Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 850–863
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Controllability of time-dependent neutral stochastic functional differential equations driven by a fractional Brownian motion

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Communicated by M. Bohner

Abstract

In this paper, we consider the controllability of certain class of non-autonomous neutral evolution stochastic functional differential equations, with time varying delays, driven by a fractional Brownian motion in a separable real Hilbert space. Sufficient conditions for controllability are obtained by employing a fixed point approach. A practical example is provided to illustrate the viability of the abstract result of this work.

Keywords: Controllability, neutral stochastic functional differential equations, evolution operator, fractional Brownian motion.

2010 MSC: 35R10, 60H15, 60G15, 60J65.

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory and plays an important role in control systems. Controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. If the system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability. A standard approach is to transform the controllability problem into a fixed-point problem for an appropriate operator in a functional space. The problem of controllability for functional differential systems has been extensively studied in many papers [3, 4, 9, 17, 21, 25]. For example, Sakthivel and Ren [24] studied the complete controllability of stochastic evolution equations with jumps. In [5], Balasubramaniam and Dauer discussed the controllability of semilinear stochastic delay evolution equations in Hilbert spaces.

It is known that fractional Brownian motion, with Hurst parameter $H \in (0, 1)$, is a generalization of Brownian motion, it reduces to Brownian motion when $H = \frac{1}{2}$. A general theory for the infinite

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doi: 10.22436/jnsa.011.06.11

Received: 2015-06-12 Revised: 2015-09-20 Accepted: 2016-10-16

dimensional stochastic differential equations driven by a fractional Brownian motion (fBm) is not yet established and just a few results have been proved. In addition, in many mathematical models the claims often display long-range memories, possibly due to extreme weather, natural disasters, in some cases, many stochastic dynamical systems depend not only on present and past states, but also contain the derivatives with delays. Neutral functional differential equations are often used to describe such systems. Very recently, neutral stochastic functional differential equations driven by fractional Brownian motion have attracted the interest of many researchers. One can see [6, 7, 12–15] and the references therein. The literature concerning the existence and qualitative properties of solutions of time-dependent functional stochastic differential equations is very restricted and limited to a very few articles. This fact is the main motivation of our work. We mention here the recent paper by Ren et al. [20] concerning the existence of mild solutions for a class of stochastic evolution equations driven by fractional Brownian motion in Hilbert space.

Motivated by the above works, this paper is concerned with the controllability results for a class of time-dependent neutral functional stochastic differential equations described in the form:

$$\begin{cases} d[x(t)+g(t,x(t-r(t)))] = [A(t)x(t)+f(t,x(t-\rho(t)))+Bu(t)]dt+\sigma(t)dB^H(t),\ t\in[0,T], \\ x(.)=\varphi(.)\in\mathcal{C}([-\tau,0],\mathbb{L}^2(\Omega,X)), \quad \tau>0, \end{cases} \tag{1.1}$$

in a real Hilbert space X with inner product $<.,.>$ and norm $\|.\|$, where $\{A(t),\ t\in[0,T]\}$ is a family of linear closed operators from a space X into X that generates an evolution system of operators $\{R(t,s),\ 0\leqslant s\leqslant t\leqslant T\}$. B^H is a fractional Brownian motion on a real and separable Hilbert space Y, $r,\ \rho:[0,+\infty)\to[0,\tau]$ $(\tau>0)$ are continuous and $f,g:[0,+\infty)\times X\to X$, $\sigma:[0,+\infty)\to\mathcal{L}_2^0(Y,X)$, are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see Section 2 below). The control function $u(.)$ taking values in $\mathbb{L}^2([0,T],U)$ of admissible control functions for a separable Hilbert space U, B is a bounded linear operator from U into X.

To the best of our knowledge, there is no paper which investigates the study of controllability for time-dependent neutral stochastic functional differential equations with delays driven by fractional Brownian motion. Thus, we will make the first attempt to study such problem in this paper.

Our results are inspired by the one in [8] where the existence and uniqueness of mild solutions to model (1.1) with $B=0$, is studied.

The rest of this paper is organized as follows. Section 2 recapitulates some notations, basic concepts, and basic results about fractional Brownian motion, Wiener integral over Hilbert spaces and recalls some preliminary results about evolution family. Section 3 gives sufficient conditions to prove the controllability result for the problem (1.1). Section 4 illustrates the efficiency of the obtained results using an example.

2. Preliminaries

2.1. *Evolution families*

In this subsection we introduce the notion of evolution family.

Definition 2.1. A set $\{R(t,s):0\leqslant s\leqslant t\leqslant T\}$ of bounded linear operators on a Hilbert space X is called an *evolution family* if

(a) $R(t,s)R(s,r)=R(t,r)$, $R(s,s)=I$ if $r\leqslant s\leqslant t$,

(b) $(t,s)\to R(t,s)x$ is strongly continuous for $t>s$.

Let $\{A(t),\ t\in[0,T]\}$ be a family of closed densely defined linear unbounded operators on the Hilbert space X and with domain $D(A(t))$ independent of t, satisfying the following conditions introduced by [1].

There exist constants $\lambda_0\geqslant 0$, $\theta\in(\frac{\pi}{2},\pi)$, $L,K\geqslant 0$, and $\mu,\nu\in(0,1]$ with $\mu+\nu>1$ such that

$$\Sigma_\theta\cup\{0\}\subset\rho(A(t)-\lambda_0), \quad \|R(\lambda,A(t)-\lambda_0)\|\leqslant\frac{K}{1+|\lambda|} \tag{2.1}$$

and

$$\|(A(t)-\lambda_0)R(\lambda, A(t)-\lambda_0)\left[R(\lambda_0, A(t))-R(\lambda_0, A(s))\right]\| \leqslant L|t-s|^{\mu}|\lambda|^{-\nu} \tag{2.2}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta$ where $\Sigma_\theta := \left\{\lambda \in \mathbb{C}-\{0\} : |\arg\lambda| \leqslant \theta\right\}$.

It is well known, that this assumption implies that there exists a unique evolution family $\{R(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$ on X such that $(t,s) \to R(t,s) \in \mathcal{L}(X)$ is continuous for $t > s$, $R(\cdot, s) \in \mathcal{C}^1((s,\infty), \mathcal{L}(X))$, $\partial_t R(t,s) = A(t)R(t,s)$, and

$$\|A(t)^k R(t,s)\| \leqslant C(t-s)^{-k} \tag{2.3}$$

for $0 < t-s \leqslant 1$, $k = 0, 1$, $0 \leqslant \alpha < \mu$, $x \in D((\lambda_0 - A(s))^\alpha)$, and a constant C depending only on the constants in (2.1)-(2.2). Moreover, $\partial_s^+ R(t,s)x = -R(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$. We say that $A(\cdot)$ generates $\{R(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$. Note that $R(t,s)$ is exponentially bounded by (2.3) with $k = 0$.

Remark 2.2. If $\{A(t),\ t \in [0,T]\}$ is a second order differential operator A, that is $A(t) = A$ for each $t \in [0,T]$, then A generates a C_0-semigroup $\{e^{At}, t \in [0,T]\}$.

2.2. *Fractional Brownian motion*

For the convenience for the reader we recall briefly here some of the basic results of fractional Brownian motion calculus. For details of this section, we refer the reader to [18] and the references therein.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A standard fractional Brownian motion (fBm) $\{\beta^H(t), t \in \mathbb{R}\}$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process with continuous sample paths such that

$$\mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \tag{2.4}$$

for $s, t \in \mathbb{R}$. It is clear that for $H = 1/2$, this process is a standard Brownian motion. In this paper, it is assumed that $H \in (\frac{1}{2}, 1)$.

This process was introduced by [11] and later studied by [16]. Its self-similar and long-range dependence make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields.

Consider a time interval $[0,T]$ with arbitrary fixed horizon T and let $\{\beta^H(t), t \in [0,T]\}$ the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. It is well known that β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s)\,d\beta(s), \tag{2.5}$$

where $\beta = \{\beta(t) :\ t \in [0,T]\}$ is a Wiener process, and $K_H(t;s)$ is the kernel given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\,du$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(,)$ denotes the Beta function. We put $K_H(t,s) = 0$ if $t \leqslant s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of the set of indicator functions $\{1_{[0;t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $1_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry.

We recall that for $\psi, \varphi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \psi(s)\varphi(t)|t-s|^{2H-2} ds dt.$$

Let us consider the operator K_H^* from $\mathcal{H}$ to $\mathbb{L}^2([0,T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r,s) dr.$$

We refer to [18] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,T])$. Moreover for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^T (K_H^* \varphi)(t) d\beta(t).$$

It follows from [18] that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. In the case $H > \frac{1}{2}$, the second partial derivative of the covariance function

$$\frac{\partial R_H}{\partial t \partial s} = \alpha_H |t-s|^{2H-2},$$

where $\alpha_H = H(2H-2)$, is integrable, and we can write

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} du dv. \tag{2.6}$$

In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^T \int_0^T |\psi(s)||\psi(t)||s-t|^{2H-2} ds dt < \infty,$$

where $\alpha_H = H(2H-1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the following inclusions (see [18]).

Lemma 2.3.

$$\mathbb{L}^2([0,T]) \subseteq \mathbb{L}^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H},$$

and for any $\varphi \in \mathbb{L}^2([0,T])$, *we have*

$$\|\psi\|^2_{|\mathcal{H}|} \leqslant 2HT^{2H-1} \int_0^T |\psi(s)|^2 ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$. where $\lambda_n \geqslant 0$ $(n = 1,2...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1,2,...)$ is a complete orthonormal basis in Y. Let $B^H = (B^H(t))$ be $Y-$ valued fbm on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance Q as

$$B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t),$$

where β_n^H are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t), x\rangle \langle B^H(s), y\rangle = R(s,t)\langle Q(x), y\rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y, X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y, X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|_{\mathcal{L}_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ is a separable Hilbert space.

Now, let $\phi(s); s \in [0, T]$ be a function with values in $\mathcal{L}_2^0(Y, X)$, such that $\sum_{n=1}^{\infty} \|K^*\phi Q^{\frac{1}{2}} e_n\|_{\mathcal{L}_2^0}^2 < \infty$. The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s)dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s)e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}(K_H^*(\phi e_n)(s)d\beta_n(s), \tag{2.7}$$

where β_n is the standard Brownian motion used to present β_n^H as in (2.5).

Now, we end this subsection by stating the following result which is fundamental to prove our result.

Lemma 2.4 ([8]). *Suppose that* $\sigma : [0, T] \to \mathcal{L}_2^0(Y, X)$ *satisfies* $\sup_{t\in[0,T]} \|\sigma(t)\|_{\mathcal{L}_2^0}^2 < \infty$, *and Suppose that* $\{R(t, s), 0 \leqslant s \leqslant t \leqslant T\}$ *is an evolution system of operators satisfying* $\|R(t, s)\| \leqslant Me^{-\beta(t-s)}$, *for some constants* $\beta > 0$ *and* $M \geqslant 1$ *for all* $t \geqslant s$. *Then, we have*

$$\mathbb{E}\|\int_0^t R(t, s)\sigma(s)dB^H(s)\|^2 \leqslant CM^2t^{2H}(\sup_{t\in[0,T]} \|\sigma(t)\|_{\mathcal{L}_2^0})^2.$$

Remark 2.5. Thanks to Lemma 2.4, the stochastic integral

$$Z(t) = \int_0^t R(t, s)\sigma(s)dB^H(s), \qquad t \in [0, T],$$

is well-defined.

3. Controllability result

Henceforth we will assume that the family $\{A(t), t \in [0, T]\}$ of linear operators generates an evolution system of operators $\{R(t, s), 0 \leqslant s \leqslant t \leqslant T\}$. In this section, we derive controllability conditions for time-dependent neutral stochastic functional differential equations with variable delays driven by a fractional Brownian motion in a real separable Hilbert space. Before starting, we introduce the concept of a mild solution of the problem (1.1) and controllability of neutral stochastic functional differential equation.

Definition 3.1. An X-valued process $\{x(t), t \in [-\tau, T]\}$, is called a mild solution of equation (1.1) if

i) $x(.) \in \mathcal{C}([-\tau, T], \mathbb{L}^2(\Omega, X))$,

ii) $x(t) = \varphi(t), -\tau \leqslant t \leqslant 0$.

iii) For arbitrary $t \in [0, T]$, we have

$$\begin{aligned} x(t) =& R(t, 0)(\varphi(0) + g(0, \varphi(-r(0)))) - g(t, x(t - r(t))) \\ & - \int_0^t AR(t, s)g(s, x(s - r(s)))ds + \int_0^t R(t, s)f(s, x(s - \rho(s)))ds \\ & + \int_0^t R(t, s)(Bu)(s)ds + \int_0^t R(t, s)\sigma(s)dB^H(s), \quad \mathbb{P} - a.s. \end{aligned} \tag{3.1}$$

Definition 3.2. The system (1.1) is said to be controllable on the interval $[-\tau, T]$, if for every initial stochastic process $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X))$, there exists a stochastic control $u \in L^2([0, T], U)$ such that the mild solution $x(.)$ of (1.1) satisfies $x(T) = x_1$, where $x_1 \in \mathbb{L}^2(\Omega, X)$ and T are the preassigned terminal state and time, respectively.

We will study the problem (1.1) under the following assumptions:

($\mathcal{H}$.1) i) The evolution family is exponentially stable, that is, there exist two constants $\beta > 0$ and $M \geqslant 1$ such that

$$\|R(t, s)\| \leqslant Me^{-\beta(t-s)}, \qquad \text{for all} \quad t \geqslant s,$$

ii) There exist a constant $M_* > 0$ such that

$$\|A^{-1}(t)\| \leqslant M_*, \qquad \text{for all} \quad t \in [0, T].$$

($\mathcal{H}$.2) The maps $f, g : [0, T] \times X \to X$ are continuous functions and there exist two positive constants C_1 and C_2, such that for all $t \in [0, T]$ and $x, y \in X$:

i) $\|f(t, x) - f(t, y)\| \vee \|g(t, x) - g(t, y)\| \leqslant C_1\|x - y\|$.

ii) $\|f(t, x)\|^2 \vee \|A^k(t)g(t, x)\|^2 \leqslant C_2(1 + \|x\|^2), \quad k = 0, 1.$

($\mathcal{H}$.3) i) There exists a positive constant L_* such that $L^*M_* < \frac{1}{\sqrt{6}}$, and

$$\|A(t)g(t, x) - A(t)g(t, y)\| \leqslant L_*\|x - y\|$$

for all $t \in [0, T]$ and $x, y \in X$.

ii) The function g is continuous in the quadratic mean sense: for all $x(.) \in \mathcal{C}([0, T], L^2(\Omega, X))$, we have

$$\lim_{t \longrightarrow s} \mathbb{E}\|g(t, x(t)) - g(s, x(s))\|^2 = 0.$$

($\mathcal{H}$.4) i) The map $\sigma : [0, T] \longrightarrow \mathcal{L}_2^0(Y, X)$ is bounded, that is : there exists a positive constant L such that $\|\sigma(t)\|_{\mathcal{L}_2^0(Y,X)} \leqslant L$ uniformly in $t \in [0, T]$.

ii) Moreover, we assume that the initial data $\varphi = \{\varphi(t) : -\tau \leqslant t \leqslant 0\}$ satisfies $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X))$.

($\mathcal{H}$.5) The linear operator W from $\mathbb{L}^2([0, T], U)$ into $\mathbb{L}^2(\Omega, X)$ defined by

$$Wu = \int_0^T R(T, s)Bu(s)ds$$

has an inverse operator W^{-1} that takes values in $L^2([0, T], U) \setminus \ker W$, where $\ker W = \{x \in L^2([0, T], U), Wx = 0\}$ (see [10, 19]), and there exists finite positive constants M_b, M_w such that $\|B\| \leqslant M_b$ and $\|W^{-1}\| \leqslant M_w$.

The main result of this paper is given in the next theorem.

Theorem 3.3. *Suppose that* ($\mathcal{H}$.1)-($\mathcal{H}$.5) *hold. Then, system* (1.1) *is controllable on* $[-\tau, T]$.

Proof. Fix $T > 0$ and let $\mathcal{B}_T := \mathcal{C}([-\tau, T], \mathbb{L}^2(\Omega, X)$ be the Banach space of all continuous functions from $[-\tau, T]$ into $\mathbb{L}^2(\Omega, X)$, equipped with the supremum norm $\|\xi\|_{\mathcal{B}_T} = \sup_{u \in [-\tau, T]} \left(\mathbb{E}\|\xi(u)\|^2\right)^{1/2}$ and let us consider the set

$$S_T = \{x \in \mathcal{B}_T : x(s) = \varphi(s), \text{ for } s \in [-\tau, 0]\}.$$

S_T is a closed subset of $\mathcal{B}_T$ provided with the norm $\|.\|_{\mathcal{B}_T}$.

Using the hypothesis $(\mathcal{H}.5)$ for an arbitrary function $x(.)$, define the stochastic control

$$\begin{aligned} u(t) =& W^{-1}\{x_1 - R(T,0)(\varphi(0) + g(0,\varphi(-r(0)))) + g(T,x(T-r(T))) \\ &+ \int_0^T AR(T,s)g(s,x(s-r(s)))ds - \int_0^T R(T,s)f(s,x(s-\rho(s))ds \\ &- \int_0^T R(T,s)\sigma(s)dB^H(s)\}(t). \end{aligned}$$

We will now show using this control that the operator ψ on $S_T(\varphi)$ defined by $\psi(x)(t) = \varphi(t)$ for $t \in [-\tau, 0]$ and for $t \in [0,T]$

$$\begin{aligned} \psi(x)(t) =& R(t,0)(\varphi(0) + g(0,\varphi(-r(0)))) - g(t,x(t-r(t))) - \int_0^t R(t,s)A(s)g(s,x(s-r(s)))ds \\ &+ \int_0^t R(t,s)f(s,x(s-\rho(s)))ds + \int_0^t R(t,v)\sigma(s)dB^H(s) \\ &+ \int_0^t R(t,v)BW^{-1}\{x_1 - R(T,0)(\varphi(0) + g(0,\varphi(-r(0)))) + g(T,x(T-r(T))) \\ &+ \int_0^T R(T,s)A(s)g(s,x(s-r(s)))ds - \int_0^T R(T,s)f(s,x(s-\rho(s))ds \\ &- \int_0^T R(T,s)\sigma(s)dB^H(s)\}dv, \end{aligned}$$

has a fixed point. This fixed point is then a solution of (1.1). Clearly, $\psi(x)(T) = x_1$, which implies that the system (1.1) is controllable.

For better readability, we break the proof into sequence of steps.

Step 1: ψ is well defined. Let $x \in S_T(\varphi)$ and $t \in [0,T]$, we are going to show that each function $\psi(x)(.)$ is continuous on $[0,T]$ in the $\mathbb{L}^2(\Omega, X)$-sense.

Let $0 < t < T$ and $|h|$ be sufficiently small. Then for any fixed $x \in S_T$, we have

$$\begin{aligned} \mathbb{E}\|\psi(x)(t+h) - \psi(x)(t)\|^2 \leqslant& 6\mathbb{E}\|(R(t+h,0) - R(t,0))(\varphi(0) + g(0,\varphi(-r(0))))\|^2 \\ &+ 6\mathbb{E}\|g(t+h,x(t+h-r(t+h))) - g(t,x(t-r(t)))\|^2 \\ &+ 6\mathbb{E}\|\int_0^{t+h} R(t+h,s)A(s)g(s,x(s-r(s))ds - \int_0^t R(t,s)A(s)g(s,x(s-r(s))ds\|^2 \\ &+ 6\mathbb{E}\|\int_0^{t+h} R(t+h,s)f(s,x(s-\rho(s)))ds - \int_0^t R(t,s)f(s,x(s-\rho(s)))ds\|^2 \\ &+ 6\mathbb{E}\|\int_0^{t+h} R(t+h,s)\sigma(s)dB^H(s) - \int_0^t R(t,s)\sigma(s)dB^H(s)\|^2 \\ &+ 6\mathbb{E}\|\int_0^{t+h} R(t+h,v)BW^{-1}\{x_1 - R(T,0)(\varphi(0) + g(0,\varphi(-r(0)))) \\ &+ g(T,x(T-r(T))) + \int_0^T R(T,s)A(s)g(s,x(s-r(s)))ds \\ &- \int_0^T R(T,s)f(s,x(s-\rho(s))ds - \int_0^T R(T,s)\sigma(s)dB^H(s)\}dv \\ &- \int_0^t R(t,v)BW^{-1}\{x_1 - R(T,0)(\varphi(0) + g(0,\varphi(-r(0)))) \\ &+ g(T,x(T-r(T))) - \int_0^T R(T,s)\sigma(s)dB^H(s)\}dv \\ =& 6\sum_{1\leqslant i\leqslant 6} \mathbb{E}\|I_i(t+h) - I_i(t)\|^2. \end{aligned}$$

From Definition 2.1, we obtain

$$\lim_{h\longrightarrow 0}(R(t+h,0)-R(t,0))(\varphi(0)+g(0,\varphi(-r(0))))=0.$$

From $(\mathcal{H}.1)$, we have

$$\|(R(t+h,0)-R(t,0))(\varphi(0)+g(0,\varphi(-r(0))))\| \leqslant Me^{-\beta t}(e^{-\beta h}+1)\|\varphi(0)+g(0,\varphi(-r(0)))\| \in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_1(t+h)-I_1(t)\|^2=0.$$

Moreover, assumption $(\mathcal{H}.2)$ ensures that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_2(t+h)-I_2(t)\|^2=0.$$

To show that the third term $I_3(h)$ is continuous, we suppose $h>0$ (similar calculus for $h<0$). We have

$$\begin{aligned}\|I_3(t+h)-I_3(t)\| \leqslant &\left\|\int_0^t (R(t+h,s)-R(t,s))A(s)g(s,x(s-r(s)))ds\right\| \\ &+\left\|\int_t^{t+h}(R(t,s)g(s,x(s-r(s)))ds\right\| \\ \leqslant & I_{31}(h)+I_{32}(h).\end{aligned}$$

By Hölder's inequality, we have

$$\mathbb{E}\|I_{31}(h)\| \leqslant t\mathbb{E}\int_0^t \|R(t+h,s)-R(t+h,s))A(s)g(s,x(s-r(s))\|^2 ds.$$

By Definition 2.1, we obtain

$$\lim_{h\longrightarrow 0}(R(t+h,s)-R(t,s))A(s)g(s,x(s-r(s)))=0.$$

From $(\mathcal{H}.1)$ and $(\mathcal{H}.2)$, we have

$$\|R(t+h,s)-R(t,s))A(s)g(s,x(s-r(s))\| \leqslant C_2Me^{-\beta(t-s)}(e^{-\beta h}+1)\|A(s)g(s,x(s-r(s))\| \in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_{31}(h)\|^2=0.$$

So, estimating as before. By using $(\mathcal{H}.1)$ and $(\mathcal{H}.2)$, we get

$$\mathbb{E}\|I_{32}(h)\|^2 \leqslant \frac{M^2C_2(1-e^{-2\beta h})}{2\beta}\int_t^{t+h}(1+\mathbb{E}\|x(s-r(s))\|^2)ds.$$

Thus,

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_{32}(h)\|^2=0.$$

For the fourth term $I_4(h)$, we suppose $h>0$ (similar calculus for $h<0$). We have

$$\begin{aligned}\|I_4(t+h)-I_4(t)\| \leqslant &\left\|\int_0^t (R(t+h,s)-R(t,s))f(s,x(s-\rho(s)))ds\right\| \\ &+\left\|\int_t^{t+h}(R(t,s)f(s,x(s-\rho(s)))ds\right\| \\ \leqslant & I_{41}(h)+I_{42}(h).\end{aligned}$$

By Hölder's inequality, we have

$$\mathbb{E}\|I_{41}(h)\| \leqslant t\mathbb{E}\int_0^t \|R(t+h,s)-R(t,s))f(s,x(s-\rho(s))\|^2 ds.$$

Again exploiting properties of Definition 2.1, we obtain

$$\lim_{h\longrightarrow 0}(R(t+h,s)-R(t,s))f(s,x(s-\rho(s)))=0,$$

and

$$\|R(t+h,s)-R(t,s))f(s,x(s-\rho(s))\| \leqslant Me^{-\beta(t-s)}(e^{-\beta h}+1)\|f(s,x(s-\rho(s))\| \in L^2(\Omega).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_{41}(h)\|^2=0.$$

On the other hand, by $(\mathcal{H}.1)$, $(\mathcal{H}.2)$, and the Hölder's inequality, we have

$$\mathbb{E}\|I_{42}(h)\| \leqslant \frac{M^2C_2(1-e^{-2\beta h})}{2\beta}\int_t^{t+h}(1+\mathbb{E}\|x(s-\rho(s))\|^2)ds.$$

Thus

$$\lim_{h\to 0} I_{42}(h)=0.$$

Now, for the term $I_5(h)$, we have

$$\begin{aligned}\|I_5(t+h)-I_5(t)\| \leqslant & \left\|\int_0^t (R(t+h,s)-R(t,s)\sigma(s)dB^H(s)\right\| \\ & + \left\|\int_t^{t+h} R(t+h,s)\sigma(s)dB^H(s)\right\| \\ \leqslant & I_{51}(h)+I_{52}(h).\end{aligned}$$

By Lemma 2.4, we get that

$$E\|I_{51}(h)\|^2 \leqslant 2Ht^{2H-1}\int_0^t \|[R(t+h,s)-R(t,s)]\sigma(s)\|^2_{\mathcal{L}_2^0}ds.$$

Since

$$\lim_{h\to 0}\|[R(t+h,s)-R(t,s)]\sigma(s)\|^2_{\mathcal{L}_2^0}=0$$

and

$$\|(R(t+h,s)-R(t,s)\sigma(s)\|_{\mathcal{L}_2^0} \leqslant MLe^{-\beta(t-s)}e^{-\beta h+1} \in \mathbb{L}^1([0,T],\, ds),$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}\|I_{51}(h)\|^2=0.$$

Again by Lemma 2.4, we get that

$$\mathbb{E}\|I_{52}(h)\|^2 \leqslant \frac{2Ht^{2H-1}LM^2(1-e^{-2\beta h})}{2\beta}.$$

Thus,

$$\lim_{h\to 0}\mathbb{E}|I_{52}(h)|^2=0.$$

For the estimation of term I_6, we have

$$
\begin{aligned}
\mathbb{E}\|I_6(h)\|^2 \leqslant & 2\mathbb{E}\|\int_t^{t+h} R(t+h,\nu)BW^{-1}\{x_1 - R(T,0)(\varphi(0)+g(0,\varphi(-r(0))))\\
& + g(T,x(T-r(T))) + \int_0^T R(T,s)A(s)g(s,x(s-r(s)))ds\\
& - \int_0^T R(T,s)f(s,x(s-\rho(s)))ds - \int_0^T R(T,s)\sigma(s)dB^H(s)\|\\
& + 2\mathbb{E}\|\int_0^t (R(t+h,\nu)-R(t,\nu))BW^{-1}\{x_1 - R(T,0)(\varphi(0)+g(0,\varphi(-r(0))))\\
& + g(T,x(T-r(T))) + \int_0^T R(T,s)A(s)g(s,x(s-r(s)))ds\\
& - \int_0^T R(T,s)f(s,x(s-\rho(s)))ds - \int_0^T R(T,s)\sigma(s)dB^H(s)\}d\nu\|\\
\leqslant & 2[\mathbb{E}\|I_{6,1}(h)\|^2 + \mathbb{E}\|I_{6,2}(h)\|^2].
\end{aligned}
$$

Let's first deal with $I_{6,1}(h)$, it follows from the conditions $(\mathcal{H}.1)$-$(\mathcal{H}.5)$ that

$$
\begin{aligned}
\mathbb{E}\|I_{6,1}(h)\|^2 \leqslant & 6M^2M_b^2M_w^2\int_t^{t+h}\{\mathbb{E}\|x_1\|^2 + M^2\mathbb{E}\|\varphi(0)+g(0,\varphi(-r(0)))\|^2\\
& + M_*^2C_2T(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2) + M^2TC_2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
& + M^2TC_2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2) + 2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\}d\nu.
\end{aligned}
$$

It results that

$$
\lim_{h\to 0}\mathbb{E}\|I_{6,1}(h)\|^2 = 0.
$$

In a similar way, we have

$$
\begin{aligned}
\mathbb{E}\|I_{6,2}(h)\|^2 \leqslant & 6M_b^2M_w^2\int_0^t\|R(t+h,\nu)-R(t,\nu)\|^2\{\mathbb{E}\|x_1\|^2\\
& + M^2\mathbb{E}\|\varphi(0)+g(0,\varphi(-r(0)))\|^2 + M_*^2C_2(1+\mathbb{E}\|x\|^2)\\
& + M^2T^2C_2(1+\mathbb{E}\|x\|^2) + M^2T^2C_2(1+\mathbb{E}\|x\|^2)\\
& + 2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\}d\nu.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\|R(t+h,\nu)-R(t,\nu)\|^2\{\mathbb{E}\|x_1\|^2 + M^2\mathbb{E}\|\varphi(0)+g(0,\varphi(-r(0)))\|^2 + M_*^2C_4(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&\quad + M^2T^2C_2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2) + M^2T^2C_2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&\quad + 2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\}\\
&\leqslant 4M^2\{\mathbb{E}\|x_1\|^2 + M^2\mathbb{E}\|\varphi(0)+g(0,\varphi(-r(0)))\|^2\\
&\quad + M_*^2C_2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2) + 2M^2T^2C_2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)\\
&\quad + 2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds\} \in L^1([0,T],ds]),
\end{aligned}
$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}\|I_{6,2}(h)\|^2=0.$$

The above arguments show that $\lim_{h\to 0}\mathbb{E}\|\psi(x)(t+h)-\psi(x)(t)\|^2=0$. Hence, we conclude that the function $t\to\psi(x)(t)$ is continuous on $[0,T]$ in the $\mathbb{L}^2$-sense.

Step 2: Now, we are going to show that ψ is a contraction mapping in $S_{T_1}(\varphi)$ with some $T_1\leqslant T$ to be specified later. Let $x,y\in S_T(\varphi)$, then for any fixed $t\in[0,T]$, we have

$$\begin{aligned}
&\mathbb{E}\|\psi(x)(t)-\psi(y)(t)\|^2\\
&\quad\leqslant 6\|A(t)^{-1}\|^2\mathbb{E}\|A(t)g(t,x(t-r(t)))-A(t)g(t,y(t-r(t)))\|^2\\
&\quad+6\mathbb{E}\|\int_0^t R(t,s)A(s)(g(s,x(s-r(s)))-g(s,y(s-r(s))))ds\|^2\\
&\quad+6\mathbb{E}\|\int_0^t R(t,s)(f(s,x(s-\rho(s)))-f(s,y(s-\rho(s))))ds\|^2\\
&\quad+6\mathbb{E}\|\int_0^t R(t,\nu)BW^{-1}[g(T,x(T-r(T)))-g(T,y(T-r(T)))]d\nu\|^2\\
&\quad+6\mathbb{E}\|\int_0^t R(t,\nu)BW^{-1}\int_0^T R(T,s)A(s)[g(s,x(s-r(s)))-g(s,y(s-r(s)))]ds]d\nu\|^2\\
&\quad+6\mathbb{E}\|\int_0^t R(t,\nu)BW^{-1}\int_0^T R(T,s)[f(s,x(s-\rho(s)))-f(s,y(s-\rho(s)))]dsd\nu\|^2.
\end{aligned}$$

By assumptions combined with Hölder's inequality, we get that

$$\begin{aligned}
\mathbb{E}\|\psi(x)(t)-\psi(y)(t)\|^2\leqslant&6L_*^2M_*^2\sup_{s\in[-\tau,t]}\mathbb{E}\|x(t-r)-y(t-r)\|^2\\
&+6M^2L_*^2\frac{1-e^{-2\beta t}}{2\beta}t\sup_{s\in[-\tau,t]}\mathbb{E}\|x(s)-y(s)\|^2\\
&+6M^2C_1^2\frac{1-e^{-2\beta t}}{2\beta}t\sup_{s\in[-\tau,t]}\mathbb{E}\|x(s)-y(s)\|^2\\
&+6tM^2M_b^2M_w^2[C_1^2\mathbb{E}\|x(T-r(T))-y(T-r(T))\|^2\\
&+L_*^2M^2T^2\sup_{s\in[-\tau,t]}\mathbb{E}\|x(s-r(s))-y(s-r(s))\|^2\\
&+T^2M^2C_1^2\sup_{s\in[-\tau,t]}\mathbb{E}\|x(s)-y(s)\|^2.
\end{aligned}$$

Hence

$$\sup_{s\in[-\tau,T]}\mathbb{E}\|\psi(x)(s)-\psi(y)(s)\|^2\leqslant\gamma(t)\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)-y(s)\|^2,$$

where

$$\begin{aligned}
\gamma(t)=&6[\|L_*^2M_*^2+M^2L_*^2\frac{1-e^{-2\beta t}}{2\beta}t+M^2C_1^2\frac{1-e^{-2\beta t}}{2\beta}t\\
&+tM^2M_b^2M_w^2(C_1^2+L_*^2M^2T^2+T^2M^2C_1^2].
\end{aligned}$$

By condition $(\mathcal{H}.3)$, we have $\gamma(0)=6L_*^2M_*^2<1$. Then there exists $0<T_1\leqslant T$ such that $0<\gamma(T_1)<1$ and ψ is a contraction mapping on S_{T_1} and therefore has a unique fixed point, which is a mild solution of equation (1.1) on $[-\tau,T_1]$. This procedure can be repeated in order to extend the solution to the entire interval $[-\tau,T]$ in finitely many steps. Clearly, $(\psi x)(T)=x_1$ which implies that the system (1.1) is controllable on $[-\tau,T]$. This completes the proof. □

4. An illustrative example

In recent years, the interest in neutral systems has been growing rapidly due to their successful applications in practical fields such as physics, chemical technology, bioengineering, and electrical networks. We consider the following stochastic partial neutral functional differential equation with finite delays τ_1 and τ_2 $(0 \leqslant \tau_i \leqslant \tau < \infty,\ i = 1, 2)$:

$$\begin{cases} d\,[u(t,\zeta) + g_1(t, u(t-\tau_1,\zeta))] = [\frac{\partial^2}{\partial^2\zeta} u(t,\zeta) + b(t,\zeta)u(t,\zeta) + f_1(t, u(t-\tau_2,\zeta)) \\ \qquad + \nu(t,\xi)]dt + \sigma(t)dB^H(t),\ 0 \leqslant t \leqslant T,\ 0 \leqslant \zeta \leqslant \pi, \\ u(t,0) = u(t,\pi) = 0, \qquad 0 \leqslant t \leqslant T, \\ u(t,\zeta) = \varphi(t,\zeta), \quad t \in [-\tau, 0],\ 0 \leqslant \zeta \leqslant \pi, \end{cases} \tag{4.1}$$

where B^H is a fractional Brownian motion, $b(t,\zeta)$ is a continuous function and is uniformly Hölder continuous in t, $f_1, g_1 : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

To study this system, we consider the space $X = L^2([0,\pi])$ and the operator $A : D(A) \subset X \longrightarrow X$ given by $Ay = y''$ with

$$D(A) = \{y \in X : y'' \in X, \quad y(0)) = y(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geqslant 0}$ on X. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$ and the corresponding normalized eigenfunctions given by

$$e_n := \sqrt{\frac{2}{\pi}} \sin nx,\ n = 1, 2, \ldots.$$

In addition, $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis in X and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, e_n > e_n$$

for $x \in X$ and $t \geqslant 0$.

Now, we define an operator $A(t) : D(A) \subset X \longrightarrow X$ by

$$A(t)x(\zeta) = Ax(\zeta) + b(t,\zeta)x(\zeta).$$

By assuming that $b(.,.)$ is continuous and that $b(t,\zeta) \leqslant -\gamma$ $(\gamma > 0)$ for every $t \in \mathbb{R}$, $\zeta \in [0,\pi]$, it follows that the system

$$\begin{cases} u'(t) = A(t)u(t), \quad t \geqslant s, \\ u(s) = x \in X, \end{cases}$$

has an associated evolution family given by

$$R(t,s)x(\zeta) = \left[T(t-s)\exp(\int_s^t b(\tau,\zeta))d\tau)x\right](\zeta).$$

From this expression, it follows that $R(t,s)$ is a compact linear operator and that for every $s, t \in [0,T]$ with $t > s$

$$\|R(t,s)\| \leqslant e^{-(\gamma+1)(t-s)}$$

In addition, $A(t)$ satisfies the assumption $\mathcal{H}_1$ (see [2, 23]).

To rewrite the initial-boundary value problem (4.1) in the abstract form we assume the following:

i) $B : U \longrightarrow X$ is a bounded linear operator defined by

$$Bu(t)(\xi) = \nu(t, \xi),\ 0 \leqslant \xi \leqslant \pi,\ u \in L^2([0, T], U).$$

ii) The operator $W : L^2([0, T], U) \longrightarrow X$ defined by

$$Wu = \int_0^T S(T-s)\nu(t, \xi)ds$$

has an inverse W^{-1} and satisfies condition $(\mathcal{H}.5)$. For the construction of the operator W and its inverse, see [19].

iii) The substitution operator $f : [0, T] \times X \longrightarrow X$ defined by $f(t, u)(.) = f_1(t, u(.))$ is continuous and we impose suitable conditions on f_1 to verify assumption $\mathcal{H}_2$.

iv) The substitution operator $g : [0, T] \times X \longrightarrow X$ defined by $g(t, u)(.) = g_1(t, u(.))$ is continuous and we impose suitable conditions on g_1 to verify assumptions $\mathcal{H}_2$ and $\mathcal{H}_3$.

If we put

$$\begin{cases} x(t)(\zeta) = x(t, \zeta),\ t \in [0, T],\ \zeta \in [0, \pi] \\ x(t, \zeta) = \varphi(t, \zeta),\ t \in [-\tau, 0],\ \zeta \in [0, \pi], \end{cases} \tag{4.2}$$

then, the problem (4.1) can be written in the abstract form

$$\begin{cases} d[x(t) + g(t, x(t - r(t)))] = [A(t)x(t) + f(t, x(t - \rho(t))]dt + \sigma(t)dB^H(t), & 0 \leqslant t \leqslant T, \\ x(t) = \varphi(t), & -\tau \leqslant t \leqslant 0. \end{cases}$$

Furthermore, if we assume that the initial data $\varphi = \{\varphi(t) : -\tau \leqslant t \leqslant 0\}$ satisfies $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X))$, thus all the assumptions of Theorem 3.3 are fulfilled. Therefore, we conclude that the system (4.1) is controllable on $[-\tau, T]$.

Acknowledgment

The authors would like to thank the referee and the editor for their useful comments on an earlier versions of this paper.

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