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Schur convexity properties for a class of symmetric functions with applications

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Abstract

In the article, we prove that the symmetric function

$$F_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} \prod_{j=1}^{r} \left(\frac{1+x_{i_j}}{1-x_{i_j}} \right)^{1/r}$$

is Schur convex, Schur multiplicatively convex and Schur harmonic convex on $[0,1)^n$, and establish several new analytic inequalities by use of the theory of majorization, where $r \in \{1, 2, \cdots, n\}$ and $i_1, i_2, \cdots i_n$ are integers.

Keywords: Schur convex, Schur multiplicatively convex, Schur harmonic convex, symmetric function.

2010 MSC: 05E05, 26B25.

1. Introduction

Throughout this paper, we use $\mathbb{R}^n$ to denote the $n-$dimensional Euclidean space $(n \geqslant 2)$, and $\mathbb{R}^n_+ = \{(x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$. In particular, we use $\mathbb{R}$ to denote $\mathbb{R}^1$.

For the sake of convenience, we use the following notation system.

Let $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n_+$ and $\alpha \in \mathbb{R}$. Then we denote by

$$\begin{aligned}
&x \pm y = (x_1 \pm y_1, x_2 \pm y_2, \cdots, x_n \pm y_n), \quad xy = (x_1y_1, x_2y_2, \cdots, x_ny_n),\\
&\alpha x = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), \quad e^x = (e^{x_1}, e^{x_2}, \cdots, e^{x_n}),\\
&\alpha \pm x = (\alpha \pm x_1, \alpha \pm x_2, \cdots, \alpha \pm x_n), \quad x^\alpha = (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha),\\
&\log x = (\log x_1, \log x_2, \cdots, \log x_n), \quad \frac{1}{x} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right),\\
&A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad G_n(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.
\end{aligned}$$

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Definition 1.1. A real-valued function F on $E \subseteq \mathbb{R}^n$ is said to be Schur convex if

$$F(x_1, x_2, \cdots, x_n) \leqslant F(y_1, y_2, \cdots, y_n)$$

for each pair of n-tuples $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in E$ such that $x \prec y$, i.e.,

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \cdots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component of x. F is said to be Schur concave if $-F$ is Schur convex.

Definition 1.2. A real-valued function F on $E \subseteq \mathbb{R}^n_+$ is said to be Schur multiplicatively convex if

$$F(x_1, x_2, \cdots, x_n) \leqslant F(y_1, y_2, \cdots, y_n)$$

for each pair of $n-$tuples $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in E$ such that $\log x \prec \log y$. F is said to be Schur multiplicatively concave if $\frac{1}{F}$ is Schur multiplicatively convex.

Definition 1.3. A real-valued function F on $E \subseteq \mathbb{R}^n_+$ is said to be Schur harmonic convex (concave) if

$$F(x_1, x_2, \cdots, x_n) \leqslant (\geqslant) F(y_1, y_2, \cdots, y_n)$$

for each pair of $n-$tuples $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in E$ such that $\frac{1}{x} \prec \frac{1}{y}$.

The Schur convexity was introduced by Schur [18] in 1923. It has many important applications in analytic inequalities [1–4, 12, 15, 21–23, 25–29], extended mean values theory [7, 16, 17, 19, 20] and other related fields. Recently, the Schur multiplicative and harmonic convexity properties were investigated in [5, 6, 8, 10, 11].

For $n \geqslant 2$, $x = (x_1, x_2, \cdots, x_n) \in [0, 1)^n$ and $r \in \{1, 2, \cdots, n\}$, the symmetric function $F_n(x, r)$ is defined by

$$F_n(x, r) = F_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} \prod_{j=1}^{r} \left(\frac{1 + x_{i_j}}{1 - x_{i_j}} \right)^{1/r}, \tag{1.1}$$

where $i_1, i_2, \cdots, i_r$ are integers.

The main purpose of this paper is to discuss the Schur convexity, Schur multiplicative convexity and Schur harmonic convexity of $F_n(x, r)$. As applications, we establish several analytic inequalities by use of the theory of majorization.

2. Lemmas

In order to prove our main result we need several lemmas, which we present in this section.

Lemma 2.1 ([14]). *Let $E \subseteq \mathbb{R}^n$ be a symmetric convex set with nonempty interior $\mathrm{int}E$ and $f : E \to \mathbb{R}$ be a continuous symmetric function. If f is differentiable on $\mathrm{int}E$, then f is Schur convex on E if and only if*

$$(x_i - x_j) \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j} \right) \geqslant 0 \tag{2.1}$$

for all $i, j = 1, 2, \cdots, n$ and $x = (x_1, \cdots, x_n) \in \mathrm{int}E$. f is Schur concave on E if and only if inequality (2.1) is reversed, where E is a symmetric set means that $x \in E$ implies $Px \in E$ for any $n \times n$ permutation matrix P.

Remark 2.2. Since f is symmetric, the Schur's condition in Lemma 2.1, i.e., (2.1) reduces to

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geqslant 0.$$

Lemma 2.3 ([13]). *Let $E \subseteq \mathbb{R}_+^n$ be a symmetric multiplicatively convex set with nonempty interior $\mathrm{int}E$ and $f : E \to \mathbb{R}_+$ be a continuous symmetric function. If f is differentiable on $\mathrm{int}E$, then f is Schur multiplicatively convex on E if and only if*

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2}\right) \geqslant 0$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathrm{int}E$, where E is a multiplicatively convex set means that $x^{1/2}y^{1/2} \in E$ whenever $x, y \in E$.

Lemma 2.4 ([5]). *Let $E \subseteq \mathbb{R}_+^n$ be a symmetric harmonic convex set with nonempty interior $\mathrm{int}E$ and $f : E \to \mathbb{R}_+$ be a continuous symmetric function. If f is differentiable on $\mathrm{int}E$, then f is Schur harmonic convex on E if and only if*

$$(x_1 - x_2)\left(x_1^2\frac{\partial f}{\partial x_1} - x_2^2\frac{\partial f}{\partial x_1}\right) \geqslant 0$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathrm{int}E$, where E is a harmonic convex set means that $2xy/(x+y) \in E$ whenever $x, y \in E$.

Lemma 2.5 ([9]). *Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \geqslant s$, then*

$$\frac{c - x}{\frac{nc}{s} - 1} = \left(\frac{c - x_1}{\frac{nc}{s} - 1}, \frac{c - x_2}{\frac{nc}{s} - 1}, \cdots, \frac{c - x_n}{\frac{nc}{s} - 1}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

Lemma 2.6 ([9]). *Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \geqslant 0$, then*

$$\frac{c + x}{\frac{nc}{s} + 1} = \left(\frac{c + x_1}{\frac{nc}{s} + 1}, \frac{c + x_2}{\frac{nc}{s} + 1}, \cdots, \frac{c + x_n}{\frac{nc}{s} + 1}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

Lemma 2.7 ([24]). *Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $0 \leqslant \lambda \leqslant 1$, then*

$$\frac{s - \lambda x}{n - \lambda} = \left(\frac{s - \lambda x_1}{n - \lambda}, \frac{s - \lambda x_2}{n - \lambda}, \cdots, \frac{s - \lambda x_n}{n - \lambda}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

Simple computations lead to Lemma 2.8 and Lemma 2.9 immediately.

Lemma 2.8. *Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $0 \leqslant \lambda \leqslant 1$, then*

$$\frac{s + \lambda x}{n + \lambda} = \left(\frac{s + \lambda x_1}{n + \lambda}, \frac{s + \lambda x_2}{n + \lambda}, \cdots, \frac{s + \lambda x_n}{n + \lambda}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

Lemma 2.9. *The function $x \to f(x) = (1+x)^{1-1/p}(1-x)^{1+1/p}$ is strictly decreasing on $[0, 1)$ if $p \geqslant 1$.*

3. Main Results

Theorem 3.1. *Let $n \geqslant 2$ and $F_n(x, r)$ be defined by (1.1). Then the following statements are true:*

(1) $F_n(x, r)$ *is Schur convex on* $[0, 1)^n$;
(2) $F_n(x, r)$ *is Schur multiplicatively convex on* $[0, 1)^n$;
(3) $F_n(x, r)$ *is Schur harmonic convex on* $[0, 1)^n$.

Proof. We clearly see that $F_n(x, r)$ is symmetric and has continuous first partial derivatives on $(0, 1)^n$. By Lemma 2.1, Remark 2.2, Lemma 2.3 and Lemma 2.4, we only need to prove that

$$(x_1 - x_2)\left(\frac{\partial F_n(x,r)}{\partial x_1} - \frac{\partial F_n(x,r)}{\partial x_2}\right) \geqslant 0, \tag{3.1}$$

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial F_n(x,r)}{\partial x_1} - x_2\frac{\partial F_n(x,r)}{\partial x_2}\right) \geqslant 0 \tag{3.2}$$

and

$$(x_1 - x_2)\left(x_1^2\frac{\partial F_n(x,r)}{\partial x_1} - x_2^2\frac{\partial F_n(x,r)}{\partial x_1}\right) \geqslant 0 \tag{3.3}$$

for all $x = (x_1, x_2, \cdots, x_n) \in (0, 1)^n$ and $r \in \{1, 2, \cdots, n\}$.

We divide the proof into four cases.

Case 1. $r = 1$. Then (1.1) leads to

$$F_n(x,1) = F_n(x_1,x_2,\cdots,x_n;1) = \sum_{i=1}^{n}\frac{1+x_i}{1-x_i},$$

$$\frac{\partial F_n(x,1)}{\partial x_i} = \frac{2}{(1-x_i)^2} \quad (i = 1, 2),$$

$$(x_1 - x_2)\left(\frac{\partial F_n(x,1)}{\partial x_1} - \frac{\partial F_n(x,1)}{\partial x_2}\right) = \frac{2(x_1-x_2)^2(2-x_1-x_2)}{(1-x_1)^2(1-x_2)^2} \geqslant 0, \tag{3.4}$$

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial F_n(x,1)}{\partial x_1} - x_2\frac{\partial F_n(x,1)}{\partial x_2}\right) = 2(\log x_1 - \log x_2)\frac{(x_1-x_2)(1-x_1x_2)}{(1-x_1)^2(1-x_2)^2} \geqslant 0 \tag{3.5}$$

and

$$(x_1 - x_2)\left(x_1^2\frac{\partial F_n(x,1)}{\partial x_1} - x_2^2\frac{\partial F_n(x,1)}{\partial x_1}\right) = 2(x_1-x_2)^2\frac{x_1+x_2-2x_1x_2}{(1-x_1)^2(1-x_2)^2} \geqslant 0. \tag{3.6}$$

Case 2. $r = n$. Then it follow from (1.1) that

$$F_n(x,n) = F_n(x_1,x_2,\cdots,x_n;n) = \prod_{i=1}^{n}\left(\frac{1+x_i}{1-x_i}\right)^{1/n},$$

$$\frac{\partial F_n(x,n)}{\partial x_i} = \frac{2F_n(x,n)}{n(1-x_i^2)} \quad (i = 1, 2),$$

$$(x_1 - x_2)\left(\frac{\partial F_n(x,n)}{\partial x_1} - \frac{\partial F_n(x,n)}{\partial x_2}\right) = \frac{2(x_1-x_2)^2(x_1+x_2)F_n(x,n)}{n(1-x_1^2)(1-x_2^2)} \geqslant 0, \tag{3.7}$$

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial F_n(x,n)}{\partial x_1} - x_2\frac{\partial F_n(x,n)}{\partial x_2}\right) = \frac{2(x_1-x_2)(\log x_1 - \log x_2)(1+x_1x_2)F_n(x,n)}{n(1-x_1^2)(1-x_2^2)} \geqslant 0 \tag{3.8}$$

and

$$(x_1 - x_2)\left(x_1^2\frac{\partial F_n(x,n)}{\partial x_1} - x_2^2\frac{\partial F_n(x,n)}{\partial x_1}\right) = \frac{2(x_1-x_2)^2(x_1+x_2)F_n(x,n)}{n(1-x_1^2)(1-x_2^2)} \geqslant 0. \tag{3.9}$$

Case 3. $n \geqslant 3$ and $r = 2$. Then from (1.1) and Lemma 2.9 we clearly see that

$$\begin{aligned} F_n(x,2) &= F_n(x_1,x_2,\cdots,x_n;2) \\ &= \left(\frac{1+x_1}{1-x_1}\right)^{1/2}\left(\frac{1+x_2}{1-x_2}\right)^{1/2} + \left(\frac{1+x_1}{1-x_1}\right)^{1/2}\sum_{3\leqslant i\leqslant n}\left(\frac{1+x_i}{1-x_i}\right)^{1/2} + F_{n-1}(x_2,x_3,\cdots,x_n;2) \end{aligned}$$

$$= \left(\frac{1+x_1}{1-x_1}\right)^{1/2}\left(\frac{1+x_2}{1-x_2}\right)^{1/2} + \left(\frac{1+x_2}{1-x_2}\right)^{1/2}\sum_{3\leqslant i\leqslant n}\left(\frac{1+x_i}{1-x_i}\right)^{1/2} + F_{n-1}(x_1, x_3, \cdots, x_n; 2),$$

$$\frac{\partial F_n(x,2)}{\partial x_1} = \left(\frac{1+x_2}{1-x_2}\right)^{1/2}\frac{(1+x_1)^{-1/2}}{(1-x_1)^{3/2}} + \frac{(1+x_1)^{-1/2}}{(1-x_1)^{3/2}}\sum_{3\leqslant i\leqslant n}\left(\frac{1+x_i}{1-x_i}\right)^{1/2},$$

$$\frac{\partial F_n(x,2)}{\partial x_2} = \left(\frac{1+x_1}{1-x_1}\right)^{1/2}\frac{(1+x_2)^{-1/2}}{(1-x_2)^{3/2}} + \frac{(1+x_2)^{-1/2}}{(1-x_2)^{3/2}}\sum_{3\leqslant i\leqslant n}\left(\frac{1+x_i}{1-x_i}\right)^{1/2},$$

$$\begin{aligned}&(x_1-x_2)\left(\frac{\partial F_n(x,2)}{\partial x_1} - \frac{\partial F_n(x,2)}{\partial x_2}\right)\\ &=\frac{(x_1-x_2)^2(x_1+x_2)}{(1+x_1)^{1/2}(1-x_1)^{3/2}(1+x_2)^{1/2}(1-x_2)^{3/2}} + (x_1-x_2)\\ &\quad\times\left[\frac{1}{(1+x_1)^{1/2}(1-x_1)^{3/2}} - \frac{1}{(1+x_2)^{1/2}(1-x_2)^{3/2}}\right]\sum_{3\leqslant i\leqslant n}\left(\frac{1+x_i}{1-x_i}\right)^{1/2} \geqslant 0,\end{aligned} \tag{3.10}$$

$$\begin{aligned}&(\log x_1 - \log x_2)\left(x_1\frac{\partial F_n(x,2)}{\partial x_1} - x_2\frac{\partial F_n(x,2)}{\partial x_2}\right)\\ &=\frac{(x_1-x_2)(\log x_1-\log x_2)(1+x_1x_2)}{(1+x_1)^{1/2}(1-x_1)^{3/2}(1+x_2)^{1/2}(1-x_2)^{3/2}} + (\log x_1 - \log x_2)\\ &\quad\times\left[\frac{x_1}{(1+x_1)^{1/2}(1-x_1)^{3/2}} - \frac{x_2}{(1+x_2)^{1/2}(1-x_2)^{3/2}}\right]\sum_{3\leqslant i\leqslant n}\left(\frac{1+x_i}{1-x_i}\right)^{1/2} \geqslant 0\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&(x_1-x_2)\left(x_1^2\frac{\partial F_n(x,2)}{\partial x_1} - x_2^2\frac{\partial F_n(x,2)}{\partial x_2}\right)\\ &=\frac{(x_1-x_2)^2(x_1+x_2)}{(1+x_1)^{1/2}(1-x_1)^{3/2}(1+x_2)^{1/2}(1-x_2)^{3/2}} + (x_1-x_2)\\ &\quad\times\left[\frac{x_1^2}{(1+x_1)^{1/2}(1-x_1)^{3/2}} - \frac{x_2^2}{(1+x_2)^{1/2}(1-x_2)^{3/2}}\right]\sum_{3\leqslant i\leqslant n}\left(\frac{1+x_i}{1-x_i}\right)^{1/2} \geqslant 0.\end{aligned} \tag{3.12}$$

Case 4. $n \geqslant 4$ and $3 \leqslant r \leqslant n-1$. Then (1.1) and Lemma 2.9 lead to

$$\begin{aligned}F_n(x,r) &= \left(\frac{1+x_1}{1-x_1}\right)^{1/r}\left(\frac{1+x_2}{1-x_2}\right)^{1/r}\sum_{3\leqslant i_1<i_2<\cdots<i_{r-2}\leqslant n}\prod_{j=1}^{r-2}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r}\\ &\quad+\left(\frac{1+x_1}{1-x_1}\right)^{1/r}\sum_{3\leqslant i_1<i_2<\cdots<i_{r-1}\leqslant n}\prod_{j=1}^{r-1}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r} + F_{n-1}(x_2, x_3, \cdots, x_n; r)\\ &= \left(\frac{1+x_1}{1-x_1}\right)^{1/r}\left(\frac{1+x_2}{1-x_2}\right)^{1/r}\sum_{3\leqslant i_1<i_2<\cdots<i_{r-2}\leqslant n}\prod_{j=1}^{r-2}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r}\\ &\quad+\left(\frac{1+x_2}{1-x_2}\right)^{1/r}\sum_{3\leqslant i_1<i_2<\cdots<i_{r-1}\leqslant n}\prod_{j=1}^{r-1}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r} + F_{n-1}(x_1, x_3, \cdots, x_n; r),\end{aligned}$$

$$\begin{aligned}\frac{\partial F_n(x,r)}{\partial x_1} =& \frac{2(1+x_1)^{1/r-1}}{r(1-x_1)^{1/r+1}}\left(\frac{1+x_2}{1-x_2}\right)^{1/r}\sum_{3\leqslant i_1<i_2\cdots<i_{r-2}\leqslant n}\prod_{j=1}^{r-2}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r}\\ &+\frac{2}{r(1+x_1)^{1-1/r}(1-x_1)^{1/r+1}}\sum_{3\leqslant i_1<i_2\cdots<i_{r-1}\leqslant n}\prod_{j=1}^{r-1}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r},\end{aligned}$$

$$\begin{aligned}\frac{\partial F_n(x,r)}{\partial x_2} =& \frac{2(1+x_2)^{1/r-1}}{r(1-x_2)^{1/r+1}}\left(\frac{1+x_1}{1-x_1}\right)^{1/r}\sum_{3\leqslant i_1<i_2\cdots<i_{r-2}\leqslant n}\prod_{j=1}^{r-2}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r}\\ &+\frac{2}{r(1+x_2)^{1-1/r}(1-x_2)^{1/r+1}}\sum_{3\leqslant i_1<i_2\cdots<i_{r-1}\leqslant n}\prod_{j=1}^{r-1}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r},\end{aligned}$$

$$\begin{aligned}&(x_1-x_2)\left(\frac{\partial F_n(x,r)}{\partial x_1}-\frac{\partial F_n(x,r)}{\partial x_2}\right)\\ &=\frac{2(x_1-x_2)}{r}\frac{x_1^2-x_2^2}{(1-x_1^2)(1-x_2^2)}\left(\frac{1+x_1}{1-x_1}\right)^{1/r}\left(\frac{1+x_2}{1-x_2}\right)^{1/r}\sum_{3\leqslant i_1<i_2\cdots<i_{r-2}\leqslant n}\prod_{j=1}^{r-2}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r}\\ &+\frac{2(x_1-x_2)}{r}\left[\frac{1}{(1+x_1)^{1-1/r}(1-x_1)^{1/r+1}}-\frac{1}{(1+x_2)^{1-1/r}(1-x_2)^{1/r+1}}\right]\\ &\times\sum_{3\leqslant i_1<i_2\cdots<i_{r-1}\leqslant n}\prod_{j=1}^{r-1}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r}\geqslant 0,\end{aligned}\tag{3.13}$$

$$\begin{aligned}&(\log x_1-\log x_2)\left(x_1\frac{\partial F_n(x,r)}{\partial x_1}-x_2\frac{\partial F_n(x,r)}{\partial x_2}\right)\\ &=\frac{2}{r}(\log x_1-\log x_2)\frac{(x_1-x_2)(1+x_1x_2)}{(1-x_1^2)(1-x_2^2)}\left(\frac{1+x_1}{1-x_1}\right)^{1/r}\left(\frac{1+x_2}{1-x_2}\right)^{1/r}\sum_{3\leqslant i_1<i_2\cdots<i_{r-2}\leqslant n}\prod_{j=1}^{r-2}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r}\\ &+\frac{2}{r}(\log x_1-\log x_2)\left[\frac{x_1}{(1+x_1)^{1-1/r}(1-x_1)^{1/r+1}}-\frac{x_2}{(1+x_2)^{1-1/r}(1-x_2)^{1/r+1}}\right]\\ &\times\sum_{3\leqslant i_1<i_2\cdots<i_{r-1}\leqslant n}\prod_{j=1}^{r-1}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r}\geqslant 0\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}&(x_1-x_2)\left(x_1^2\frac{\partial F_n(x,r)}{\partial x_1}-x_2^2\frac{\partial F_n(x,r)}{\partial x_1}\right)\\ &=\frac{2(x_1-x_2)}{r}\frac{x_1^2-x_2^2}{(1-x_1^2)(1-x_2^2)}\left(\frac{1+x_1}{1-x_1}\right)^{1/r}\left(\frac{1+x_2}{1-x_2}\right)^{1/r}\sum_{3\leqslant i_1<i_2\cdots<i_{r-2}\leqslant n}\prod_{j=1}^{r-2}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r}\\ &+\frac{2(x_1-x_2)}{r}\left[\frac{x_1^2}{(1+x_1)^{1-1/r}(1-x_1)^{1/r+1}}-\frac{x_2^2}{(1+x_2)^{1-1/r}(1-x_2)^{1/r+1}}\right]\\ &\times\sum_{3\leqslant i_1<i_2\cdots<i_{r-1}\leqslant n}\prod_{j=1}^{r-1}\left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{1/r}\geqslant 0.\end{aligned}\tag{3.15}$$

Therefore, inequality (3.1) follows from (3.4), (3.7), (3.10) and (3.13), inequality (3.2) follows from (3.5), (3.8), (3.11) and (3.14), and inequality (3.3) follows from (3.6), (3.9), (3.12) and (3.15). □

4. Applications

In this section, we establish several analytic inequalities by use of Theorem 3.1 and the theory of majorization.

From Theorem 3.1(1) and Lemmas 2.5–2.8, we get Corollary 4.1 immediately.

Corollary 4.1. *If* $x = (x_1, x_2, \cdots, x_n) \in [0,1)^n$ *with* $\sum_{i=1}^{n} x_i = s$, $r \in \{1, 2, \cdots, n\}$ *and* $F_n(x, r)$ *be defined by* (1.1), *then we have*

(1) $F_n(x,r) \geqslant F_n\left(\frac{c-x}{\frac{nc}{s}-1}, r\right)$ *for* $c \geqslant s$;

(2) $F_n(x,r) \geqslant F_n\left(\frac{c+x}{\frac{nc}{s}+1}, r\right)$ *for* $c \geqslant 0$;

(3) $F_n(x,r) \geqslant F_n\left(\frac{s-\lambda x}{n-\lambda}, r\right)$ *for* $0 \leqslant \lambda \leqslant 1$;

(4) $F_n(x,r) \geqslant F_n\left(\frac{s+\lambda x}{n+\lambda}, r\right)$ *for* $0 \leqslant \lambda \leqslant 1$.

Let $x = (x_1, x_2, \cdots, x_n) \in [0,1)^n$. Then from (1.1) and Theorem 3.1(1) together with the fact

$$(A_n(x), A_n(x), \cdots, A_n(x)) \prec (x_1, x_2, \cdots, x_n),$$

we get Corollary 4.2.

Corollary 4.2. *If* $x = (x_1, x_2, \cdots, x_n) \in [0,1)^n$, $r \in \{1, 2, \cdots, n\}$ *and* $F_n(x, r)$ *be defined by* (1.1), *then*

$$F_n(x,r) \geqslant \frac{n!}{r!(n-r)!} \frac{A_n(1+x)}{A_n(1-x)}.$$

Let $r = 1$ and $r = n$ in Corollary 4.2, respectively. Then we get Corollary 4.3 immediately.

Corollary 4.3. *If* $x = (x_1, x_2, \cdots, x_n) \in [0,1)^n$, *then*

$$A_n\left(\frac{1+x}{1-x}\right) \geqslant \frac{A_n(1+x)}{A_n(1-x)}, \quad G_n\left(\frac{1+x}{1-x}\right) \geqslant \frac{G_n(1+x)}{G_n(1-x)}.$$

Corollary 4.4. *Let* $\mathcal{A} = A_1A_2 \cdots A_{n+1}$ *be a* n*-dimensional simplex in* $\mathbb{R}^n$, P *be an arbitrary point in the interior of* $\mathcal{A}$, B_i *be the intersection point of straight line* A_iP *and hyperplane* $\sum_i = A_1A_2 \cdots A_{i-1}A_{i+1} \cdots A_{n+1}$ $(i = 1, 2, \cdots, n+1)$ *and* $r \in \{1, 2, \cdots, n+1\}$. *Then*

$$\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n+1} \prod_{j=1}^{r} \left(1 + \frac{2PB_{i_j}}{PA_{i_j}}\right)^{1/r} \geqslant \frac{(n+2)!}{n[r!(n+1-r)!]},$$

$$\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n+1} \prod_{j=1}^{r} \left(1 + \frac{2PA_{i_j}}{PB_{i_j}}\right)^{1/r} \geqslant \frac{(n+1)!(2n+1)}{r!(n+1-r)!}.$$

Proof. We clearly see that

$$\sum_{i=1}^{n+1} \frac{PB_i}{A_iB_i} = 1, \quad \sum_{i=1}^{n+1} \frac{PA_i}{A_iB_i} = n,$$

$$\left(\frac{1}{n+1}, \frac{1}{n+1}, \cdots, \frac{1}{n+1}\right) \prec \left(\frac{PB_1}{A_1B_1}, \frac{PB_2}{A_2B_2}, \cdots, \frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right), \tag{4.1}$$

$$\left(\frac{n}{n+1}, \frac{n}{n+1}, \cdots, \frac{n}{n+1}\right) \prec \left(\frac{PA_1}{A_1B_1}, \frac{PA_2}{A_2B_2}, \cdots, \frac{PA_{n+1}}{A_{n+1}B_{n+1}}\right). \tag{4.2}$$

Therefore, Corollary 4.4 follows from (1.1), (4.1), (4.2) and Theorem 3.1(1). □

Corollary 4.5. *Suppose that* $A = (a_{ij})_{n\times n}$ *is a complex matrix, and* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *are the eigenvalues of* A. *If* A *is a positive definite Hermitian matrix, then one has*

$$\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n} \prod_{j=1}^{r} \left(\frac{trA+\lambda_{i_j}}{trA-\lambda_{i_j}}\right)^{1/r} \geqslant \frac{(n+1)!}{r!(n-r)!(n-1)},$$

$$\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n} \prod_{j=1}^{r} \left(\frac{trA+\lambda_{i_j}}{trA-\lambda_{i_j}}\right)^{1/r} \geqslant \frac{n!}{r!(n-r)!}\frac{trA+\sqrt[n]{detA}}{trA-\sqrt[n]{detA}},$$

$$\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n} \prod_{j=1}^{r} \left(1+\frac{2}{\lambda_{i_j}}\right)^{1/r} \geqslant \frac{n!}{r!(n-r)!}(1+\frac{2n}{trA}).$$

Proof. We clearly see that

$$\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right) \prec \left(\frac{\lambda_1}{trA}, \frac{\lambda_2}{trA}, \cdots, \frac{\lambda_n}{trA}\right), \tag{4.3}$$

$$\log\left(\frac{\sqrt[n]{detA}}{trA}, \frac{\sqrt[n]{detA}}{trA}, \cdots, \frac{\sqrt[n]{detA}}{trA}\right) \prec \log\left(\frac{\lambda_1}{trA}, \frac{\lambda_2}{trA}, \cdots, \frac{\lambda_n}{trA}\right), \tag{4.4}$$

$$\left(\frac{tr(A+I)}{n}, \frac{tr(A+I)}{n}, \cdots, \frac{tr(A+I)}{n}\right) \prec (1+\lambda_1, 1+\lambda_2, \cdots, 1+\lambda_n), \tag{4.5}$$

$$\frac{\lambda_i}{trA}, \frac{1}{1+\lambda_i}, \frac{\sqrt[n]{detA}}{trA}, \frac{n}{tr(A+I)} \in (0,1). \tag{4.6}$$

Therefore, the first inequality of Corollary 4.5 follows from Theorem 3.1(1), (4.3) and (4.6), the second inequality of Corollary 4.5 follows from Theorem 3.1(2), (4.4) and (4.6), and the last inequality of Corollary 4.5 follows from Theorem 3.1(3), (4.5) and (4.6). □

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