is a set of all homogeneous polynomials of degree k. If $P_k \in \mathcal{P}$ $(k \geqslant 1)$, then*

$$P_k(x) = P_0(x) + |x|^2 P_1(x) + \ldots + |x|^{2k_1} P_{k_1}(x),$$

where $k_1 = [\frac{k}{2}]$ and P_j is a homogeneous polynomial harmonic of degree $k - 2j, j = 0, 1, \ldots, k_1$.

Lemma 3.3. *Let* j, m *be positive integers with* $j > m$ *and* u *be a homogeneous harmonic function on* $\mathbb{R}^n_+$. *Then*

$$\Delta^j_\nu(|x|^{2m}u(x)) = 0.$$

Lemma 3.4. *Let* k, ℓ *be positive integers,* $k_1 = [\frac{k}{2}]$ *and* P_k *be a homogeneous polynomial of degree* k. *Then* P_k *is polyharmonic of degree* ℓ *if and only if*

$$P_k(x) = P_0(x) + |x|^2 P_1(x) + \ldots + |x|^{2m} P_m(x),$$

where $m = \min(\ell - 1, k_1)$ *and* P_j *is a homogeneous harmonic polynomial of degree* $k - 2j, j = 0, 1, \ldots, k_1$.

Proof. If $k = 1$, then the lemma is obvious. Let $k > 2$. If $\ell - 1 \geqslant k_1$ then $2\ell > k$. Hence the lemma follows from Lemma 3.2 and the fact that a homogeneous polynomial of degree k is polyharmonic of degree ℓ. Let $\ell - 1 < k_1$. If $P_k(x) = P_0(x) + |x|^2 P_1(x) + \ldots + |x|^{2m} P_m(x)$, then $\Delta^\ell_\nu P_k(x) = 0$ by Lemma 3.3 since $m = \ell - 1$. Conversely, we assume that P_k is polyharmonic of degree ℓ. Since P_k is a homogeneous polynomial of degree k, by Lemma 3.2

$$P_k(x) = P_0(x) + |x|^2 P_1(x) + \ldots + |x|^{2k_1} P_{k_1}(x),$$

where P_j is a homogeneous harmonic polynomial of degree $k - 2j, j = 0, 1, \ldots, k_1$. By the assumption, $\Delta^\ell_\nu P_k(x) = \Delta^{\ell+1}_\nu P_k(x) = \ldots = \Delta^{k_1}_\nu P_k(x) = 0$. By Lemma 3.3, we see that $0 = \Delta^{k_1}_\nu P_k(x) = c(k - 2k_1, k_1) P_{k_1}(x)$. Since $c(k - 2k_1, k_1) \neq 0$, this implies that $P_{k_1} = 0$. By repeating the same arguments we obtain that $P_\ell = P_{\ell+1} = \ldots = P_{k_1} = 0$. This proves the lemma. □

Corollary 3.5. *Let* P_k *be a homogeneous polynomial of degree* 2ℓ. *Then* P_k *is polyharmonic of degree* ℓ *if and only if*

$$\int_{S^n_+} P_k(x) x_n^\nu dx = 0.$$

Proof. By Lemma 3.2 we may write

$$P_k(x) = P_0(x) + |x|^2 P_1(x) + \ldots + |x|^{2(\ell-1)} P_{\ell-1}(x) + c_\ell |x|^{2\ell},$$

where P_j is a homogeneous harmonic polynomial of degree $2\ell - 2j, j = 0, 1, \ldots, \ell - 1$ and c_ℓ is a constant. By harmonically of P_j and $P_j(0) = 0$, we have

$$\int_{S^n_+} P_k(x) x_n^\nu dx = c_\ell \sigma_n,$$

where $\sigma_n = \int_{S^n_+} x_n^\nu dx$. □

Lemma 3.6. *Let* $m + n + \nu \in \mathbb{N}$. *If*

$$\lim_{|x| \to 0} \frac{P_k(x)}{|x|^{n+m+\nu}} = 0,$$

then $a_k = 0$ *for all* k *with* $|k| = n + m + \nu$.

Proof. Let us take $P_k(x) = \sum\limits_{|k|=m+n+\nu} a_k x^k$. By assumption for θ with $|\theta| = 1$ we have

$$0 = \lim_{t \to +0} \frac{P_k(t\theta)}{|t\theta|^{n+\nu+m}} = \lim_{t \to +0} \frac{t^{n+m+\nu} P_k(\theta)}{t^{n+m+\nu}} = P_k(\theta).$$

For $x \neq 0$, by putting $t = |x|$ and $\theta = \frac{x}{|x|}$, we get,

$$P_k(x) = P_k(t\theta) = t^{m+n+\nu} P_k(\theta) = 0.$$

Since it is clear that $P_k(0) = 0$, we conclude that $P_k(x)$ is identically 0, and hence $a_k = 0$ for all k with $|k| = n + m + \nu$, (see [13]). □

Theorem 3.7. *Let P_k be homogeneous harmonic polynomial of degree k. Then we get*

$$F_\nu\Big[\, p.v\, \frac{P_k(x)}{|x|^{n+m+\nu}}\Big](y) = 2^{\frac{-n-\nu}{2}} i^m \Gamma(\tfrac{m}{2})\Gamma\big(\tfrac{n+m+\nu}{2}\big)^{-1} P_k(y)|y|^{-m}.$$

Further, we define the higher order Riesz-Bessel transforms R_ν^m as follows:

Definition 3.8. Let T^y be the Bessel generalized shift operator and let f be a Schwartz function on $\mathbb{R}_+^n$. We define the higher order Riesz-Bessel transforms R_ν^m of order m with respect to Bessel generalized shift operator as

$$\begin{aligned} R_\nu^m(f)(x) &= c_m(n,\nu)\Big[\, p.v\, \frac{P_k(y)}{|y|^{n+m+\nu}} \otimes f\Big](x) \\ &= c_m(n,\nu) \lim_{\varepsilon\to 0} \int\limits_{0<\varepsilon<|x|} \frac{P_k(y)}{|y|^{n+m+\nu}} T^y f(x) d\mu_\nu(y), \end{aligned}$$

where $c_m(n,\gamma) = 2^{\frac{n+\nu}{2}}\Gamma(\frac{n+m+\nu}{2})\big[\Gamma(\frac{m}{2})\big]^{-1}(m = 1,2,\dots,n)$ and $P_k(x)$ is a homogeneous polynomial of degree k in $\mathbb{R}_+^n$ which satisfies $\Delta_\nu P_k = 0$ (see [6, 7]).

4. The higher order Riesz Bessel operator in $BL^m_{p,\nu}(\mathbb{R}_+^n)$ spaces

In this section, we prove an estimate for the singular integral operator with convolution type generated by generalized shift operator in the Beppo-Levi spaces. Let us start to give the following $L_{p,\nu}$ boundedness of the higher order Riesz-Bessel transform in the $L_{p,\nu}(\mathbb{R}_+^n)$ space.

Theorem 4.1. *The higher order Riesz-Bessel transforms generated by Bessel generalized shift operator are bounded operator from $L_{p,\nu}(\mathbb{R}_+^n)$ into itself for all $1 < p < \infty$*

$$\|R_\nu^m f\|_{p,\nu} \leqslant A_p \|f\|_{p,\nu}.$$

Moreover, the relationship between partial derivatives of the Riesz potential of order one and the Riesz transforms is given for $f \in L_p$ $(p < n)$ as follows

$$D_j(I_m f) = (1-n)c_n^{-1} R_j f, \qquad j = 1,2,\dots,n.$$

In this section we are concerned with relationship between partial derivatives of the Riesz potentials of order m and higher order Riesz-Bessel transforms. Since the Riesz-Bessel transforms are bounded operators on $L_{p,\nu}(\mathbb{R}_+^n)$, it can be easily seen that $R_\nu^m \in L_{p,\nu}$ for $f \in L_{p,\nu}(\mathbb{R}_+^n)$. For $f \in L_{p,\nu}(\mathbb{R}_+^n)$, it follows from Theorem 3.7

$$F_\nu[R_\nu^m(f)](\zeta) = i^m P_k(\zeta)|\zeta|^{-m} F_\nu[f](\zeta) \tag{4.1}$$

and

$$F_\nu\big[K_{m,\nu}(x)\big] = 2^{\frac{-n-\nu}{2}} i^m \Gamma(\tfrac{m}{2})\Gamma\big(\tfrac{n+m+\nu}{2}\big)^{-1} P_k(x)|x|^{-m}$$

for $f \in L_{p,\nu}(\mathbb{R}_+^n)$, where a multi-index $\nu = (\nu_1,\dots,\nu_n)$, and $R_\nu^m = R_{\nu_1}^m \dots R_{\nu_n}^m$. For functions f such that $D_{x'}^{\alpha'} B_n^{\alpha_n} f \in L_{p,\nu}(\mathbb{R}_+^n)$ for any $|\alpha| = m$, we set $D_{x'}^{\alpha'} B_n^{\alpha_n} f = R_\nu^m D_{x'}^{\alpha'} B_n^{\alpha_n} f$.

Proposition 4.2. *Assume that f is a class of $S(\mathbb{R}_+^n)$ and has compact support. Let Δ_ν be the Laplace-Bessel differential operator. Then the following inequality holds*

$$\Big\| \sum_{i,j=1}^{n-1} \frac{\partial^2 f}{\partial_{x_i}\partial_{x_j}} \Big\|_{p,\nu} + \|B_n f\|_{p,\nu} \leqslant A_p \|\Delta_\nu f\|_{p,\nu}.$$

Here, A_p is a constant independent of f.

In (4.1), we set $m = 1$. Then this proposition is an immediate consequence of the $L_{p,\nu}(\mathbb{R}^n_+)$ boundedness of the Riesz-Bessel transforms generated by generalized shift operator and the identity

$$\partial_{x_i}(\partial_{x_j} f) + B_n f = -R_{\nu_i} R_{\nu_j} \Delta_\nu f. \tag{4.2}$$

To prove (4.2), we use the Fourier-Bessel transform. Since $F_\nu[f](x)$ is the Fourier-Bessel transform of f, we consider the Fourier-Bessel transform of $\partial_{x_i} f$ and $B_n f$. By (2.2)-(2.5), we have

$$\begin{aligned} F_\nu\left[\partial_{x_i}(\partial_{x_i} f) + B_n f\right](x) &= F_\nu\left(D^{2\alpha'}_{x'} B^{\alpha_n}_n f\right)(x) \\ &= \sum_{i=1}^{n} (-1)^{|\alpha|} x_i^2 F_\nu f(x) \\ &= \sum_{i=1}^{n} (-1)^{|\alpha|} \frac{x_i}{|x|} \frac{x_i}{|x|} |x|^2 F_\nu f(x) \\ &= \sum_{i=1}^{n} (-1)^{|\alpha|} F_\nu [R_{\nu_i} R_{\nu_i}] F_\nu (\Delta^\alpha_\nu f)(x), \end{aligned}$$

which gives (4.2). Thus, we obtain

$$\|\partial_{x_i}(\partial_{x_j} f)\|_{p,\nu} + \| B^\alpha_n f \|_{p,\nu} \leqslant \|R_{\nu_i} R_{\nu_j} \Delta^\alpha_\nu f\|_{p,\nu} = C\|\Delta^\alpha_\nu f\|_{p,\nu}.$$

Using the Fourier-Bessel transformations, we have

$$P_k(|x|^2) F_\nu[(\Delta_\nu) f](x) \quad = -|x|^2 F_\nu [P_k(\Delta_\nu) f](x).$$

Corollary 4.3. *Suppose that P_k is a homogeneous elliptic polynomial of degree k and f is k-times continuously differentiable with compact support. Then we have the priori estimate*

$$\left|\left|\Delta^\alpha_\nu f\right|\right|_{p,\nu} \leqslant A_{p,\alpha} \left|\left|P_k(\Delta_\nu) f\right|\right|_{p,\nu}, \qquad 1 < p < \infty.$$

To prove this inequality, we note that the following relation between Fourier-Bessel transform of $\Delta^\alpha_\nu f$ and $P_k(\Delta_\nu) f$ holds

$$P_k(|x|^2) F_\nu\left[\Delta^\alpha_\nu f\right](x) = -|x|^2 F_\nu\left[P_k(\Delta_\nu) f\right](x).$$

Since $P_k(|x|^2)$ is non-vanishing except the origin and let $-\frac{|x|^2}{P_k(|x|^2)}$ be homogeneous of degree zero and indefinitely differentiable on the unit sphere. Then we get

$$\Delta^\alpha_\nu f = R_\nu\left(P_k(\Delta_\nu) f\right).$$

In this section, we are concerned with relationship between partial derivatives of the Riesz-Bessel potentials of order m and higher order Riesz-Bessel transforms. Following S. G. Samko[20], for a multi-index with $|\alpha| = m$ we set $R^m = R_1^{m_1} \dots R_n^{m_n}$. We call $R^m_\nu (|\alpha| = m)$ the high order Riesz-Bessel transforms of degree m. For functions f such that $D^{\alpha'}_{x'} B^{\alpha_n}_n f \in L_{p,\nu}(\mathbb{R}^n_+)$ for any $|\alpha| = m$, we set $D^{\alpha'}_{x'} B^{\alpha_n}_n f = R^m_\nu D^{\alpha'}_{x'} B^{\alpha_n}_n f$

Lemma 4.4. *For $f \in \mathcal{S}(\mathbb{R}^n_+)$, $F_\nu(\Delta^\alpha_\nu f) = (-1)^\alpha |x|^{2\alpha} F_\nu f$ equality holds.*

Moreover, relationship between partial derivatives of the B-Riesz potential of order α and the Riesz-Bessel transforms is given as follows.

Theorem 4.5. *Let $k = [m - \frac{n+\nu}{p}]$ and for a multi-index α with $|\alpha| = m$ and $0 < m < n + \nu$. Then there is a constant $c^{-1}_{m,\nu}$ such that for all f in $BL^m_{p,\nu}(\mathbb{R}^n_+)$ we have $\Delta^\alpha_\nu(I_{m,\nu} f) = c^{-1}_{m,\nu} R^\alpha_\nu f$.*

Proof. Let $m-n-\nu$ nonnegative even numbers. For a positive integer α, the Riesz-Bessel potential kernel $K_{m,\nu}(x)$ of order m is given by $K_{m,\nu}(x)=|x|^{m-n-\nu}$ we obtain

$$F_\nu\left[K_{m,\nu}(x)\right](x)=2^{\frac{-m-n-\nu}{2}}\Gamma\left(\frac{m}{2}\right)^{-1}\Gamma\left(\frac{m+n+\nu}{2}\right)|x|^{-m}.$$

Now we consider Fourier-Bessel transform of B-generalized shift operator. We get

$$\begin{aligned}F_\nu[T^y f(x)](\zeta)&=\int_{R_n^+}T^y f(x)j_{\nu-\frac{1}{2}}(x_n\zeta_n)(x')^\nu dx\\&=\int_{R_n^+}T^y\left[j_{\nu-\frac{1}{2}}(x_n\zeta_n)\right]f(x)(x')^\nu dx\\&=\int_{R_n^+}j_{\nu-\frac{1}{2}}(x_n\zeta_n)j_{\nu-\frac{1}{2}}(z_n\zeta_n)f(x)(x')^\nu dx\\&=j_{\nu-\frac{1}{2}}(z_n\zeta_n))\int_{R_n^+}f(x)j_{\nu-\frac{1}{2}}(x_n\zeta_n))(x')^\nu dx\\&=j_{\nu-\frac{1}{2}}(z_n\zeta_n))F_\nu[f](\zeta),\end{aligned}$$

where $T^y(j_p(\sqrt{\lambda}x)=j_p(\sqrt{\lambda}y)j_p(\sqrt{\lambda}x)$. By using (4.1) and (2.4), and if we consider that

$$F_\nu(\Delta_\nu^\alpha I_{m,\nu}f)(x)=F_\nu(\Delta_\nu^\alpha(K_{m,\nu}\otimes T^y f))(x)=(-1)^{|\alpha|}|x|^{2\alpha}F_\nu[I_{m,\nu}f](x),$$

then we have

$$\begin{aligned}F_\nu(\Delta_\nu^\alpha I_{m,\nu}f)(x)&=F_\nu(\Delta_\nu^\alpha(K_{m,\nu}\otimes T^y f)(x))=(-1)^\alpha|x|^{2\alpha}F_\nu(K_{m,\nu})F_\nu(T^y f)\\&=(-1)^\alpha|x|^{2\alpha}2^{\frac{-m-n-\nu}{2}}\frac{\Gamma(\frac{m}{2})}{\Gamma\left(\frac{n+m+\nu}{2}\right)}\frac{P_k(x)}{|x|^\alpha}j_{\nu-\frac{1}{2}}(x_ny_n))F_\nu f(x)\\&=(-1)^\alpha|x|^{2\alpha}j_{\nu-\frac{1}{2}}(x_ny_n)2^{\frac{-m-n-\nu}{2}}i^\alpha\frac{\Gamma(\frac{m}{2})}{\Gamma\left(\frac{n+m+\nu}{2}\right)}\frac{P_k(x)}{|x|^\alpha}F_\nu f(x)\\&=i^{2\alpha}|x|^{2\alpha}j_{\nu-\frac{1}{2}}(x_ny_n)2^{\frac{-m-n-\nu}{2}}\frac{\Gamma(\frac{m}{2})}{\Gamma\left(\frac{n+m+\nu}{2}\right)}P_k(x)|x|^{-\alpha}F_\nu f(x)\\&=i^\alpha j_{\nu-\frac{1}{2}}(x_ny_n)2^{\frac{-m-n-\nu}{2}}\frac{\Gamma(\frac{m}{2})}{\Gamma\left(\frac{n+m+\nu}{2}\right)}F_\nu[R_\nu^\alpha](f).\end{aligned}$$

If we choose $c_{m,\nu}$ as follows

$$c_{m,\nu}=\left[i^\alpha j_{\nu-\frac{1}{2}}(x_ny_n)2^{\frac{-m-n-\nu}{2}}\frac{\Gamma(\frac{m}{2})}{\Gamma\left(\frac{n+m+\nu}{2}\right)}\right]^{-1}$$

and we consider the Fourier-Bessel transform of higher order Riesz-Bessel transform $R_\nu^\alpha(f)$ then we obtain $\Delta_\nu^\alpha(I_{m,\nu}f)=c_{m,\nu}^{-1}R_\nu^\alpha f$ which proves Theorem 4.5. □

The key point is the following characterization:

Theorem 4.6. *Let $1<p<\infty$ and $k=[m-\frac{n+\nu}{p}]$ and for a multi-index α with $|\alpha|=m$ and $0<m<n+\nu$. Then $f\in S_+(\mathbb{R}_+^n)$ belongs to $BL_{p,\nu}^m(\mathbb{R}_+^n)$ if and only if $\Delta_\nu^\alpha I_{m,\nu}f\in L_{p,\nu}(\mathbb{R}_+^n)$ and $R_\nu^\alpha f\in BL_{p,\nu}^m(\mathbb{R}_+^n)$. Moreover, there exists a positive constant C such that, for all $f\in BL_{p,\nu}^m(\mathbb{R}_+^n)$,*

$$\|\Delta_\nu^\alpha I_{m,\nu}f\|_{L_{p,\nu}(\mathbb{R}^n)}\leqslant\|\mu_m\|_{M_m(\mathbb{R}_+^n)}\|R_\nu^\alpha f\|_{BL_{p,\nu}^m(\mathbb{R}_+^n)}\leqslant C\|f\|_{BL_{p,\nu}^m(\mathbb{R}_+^n)},$$

where

$$\|\mu_m\|_{M_m(\mathbb{R}_+^n)}:=\sup_{\varphi\in C_c(\mathbb{R}_+^n),\|\varphi\|_\infty\leqslant1}\int_{\mathbb{R}_+^n}\varphi d\mu_m.$$

Proof. We can state that there exists a finite signed measure μ_m for which

$$F_\nu(\mu_m) = i^\alpha j_{\nu-\frac{1}{2}}(x_n y_n) 2^{\frac{-m-n-\nu}{2}} \frac{\Gamma(\frac{m}{2})}{\Gamma\left(\frac{n+m+\nu}{2}\right)}$$

and therefore inverting the Fourier-Bessel transform we have

$$(\Delta_\nu^\alpha I_{m,\nu} f)(x) = \mu_m \otimes R_\nu^\alpha f,$$

where R_ν^m is the Riesz-Bessel transform (see Chapter V, Section 3.2, Lemma 2 in Stein [21]). Another application of Young's inequality yields

$$\|\Delta_\nu^\alpha I_{m,\nu} f\|_{L_{p,\nu}(\mathbb{R}^n)} \leqslant \|\mu_m\|_{M_m(\mathbb{R}^n_+)} \|R_\nu^\alpha f\|_{BL^m_{p,\nu}(\mathbb{R}^n_+)},$$

where

$$\|\mu_m\|_{M_m(\mathbb{R}^n_+)} := \sup_{\varphi \in C_c(\mathbb{R}^n_+), \|\varphi\|_\infty \leqslant 1} \int_{\mathbb{R}^n_+} \varphi \, d\mu_m.$$

The previous inequality, along with the boundedness of the Riesz-Bessel transform $R_\nu^m : BL^m_{p,\nu}(\mathbb{R}^n_+) \to BL^m_{p,\nu}(\mathbb{R}^n_+)$ for $1 < p < \infty$, and $|\alpha| = m$ implies the desired result. □

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