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Superstability of Kannappan's and Van vleck's functional equations

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Abstract

In this paper, we prove the superstability theorems of the functional equations

$$\mu(y)f(x\sigma(y)z_0) \pm f(xyz_0) = 2f(x)f(y),\ x,y \in S, \quad \mu(y)f(\sigma(y)xz_0) \pm f(xyz_0) = 2f(x)f(y),\ x,y \in S,$$

where S is a semigroup, σ is an involutive morphism of S, and $\mu : S \longrightarrow \mathbb{C}$ is a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$, and z_0 is in the center of S.

Keywords: Hyers-Ulam stability, semigroup, d'Alembert's equation, Van Vleck's equation, Kannappan's equation, involution, automorpnism, multiplicative function.

2010 MSC: 39B32, 39B82.

1. Introduction

Van Vleck [31, 32] studied the continuous solutions f: $\mathbb{R} \longrightarrow \mathbb{R}$, $f \neq 0$ of the following functional equation

$$f(x-y+z_0) - f(x+y+z_0) = 2f(x)f(y),\ x,y \in \mathbb{R},$$

where $z_0 > 0$ is fixed. He showed that any continuous solution with minimal period $4z_0$ has to be the sine function $f(x) = \sin(\frac{\pi}{2z_0}x) = \cos(\frac{\pi}{2z_0}(x-z_0))$, $x \in \mathbb{R}$. Kannappan [18] proved that any solution f: $\mathbb{R} \longrightarrow \mathbb{C}$ of the functional equation

$$f(x+y+z_0) + f(x-y+z_0) = 2f(x)f(y),\ x,y \in \mathbb{R}$$

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is periodic, if $z_0 \neq 0$. Furthermore, the periodic solutions have the form $f(x) = g(x - z_0)$ where g is a periodic solution of d'Alembert functional equation

$$g(x+y) + g(x-y) = 2g(x)g(y), \; x, y \in \mathbb{R}.$$

Stetkær [24, Exercise 9.18] found the complex-valued solutions of the functional equation

$$f(xy^{-1}z_0) - f(xyz_0) = 2f(x)f(y), \; x, y \in G$$

on groups G, where z_0 is a fixed element in the center of G.

Perkins and Sahoo [21] replaced the group inversion by an involutive anti-automorphism $\sigma\colon G \longrightarrow G$ and they obtained the abelian, complex-valued solutions of the functional equation

$$f(x\sigma(y)z_0) - f(xyz_0) = 2f(x)f(y), \; x, y \in G. \tag{1.1}$$

Stetkær [26] extends the results of Perkins and Sahoo [21] about equation (1.1) to the more general case where G is a semigroup and the solutions are not assumed to be abelian and z_0 is a fixed element in the center of G.

In 1979, a type of stability was observed by Baker et al. [9]. Indeed, they proved that if a function is approximately exponential, then it is either a true exponential function or bounded. Then the exponential functional equation is said to be superstable. This result was the first result concerning the superstability phenomenon of functional equations. Later, Baker [1] generalized this result as follows: Let $(S, .)$ be an arbitrary semigroup, and let $f : S \longrightarrow \mathbb{C}$. Assume that f is an approximately exponential function, i.e., there exists a nonnegative number δ such that $|f(xy) - f(x)f(y)| \leqslant \delta$ for all $x, y \in S$. Then f is either bounded or f is a multiplicative function. The result of Baker et al. [2] was generalized by Székelyhidi [28–30] in another way. We refer also to [5, 11, 16, 17, 19, 20, 22, 23] for other results concerning the stability and the superstability of functional equations.

Throughout this paper S denotes a semigroup with an involutive morphism $\sigma\colon S \longrightarrow S$. That is σ an involutive anti-automorphism: $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$ or σ an involutive automorphism: $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$. Let $\mu\colon S \longrightarrow \mathbb{C}$ denotes a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$ and $|\mu(x)| \leqslant M$ for all $x \in S$ and for same $M > 0$. In all proofs of the results of this paper we use without explicit mentioning the assumption that z_0 is contained in the center of S and its consequence $\sigma(z_0)$ is contained in the center of S.

In the present paper, we consider the following functional equations which are solved recently by Bouikhalene and Elqorachi [3] and Elqorachi and Redouani [12]. The equations are the Kannappan's functional equation

$$f(xyz_0) + \mu(y)f(x\sigma(y)z_0) = 2f(x)f(y), \; x, y \in S, \tag{1.2}$$

the Van Vleck functional equation

$$\mu(y)f(x\sigma(y)z_0) - f(xyz_0) = 2f(x)f(y), \; x, y \in S, \tag{1.3}$$

a variant of Van Vleck functional equation

$$\mu(y)f(\sigma(y)xz_0) - f(xyz_0) = 2f(x)f(y), \; x, y \in S, \tag{1.4}$$

and a variant of Kannappan's functional equation

$$f(xyz_0) + \mu(y)f(\sigma(y)xz_0) = 2f(x)f(y), \; x, y \in S. \tag{1.5}$$

The results of this paper are organized as follows. In Section 2, we prove the superstability of (1.2). In Section 3, we prove the superstability of (1.3). In Section 4 and 5 we prove the superstability of the functional equations (1.4) and (1.5), respectively.

2. The superstability of Kannappan's functional equation (1.2)

In this section we obtain the superstability result of equation (1.2) on semigroups not necessarily abelian. The following Lemma will be used later.

Lemma 2.1. *Let σ be an involutive morphism of a semigroup S. Let μ be a bounded multiplicative function on S such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. Let $\delta > 0$ be fixed. If $f : S \longrightarrow \mathbb{C}$ is an unbounded function such that*

$$|f(xyz_0) + \mu(y)f(x\sigma(y)z_0) - 2f(x)f(y)| \leqslant \delta \tag{2.1}$$

for all $x, y \in S$, then, for all $x \in S$ we have

$$f(x) = \mu(x)f(\sigma(x)), \tag{2.2}$$

$$|f(x\sigma(z_0)z_0) - \mu(\sigma(z_0))f(z_0)f(x)| \leqslant \frac{\delta}{2}, \tag{2.3}$$

$$|f(xz_0^2) - f(x)f(z_0)| \leqslant \frac{M\delta}{2} + \delta, \tag{2.4}$$

$$f(z_0) \neq 0. \tag{2.5}$$

The function g defined by

$$g(x) = \frac{f(xz_0)}{f(z_0)},\ x \in S \tag{2.6}$$

is unbounded on S and satisfies

$$|g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y)| \leqslant \frac{(2+M)\delta}{|f(z_0)|^2}, \tag{2.7}$$

$$|g(xz_0) - g(x)g(z_0)| \leqslant \frac{(2+M)\delta}{|f(z_0)|^2} + \frac{(1+M)\delta}{|f(z_0)|} \tag{2.8}$$

for all $x, y \in S$. Furthermore, g is a non-zero solution of d'Alembert's functional equation

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)g(y),\ x, y \in S, \tag{2.9}$$

and satisfies the condition

$$g(xz_0) = g(z_0)g(x) \text{ for all } x \in S. \tag{2.10}$$

Proof.
Equation (2.2): Replacing y by $\sigma(y)$ in (2.1) and multiplying the result obtained by $\mu(y)$ and using that $\mu(y\sigma(y)) = 1$, $|\mu(y)| \leqslant M$ we get

$$|\mu(y)f(x\sigma(y)z_0) + f(xyz_0) - 2\mu(y)f(x)f(\sigma(y))| \leqslant M\delta.$$

Subtracting resulting inequalities we find after using the triangle inequality that

$$|f(x)(f(y) - \mu(y)f(\sigma(y)))| \leqslant \frac{\delta + M\delta}{2}.$$

Since f is assumed to be unbounded then $\mu(y)f(\sigma(y)) = f(y)$ for all $y \in S$.
Equation (2.3): By setting x by $\sigma(z_0)$ in (2.1) we get

$$|f(\sigma(z_0)yz_0) + \mu(y)f(\sigma(z_0)\sigma(y)z_0) - 2f(\sigma(z_0))f(y)| \leqslant \delta.$$

By using (2.2) and $\mu(y\sigma(y)) = 1$ we get

$$|f(\sigma(z_0)yz_0) + f(\sigma(z_0)yz_0) - 2\mu(\sigma(z_0))f(y)f(z_0)| \leqslant \delta,$$

which proves (2.3).

Equation (2.4): Putting $y = z_0$ in (2.1) we get

$$|f(xz_o^2) + \mu(z_0)f(x\sigma(z_0)z_0) - 2f(x)f(z_0)| \leqslant \delta.$$

From equation (2.3), the triangle inequality, and that $\mu(x\sigma(x)) = 1$, $|\mu(x)| \leqslant M$ for all $x \in S$ we obtain (2.4).

Equation (2.5): Assume that $f(z_0) = 0$. Replacing x by xz_0, y by yz_0 in (2.1) we get

$$|f(xyz_0^3) + \mu(yz_0)f(xz_0\sigma(yz_0)z_0) - 2f(xz_0)f(yz_0)| \leqslant \delta. \tag{2.11}$$

From (2.3) and (2.4) we get

$$|f(xz_0\sigma(z_0)\sigma(y)z_0) - \mu(\sigma(z_0))f(z_0)f(xz_0\sigma(y))| \leqslant \frac{\delta}{2}, \quad |f(xyz_0^3) - f(z_0)f(xyz_0)| \leqslant \frac{3\delta}{2}.$$

Since $f(z_0) = 0$, then we get

$$|f(xz_0\sigma(z_0)\sigma(y)z_0)| \leqslant \frac{\delta}{2}, \quad |f(xyz_0^3)| \leqslant \frac{M\delta}{2} + \delta.$$

From (2.11) we conclude that the function $x \longmapsto f(xz_0)$ is a bounded function on S, then the functions $(x, y) \longmapsto f(xyz_0)$; $(x, y) \longmapsto f(x\sigma(y)z_0)$ are bounded on $S \times S$. So, from (2.1) and the triangle inequality we deduce that f is a bounded function, which contradict the assumption that f is an unbounded function on S and this proves (2.5).

Equation (2.7): First we show that the function g defined by (2.6) is unbounded. If g is bounded, then the function $x \longmapsto f(xz_0)$ is also bounded. From (2.1) and the triangle inequality we get that the function $(x, y) \longrightarrow f(x)f(y)$ is bounded on $S \times S$ and this implies that f is bounded. This contradict the assumption that f is unbounded on S. From the inequalities (2.1), (2.3), (2.4), (2.5), (2.6) and the fact that $\mu(x\sigma(x)) = 1$ for all $x \in S$ we get

$$\begin{aligned}
&(f(z_0))^2[g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y)]\\
&\quad = f(z_0)f(xyz_0) + \mu(y)f(z_0)f(x\sigma(y)z_0) - 2f(xz_0)f(yz_0)\\
&\quad = f(z_0)f(xyz_0) - f(xyz_0^3) + \mu(yz_0)[\mu(\sigma(z_0))f(z_0)f(x\sigma(y)z_0) - f(x\sigma(y)z_0\sigma(z_0)z_0)]\\
&\qquad + \mu(yz_0)f(xz_0\sigma(yz_0)z_0) + f(xyz_0^3) - 2f(xz_0)f(yz_0) \leqslant (2 + M)\delta,
\end{aligned}$$

which proves (2.7). Since g is unbounded and satisfies the inequality (2.7) then from [4], we deduce that g satisfies μ-d'Alembert's functional equation (2.9).

Equation (2.8): For all $x \in S$, we have

$$g(xz_0) - g(x)g(z_0) = \frac{f(xz_0^2)}{f(z_0)} - \frac{f(z_0^2)f(xz_0)}{(f(z_0))^2} = \frac{f(xz_0^2)f(z_0) - f(z_0^2)f(xz_0)}{(f(z_0))^2}.$$

By replacing x by xz_0^2 and y by z_0 in (2.1) we get

$$|f(xz_0^4) + \mu(z_0)f(xz_0^2\sigma(z_0)z_0) - 2f(xz_0^2)f(z_0)| \leqslant \delta.$$

By replacing x by xz_0 and y by z_0^2 in (2.1) we get

$$|f(xz_0^4) + \mu(z_0^2)f(xz_0^2\sigma(z_0^2)) - 2f(z_0^2)f(xz_0)| \leqslant \delta.$$

By using the fact that $\mu(y\sigma(y)) = 1$ we get

$$\begin{aligned}
&2f(z_0^2)f(xz_0) - 2f(xz_0^2)f(z_0)\\
&\quad = [2f(z_0^2)f(xz_0) - f(xz_0^4) - \mu(z_0^2)f(xz_0^2\sigma(z_0^2))]\\
&\qquad - [2f(xz_0^2)f(z_0) - f(xz_0^4) - \mu(z_0)f(xz_0^3\sigma(z_0))] + \mu(z_0^2)[f(xz_0^2\sigma(z_0^2)) - \mu(\sigma(z_0))f(x\sigma(z_0)z_0)f(z_0)]\\
&\qquad + \mu(z_0)[f(x\sigma(z_0)z_0)f(z_0) - \mu(\sigma(z_0))f(x)(f(z_0))^2] - \mu(z_0)[f(xz_0^3\sigma(z_0)) - \mu(\sigma(z_0))f(xz_0^2)f(z_0)]\\
&\qquad - [f(xz_0^2)f(z_0) - f(x)(f(z_0))^2].
\end{aligned}$$

From inequalities (2.1), (2.2), (2.3), and the above relations we get

$$|2f(z_0^2)f(xz_0)-2f(xz_0^2)f(z_0)| \leqslant (1+M+M|f(z_0)|+|f(z_0)|)\delta,$$

from which we deduce (2.8).

Equation (2.10):

$$\begin{aligned}2|g(y)||g(xz_0)-g(x)g(z_0)| &= |2g(y)g(xz_0)-2g(x)g(y)g(z_0)|\\ &= |[g(xyz_0)+\mu(y)g(x\sigma(y)z_0)]-g(z_0)[g(xy)+\mu(y)g(x\sigma(y))]|\\ &= |g(xyz_0)-g(z_0)g(xy)+\mu(y)[g(x\sigma(y)z_0)-g(z_0)g(x\sigma(y))]|\\ &\leqslant |g(xyz_0)-g(xy)g(z_0)|+M|g(x\sigma(y)z_0)-g(x\sigma(y))g(z_0)|.\end{aligned}$$

In view of inequality (2.8), we get that the function $(x,y)\longmapsto g(y)(g(xz_0)-g(x)g(z_0))$ is bounded. Since g is an unbounded function on S then we get $g(xz_0)=g(x)g(z_0)$ for all $x\in S$. This completes the proof. □

The following theorem is the main result of the present section.

Theorem 2.2. *Let σ be an involutive morphism of S. Let μ be a bounded multiplicative function such that $\mu(x\sigma(x))=1$ for all $x\in S$. Let $\delta>0$ be fixed. If $f:S\longrightarrow\mathbb{C}$ satisfies the inequality*

$$|f(xyz_0)+\mu(y)f(x\sigma(y)z_0)-2f(x)f(y)|\leqslant\delta \tag{2.12}$$

for all $x,y\in S$, then either f is bounded or f is a solution of Kannappan's functional equation (1.2).

Proof. Assume that f is an unbounded solution of (2.12). By replacing y by yz_0 in (2.12) we get

$$|f(xyz_0^2)+\mu(yz_0)f(x\sigma(z_0)\sigma(y)z_0)-2f(x)f(yz_0)|\leqslant\delta.$$

From (2.3), (2.4), $\mu(x\sigma(x))=1$, $|\mu(x)|\leqslant M$ for all $x\in S$ and the triangle inequality we get

$$|f(z_0)f(xy)+\mu(y)f(z_0)f(x\sigma(y))-2f(x)f(yz_0)|\leqslant(1+\frac{M}{2}+\frac{M^2}{2})\delta \tag{2.13}$$

for all $x,y\in S$. Since from (2.5) we have $f(z_0)\neq0$, then the inequality (2.13) can be written as follows

$$|f(xy)+\mu(y)f(x\sigma(y))-2f(x)g(y)|\leqslant\frac{3\delta}{|f(z_0)|}$$

for all $x,y\in S$, where g is the function defined in Lemma 2.1 by the formulas (2.6). Now, from [4, Theorem 2.2(b)], we conclude that f,g are solutions of μ-Wilson's functional equation

$$f(xy)+\mu(y)f(x\sigma(y))=2f(x)g(y) \tag{2.14}$$

for all $x,y\in S$. By replacing x by z_0 in (2.14) we get $f(z_0y)+\mu(y)f(z_0\sigma(y))=2g(y)f(z_0)$. Since $\mu(y)f(\sigma(y))=f(y)$ and $\mu(y\sigma(y))=1$ then we get

$$f(z_0y)+\mu(y)f(z_0\sigma(y))=f(yz_0)+\mu(z_0)f(y\sigma(z_0))=2f(y)g(z_0).$$

Then we have $f(y)g(z_0)=g(y)f(z_0)$. Since $f(z_0)\neq0$, then we have $g=\frac{g(z_0)}{f(z_0)}f$.

For all $x,y\in S$ we have

$$f(xyz_0)+\mu(y)f(x\sigma(y)z_0)=[f(xz_0y)+\mu(y)f(xz_0\sigma(y))]=2f(xz_0)g(y)=2\beta f(x)f(y), \tag{2.15}$$

where $\beta=\frac{(g(z_0))^2}{f(z_0)}$. Substituting this into (2.1) we obtain $|2(\beta-1)f(y)f(x)|\leqslant\delta$ for all $x,y\in S$. Since f is assumed to be unbounded then we deduce that $\beta=1$ and then from (2.15) we deduce that f is a solution of (1.2). This completes the proof. □

3. The superstability of the Van Vleck's functional equation (1.3)

In the present section we prove the superstability theorem of the Van Vleck's functional equation (1.3) on semigroups. First, we prove the following useful lemma.

Lemma 3.1. *Let σ be an involutive morphism of* S*. Let μ be a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. Let $\delta > 0$ be fixed. If $f : S \longrightarrow \mathbb{C}$ is an unbounded function which satisfies the following inequality*

$$|\mu(y)f(x\sigma(y)z_0) - f(xyz_0) - 2f(x)f(y)| \leqslant \delta \tag{3.1}$$

for all $x, y \in S$, then, for all $x \in S$ we have

$$f(x) = -\mu(x)f(\sigma(x)), \tag{3.2}$$

$$|f(x\sigma(z_0)z_0) - \mu(\sigma(z_0))f(x)f(z_0)| \leqslant \frac{\delta}{2}, \tag{3.3}$$

$$|f(xz_0^2) + f(x)f(z_0)| \leqslant \frac{M\delta}{2} + \delta, \tag{3.4}$$

$$f(z_0) \neq 0, \tag{3.5}$$

$$f(z_0^2) = 0, \tag{3.6}$$

$$|f(xz_0) - \mu(\sigma(z_0))f(\sigma(x)z_0)| \leqslant \frac{3\delta + \frac{3}{2}M\delta + \frac{1}{2}M^2\delta}{|f(z_0)|}. \tag{3.7}$$

The function g *defined by*

$$g(x) = \frac{f(xz_0)}{f(z_0)}, \ x \in S \tag{3.8}$$

is unbounded on S *and satisfies the following inequality*

$$|g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y)| \leqslant \frac{(2+M)\delta}{|f(z_0)|^2} \text{ for all } x, y \in S. \tag{3.9}$$

Furthermore,

1) $g(z_0) = 0$; $g(z_0^2) \neq 0$;
2) g *is an abelian solution of μ-d'Alembert's functional equation (2.9);*
3) f, g *are solutions of μ-Wilson's functional equation (2.14).*

Proof.

Equation (3.2): By replacing y by $\sigma(y)$ in (3.1) and multiplying the resulting inequality by $\mu(y)$ and using $\mu(y\sigma(y)) = 1$ and $|\mu(y)| \leqslant M$ we get

$$|f(xyz_0) - \mu(y)f(x\sigma(y)z_0) - 2\mu(y)f(x)f(\sigma(y))| \leqslant M\delta \tag{3.10}$$

for all $x, y \in S$. By adding the result of (3.1) and (3.10) and using the triangle inequality we obtain $|2f(x)(f(y) + \mu(y)f(\sigma(y)))| \leqslant (1+M)\delta$ for all $x, y \in S$. Since f is assumed to be unbounded then we get (3.2).

Equation (3.3): By replacing x by $\sigma(z_0)$ in (3.1) we have

$$|\mu(y)f(\sigma(z_0)\sigma(y)z_0) - f(\sigma(z_0)yz_0) - 2f(\sigma(z_0))f(y)| \leqslant \delta \tag{3.11}$$

for all $y \in S$. By using (3.2) and the fact that $\mu(y\sigma(y)) = 1$ we have $f(\sigma(z_0)\sigma(y)z_0) = -\mu(\sigma(y))f(y\sigma(z_0)z_0)$ and $f(\sigma(z_0)) = -\mu(\sigma(z_0))f(z_0)$. So, equation (3.11) can be written as follows

$$|-f(y\sigma(z_0)z_0) - f(y\sigma(z_0)z_0) + 2\mu(\sigma(z_0))f(y)f(z_0)| \leqslant \delta$$

for all $y \in S$, which proves (3.3).

Equation (3.4): Taking $y = z_0$ in (3.1) we get

$$|\mu(z_0)f(x\sigma(z_0)z_0) - f(xz_0^2) - 2f(x)f(z_0)| \leqslant \delta \tag{3.12}$$

for all $x \in S$. Since

$$|f(xz_0^2) + f(x)f(z_0)| = |f(xz_0^2) + 2f(x)f(z_0) - \mu(z_0)f(x\sigma(z_0)z_0) + \mu(z_0)[f(x\sigma(z_0)z_0) - \mu(\sigma(z_0))f(x)f(z_0)]|,$$

then from (3.3), (3.12) and the triangle inequality we get (3.4).

Equation (3.5): f is assumed to be an unbounded solution of the inequality (3.1) then $f \neq 0$. Now assume that $f(z_0) = 0$. By replacing x by xz_0 in (3.1) we get

$$|\mu(y)f(x\sigma(y)z_0^2) - f(xyz_0^2) - 2f(y)f(xz_0)| \leqslant \delta.$$

For all $x, y \in S$, we have

$$\begin{aligned} 2f(y)f(xz_0) &= 2f(y)f(xz_0) + f(xyz_0^2) - \mu(y)f(x\sigma(y)z_0^2) - (f(xyz_0^2) + f(xy)f(z_0)) \\ &\quad + \mu(y)[f(x\sigma(y)z_0^2) + f(x\sigma(y))f(z_0)] + f(xy)f(z_0) - \mu(y)f(x\sigma(y))f(z_0). \end{aligned}$$

So, using (3.4), (3.1), $f(z_0) = 0$, and the triangle inequality we get that $y \longmapsto f(y)f(xz_0)$ is a bounded function on S, since f is unbounded then we obtain $f(xz_0) = 0$ for all $x \in S$. By substituting this into (3.1) we get f a bounded function on S and this contradicts the assumption that f is an unbounded function. So, we have (3.5).

Equation (3.9): By using similar computation used in the above section, the function g defined by (3.8) is an unbounded function on S. Furthermore,

$$\begin{aligned} &f(z_0)^2[g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)g(y)] \\ &\quad = f(z_0)f(xyz_0) + \mu(y)f(z_0)f(x\sigma(y)z_0) - 2f(xz_0)f(yz_0) \\ &\quad = f(xyz_0)f(z_0) + f(xyz_0^3) + \mu(yz_0)[\mu(\sigma(z_0))f(x\sigma(y)z_0)f(z_0) - f(x\sigma(y)z_0\sigma(z_0)z_0)] \\ &\qquad + \mu(yz_0)f(xz_0\sigma(yz_0)z_0) - f(xyz_0^3) - 2f(xz_0)f(yz_0). \end{aligned}$$

So, using (3.3), (3.4), and (3.1) we get (3.9).

Equation (3.6): Since g is unbounded so, from [4] g satisfies the μ-d'Alembert's functional equation (2.9). From (3.3), (3.4), the triangle inequality, and the fact that $\mu(y\sigma(y)) = 1$ we have

$$|\mu(z_0)f(x\sigma(z_0)z_0) + f(xz_0^2)| \leqslant (1 + M)\delta \tag{3.13}$$

for all $x, y \in S$. Since $g = \frac{f(xz_0)}{f(z_0)}$, the inequality (3.13) can be written as follows

$$|\mu(z_0)f(z_0)g(x\sigma(z_0)) + f(z_0)g(xz_0)| \leqslant (1 + M)\delta.$$

On the other hand g is a solution of μ-d'Alembert's functional equation (2.9) then we get $|2g(x)g(z_0)| \leqslant \frac{2\delta}{|f(z_0)|}$ for all $x \in S$. Since g is unbounded then we deduce that $g(z_0) = 0$. That is $f(z_0^2) = 0$. This proves (3.6).

Equation (3.7): By replacing x by z_0^2 in (3.1), and using (3.6) we obtain

$$|\mu(y)f(\sigma(y)z_0^3) - f(yz_0^3)| \leqslant \delta \tag{3.14}$$

for all $y \in S$. Since,

$$\begin{aligned} \mu(y)f(\sigma(y)z_0^3) - f(yz_0^3) &= \mu(y)(f(\sigma(y)z_0^3) + f(\sigma(y)z_0)f(z_0)) \\ &\quad - (f(yz_0^3) + f(yz_0)f(z_0)) - (\mu(y)f(\sigma(y)z_0) - f(yz_0))f(z_0). \end{aligned}$$

Then from (3.14), (3.4), and the triangle inequality we get (3.7).

From (3.1), (3.3), (3.4), the triangle inequality, and that $\mu(y\sigma(y)) = 1$ we get

$$\begin{aligned} &|f(z_0)f(xy) + \mu(y)f(z_0)f(x\sigma(y)) - 2f(x)f(yz_0)| \\ &\quad \leqslant |f(z_0)f(xy) + f(xyz_0^2)| + |\mu(yz_0)(\mu(\sigma(z_0))f(z_0)f(x\sigma(y)) - f(x\sigma(y)\sigma(z_0)z_0))| \\ &\quad\quad + |\mu(yz_0)f(x\sigma(y)\sigma(z_0)z_0) - f(xyz_0^2) - 2f(x)f(yz_0)| \leqslant (2+M)\delta \end{aligned} \tag{3.15}$$

for all $x, y \in S$. Since from (3.5) we have $f(z_0) \neq 0$. Then the inequality (3.15) can be written as follows

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)g(y)| \leqslant \frac{3\delta}{|f(z_0)|}$$

for all $x, y \in S$ and where g is the function defined in Lemma 3.1, (3.8). Now, by using [4, Theorem 2.2(b)] we conclude that f, g are solutions of μ-Wilson's functional equation (2.14). This completes the proof. □

The main result of the present section is the following.

Theorem 3.2. *Let σ be an involutive morphism of S. Let μ be a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. Let $\delta > 0$ be fixed. If $f : S \longrightarrow \mathbb{C}$ satisfies the following inequality*

$$|\mu(y)f(x\sigma(y)z_0) - f(xyz_0) - 2f(x)f(y)| \leqslant \delta \tag{3.16}$$

for all $x, y \in S$, then either f is bounded or f is a solution of Van Vleck's functional equation (1.3).

Proof. Assume that f is an unbounded solution of (3.16). From Lemma 3.1, the pair f, g is a solution of μ-Wilson's functional equation (2.14). Taking $y = z_0$ in (2.14) and using $g(z_0) = 0$ (see Lemma 3.1) we get

$$f(xz_0) + \mu(z_0)f(x\sigma(z_0)) = 0. \tag{3.17}$$

By replacing y by $z_0\sigma(z_0)$ in (2.14) and using that $\mu(z_0\sigma(z_0)) = 1$ we obtain

$$f(xz_0\sigma(z_0)) + f(x\sigma(z_0)z_0) = 2f(x)g(z_0\sigma(z_0)) = 2f(xz_0\sigma(z_0)).$$

That is,

$$f(xz_0\sigma(z_0)) = f(x)g(z_0\sigma(z_0)). \tag{3.18}$$

Now from (3.3) and (3.18), we get

$$|f(x)(g(z_0\sigma(z_0)) - \mu(\sigma(z_0))f(z_0))| \leqslant \frac{\delta}{2}$$

for all $x \in S$. Since f is assumed to be unbounded then we get

$$g(z_0\sigma(z_0)) = \mu(\sigma(z_0))f(z_0). \tag{3.19}$$

The function g satisfies μ-d'Alembert's functional equation (2.9) and $g(z_0) = 0$ then we have $g(yz_0) = -\mu(z_0)g(y\sigma(z_0))$ for all $y \in S$. So, by using the definition of g, equations (3.18), (3.19), and that $\mu(y\sigma(y)) = 1$ we have

$$g(yz_0) = -\mu(z_0)g(y\sigma(z_0)) = \frac{-\mu(z_0)f(y\sigma(z_0)z_0)}{f(z_0)} = \frac{-\mu(z_0)f(y)g(\sigma(z_0)z_0)}{f(z_0)} = \frac{-f(y)f(z_0)}{f(z_0)} = -f(y). \tag{3.20}$$

Finally, from (2.14), (3.17), and (3.20) we have

$$\begin{aligned} \mu(y)f(x\sigma(y)z_0) - f(xyz_0) &= -\mu(yz_0)f(x\sigma(y)\sigma(z_0)) - f(xyz_0) \\ &= -[\mu(yz_0)f(x\sigma(yz_0)) + f(xyz_0)] = -2f(x)g(yz_0) = 2f(x)f(y) \end{aligned}$$

for all $x, y \in S$. That is f is a solution of Van Vleck's functional equation (1.3). This completes the proof. □

4. The superstability of a variant of Van Vleck's functional equation (1.4)

In this section, we obtain the superstability of the variant of Van Vleck's functional equation (1.4) on semigroups. The following useful lemma will be used later.

Lemma 4.1. *Let S be a semigroup, let σ be an involutive morphism of S. Let μ be a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. Let $\delta > 0$ be fixed. If $f : S \longrightarrow \mathbb{C}$ is an unbounded function which satisfies the following inequality*

$$|\mu(y)f(\sigma(y)xz_0) - f(xyz_0) - 2f(x)f(y)| \leqslant \delta \tag{4.1}$$

for all $x, y \in S$, then, for all $x, y \in S$ we have

$$f(x) = -\mu(x)f(\sigma(x)), \tag{4.2}$$

$$|\mu(x)f(\sigma(x)y) + \mu(y)f(\sigma(y)x)| \leqslant \frac{2\delta + M}{|f(z_0)|}, \tag{4.3}$$

$$|f(x\sigma(z_0)z_0) - \mu(\sigma(z_0))f(x)f(z_0)| \leqslant \frac{\delta}{2}, \tag{4.4}$$

$$|f(xz_0^2) + f(x)f(z_0)| \leqslant \frac{M\delta}{2} + \delta, \tag{4.5}$$

$$f(z_0) \neq 0, \tag{4.6}$$

$$f(x\sigma(z_0)) = \mu(x)f(\sigma(x)\sigma(z_0)), \tag{4.7}$$

$$|f(xz_0) - \mu(x)f(\sigma(x)z_0)| \leqslant \frac{2\delta + M}{|f(z_0)|} + \frac{M(2\delta + M)}{|f(z_0)|}. \tag{4.8}$$

The function g defined by

$$g(x) = \frac{f(xz_0)}{f(z_0)}, \; x \in S \tag{4.9}$$

is unbounded on S and satisfies the following inequality

$$|g(xy) + \mu(y)g(\sigma(y)x) - 2g(x)g(y)| \leqslant \frac{3\delta}{|f(z_0)|^2} \tag{4.10}$$

for all $x, y \in S$. Furthermore, g satisfies the variant of d'Alembert's functional equation

$$g(xy) + \mu(y)g(\sigma(y)x) = 2g(x)g(y), \; x, y \in S, \tag{4.11}$$

and we have $g(z_0^2) \neq 0$ and $g(z_0) = 0$.

Proof. Let f: $S \longrightarrow \mathbb{C}$ be an unbounded function which satisfies (4.1).

Equation (4.6): We prove that $f(z_0) \neq 0$ by contradiction. Assume that $f(z_0) = 0$. By replacing y by z_0 and x by $\sigma(y)x$ in (4.1) we get

$$|\mu(z_0)f(\sigma(y)x\sigma(z_0)z_0) - f(\sigma(y)xz_0^2)| \leqslant \delta. \tag{4.12}$$

Replacing y by yz_0 in (4.1) we have

$$|\mu(yz_0)f(\sigma(y)x\sigma(z_0)z_0) - f(xyz_0^2) - 2f(x)f(yz_0)| \leqslant \delta. \tag{4.13}$$

By replacing x by xz_0 in (4.1) we obtain

$$|\mu(y)f(\sigma(y)xz_0^2) - f(xyz_0^2) - 2f(y)f(xz_0)| \leqslant \delta. \tag{4.14}$$

By subtracting the result of equation (4.14) from the result of (4.13) and using the triangle inequality, we get after computation that

$$|\mu(y)(\mu(z_0)f(\sigma(y)x\sigma(z_0)z_0) - f(\sigma(y)xz_0^2)) - 2[f(x)f(yz_0) - f(y)f(xz_0)]| \leqslant 2\delta. \tag{4.15}$$

From (4.12), (4.15), and the triangle inequality we get

$$|f(x)f(yz_0) - f(y)f(xz_0)| \leqslant \frac{M\delta}{2} + \delta. \tag{4.16}$$

Since f is assumed to be unbounded function on S then $f \neq 0$. Let $y_0 \in S$ such that $f(y_0) \neq 0$. Equation (4.16) can be written as follows

$$|f(xz_0) - \alpha f(x)| \leqslant \frac{\delta}{|f(y_0)|} + \frac{M\delta}{2|f(y_0)|}, \tag{4.17}$$

where $\alpha = \frac{f(y_0z_0)}{f(y_0)}$. Of course $\alpha \neq 0$ because if $\alpha = 0$ then by using (4.17) we deduce that the function $x \longmapsto f(xz_0)$ is bounded and from (4.1) and the triangle inequality we get f bounded which contradicts the assumption that f is an unbounded function on S.

From (4.17) and the triangle inequality, the inequality (4.1) can be written as follows

$$|\mu(y)f(\sigma(y)x) - f(xy) - \frac{2}{\alpha}f(x)f(y)| \leqslant \frac{\frac{2\delta}{|f(y_0)|} + \frac{M\delta}{|f(y_0)|}}{|\alpha|} + \frac{\delta}{|\alpha|} = N \tag{4.18}$$

for all $x, y \in S$. By replacing y by z_0 in (4.18) and using that $f(z_0) = 0$ we get

$$|\mu(z_0)f(\sigma(z_0)x) - f(xz_0)| \leqslant N \text{ for all } x \in S. \tag{4.19}$$

Replacing y by x and x by z_0 in (4.18) we get

$$|\mu(x)f(\sigma(x)z_0) - f(z_0x)| \leqslant N \text{ for all } x \in S. \tag{4.20}$$

Subtracting the result of (4.19) from the result of (4.20) and using the triangle inequality we get

$$|\mu(x)f(\sigma(x)z_0) - \mu(z_0)f(\sigma(z_0)x)| \leqslant 2M \text{ for all } x \in S. \tag{4.21}$$

By interchanging x with y in (4.18) we get

$$|\mu(x)f(\sigma(x)y) - f(yx) - \frac{2}{\alpha}f(x)f(y)| \leqslant N. \tag{4.22}$$

Replacing y by $\sigma(y)$ in (4.18) and multiplying the result by $\mu(y)$ and using that $\mu(y\sigma(y)) = 1$ we get

$$|f(yx) - \mu(y)f(x\sigma(y)) - \frac{2}{\alpha}f(x)\mu(y)f(\sigma(y))| \leqslant NM. \tag{4.23}$$

By adding the results of (4.23) and (4.22) and using the triangle inequality we have

$$|\mu(x)f(\sigma(x)y) - \mu(y)f(x\sigma(y)) - \frac{2}{\alpha}f(x)[f(y) + \mu(y)f(\sigma(y))]| \leqslant 2M. \tag{4.24}$$

By replacing x by $\sigma(x)$ in (4.24) and multiplying the result obtained by $\mu(x)$ and using that $\mu(x\sigma(x)) = 1$ we get

$$|f(xy) - \mu(xy)f(\sigma(x)\sigma(y)) - \frac{2}{\alpha}\mu(x)f(\sigma(x))[f(y) + \mu(y)f(\sigma(y))]| \leqslant N + NM. \tag{4.25}$$

If we replace y by $\sigma(y)$ in (4.24) and multiplying the result by $\mu(y)$ and using that $\mu(y\sigma(y)) = 1$ we get

$$|\mu(xy)f(\sigma(x)\sigma(y)) - f(xy) - \frac{2}{\alpha}f(x)[f(y) + \mu(y)f(\sigma(y))]| \leqslant N + NM. \tag{4.26}$$

Now, by adding the results of (4.25) and (4.26) and using the triangle inequality we have

$$|[f(x) + \mu(x)f(\sigma(x))][f(\sigma(y)) + \mu(y)f(\sigma(y))]| \leqslant \frac{((N + NM) + M(N + NM))|\alpha|}{2}.$$

That is $x \longmapsto f(x) + \mu(x)f(\sigma(x))$ is a bounded function on S. So, the function

$$x \longmapsto f(\sigma(x)z_0) + \mu(x\sigma(z_0))f(x\sigma(z_0))$$

is also a bounded function on S. Multiplying (4.21) by $\mu(\sigma(z_0))$ we deduce that $x \longmapsto \mu(x\sigma(z_0))f(\sigma(x)z_0) - f(x\sigma(z_0))$ is a bounded function on S. By using triangle inequality, the function $x \longmapsto f(x\sigma(z_0))$ is a bounded function on S and consequently the function $x \longmapsto (f \circ \sigma)(xz_0)$. Since $x \longmapsto f(xz_0) + \mu(xz_0)f \circ \sigma(xz_0)$ is bounded, then we get that f is a bounded function on S which contradicts the assumption that f is an unbounded function on S and this proves (4.6).

Equation (4.3): If we replace y by yz_0 in (4.1) we get

$$|\mu(yz_0)f(\sigma(y)x\sigma(z_0)z_0) - f(xyz_0^2) - 2f(x)f(yz_0)| \leqslant \delta. \tag{4.27}$$

Replacing x by xz_0 in (4.1) we get

$$|\mu(y)f(\sigma(y)xz_0^2) - f(xyz_0^2) - 2f(y)f(xz_0)| \leqslant \delta. \tag{4.28}$$

By subtracting the result of (4.28) from the result of (4.27) and using the triangle inequality we deduce that

$$|\mu(yz_0f(\sigma(y)x\sigma(z_0)z_0) - \mu(y)f(\sigma(y)xz_0^2) - 2[f(x)f(yz_0) - f(y)f(xz_0)]| \leqslant 2\delta. \tag{4.29}$$

Replacing y by z_0 and x by $\sigma(y)x$ in (4.1) and multiplying the result obtained by $\mu(y)$ we get

$$|\mu(yz_0)f(\sigma(y)x\sigma(z_0)z_0) - \mu(y)f(\sigma(y)xz_0^2) - 2\mu(y)f(\sigma(y)x)f(z_0)| \leqslant M\delta. \tag{4.30}$$

By subtracting the result of (4.29) from the result of (4.30) and using the triangle inequality we obtain

$$|\mu(y)f(\sigma(y)x)f(z_0) - [f(x)f(yz_0) - f(y)f(xz_0)]| \leqslant \frac{M\delta + 2\delta}{2}. \tag{4.31}$$

By interchanging x and y in (4.31) we have

$$|\mu(x)f(\sigma(x)y)f(z_0) - [f(y)f(xz_0) - f(x)f(yz_0)]| \leqslant \frac{M\delta + 2\delta}{2}. \tag{4.32}$$

By adding the result of (4.31) and the result of (4.32) and using the triangle inequality we get (4.3).

Equation (4.7): Replacing x by $x\sigma(z_0)$ in (4.1) we get

$$|\mu(y)f(\sigma(y)x\sigma(z_0)z_0) - f(xy\sigma(z_0)z_0) - 2f(y)f(x\sigma(z_0))| \leqslant \delta. \tag{4.33}$$

Replacing y by yz_0 and x by xz_0 in (4.3) and multiplying the result by $\mu(\sigma(z_0))$ and using that $\mu(y\sigma(y)) = 1$ we obtain

$$|\mu(x)f(\sigma(x)y\sigma(z_0)z_0) + \mu(y)f(\sigma(y)x\sigma(z_0)z_0)| \leqslant \frac{2\delta + M\delta}{|f(z_0)|}. \tag{4.34}$$

By subtracting the result of (4.33) from the result of (4.34) and using the triangle inequality we get

$$|\mu(x)f(\sigma(x)y\sigma(z_0)z_0) + f(xy\sigma(z_0)z_0) + 2f(y)f(x\sigma(z_0))| \leqslant \frac{2\delta + M\delta}{|f(z_0)|} + \delta. \tag{4.35}$$

Replacing x by $\sigma(x)$ in (4.35) and multiplying the result by $\mu(x)$ and using that $\mu(x\sigma(x)) = 1$ we get

$$|f(xy\sigma(z_0)z_0) + \mu(x)f(\sigma(x)y\sigma(z_0)z_0) + 2\mu(x)f(y)f(\sigma(x)\sigma(z_0))| \leqslant \frac{2\delta + M\delta}{|f(z_0)|} + \delta. \tag{4.36}$$

Subtracting the result of (4.36) from the result of (4.35) and using the triangle inequality we get

$$|2f(y)[f(x\sigma(z_0)) - \mu(x)f(\sigma(x)\sigma(z_0))]| \leqslant \frac{4\delta + 2M\delta}{|f(z_0)|} + 2\delta.$$

Since f is assumed to be unbounded then we have (4.7).

Equation (4.8): If we replace y by $\sigma(z_0)$ in (4.3) we obtain

$$|\mu(\sigma(z_0))f(xz_0)+\mu(x)f(\sigma(x)\sigma(z_0))\leqslant \frac{2\delta+M}{|f(z_0)|}. \tag{4.37}$$

In view of (4.7), the inequality (4.37) can be written as follows

$$|\mu(\sigma(z_0))f(xz_0)+f(x\sigma(z_0))|\leqslant \frac{2\delta+M}{|f(z_0)|}. \tag{4.38}$$

Replacing y by z_0 in (4.3) we get

$$|\mu(z_0)f(x\sigma(z_0))+\mu(x)f(\sigma(x)z_0)|\leqslant \frac{2\delta+M}{|f(z_0)|}. \tag{4.39}$$

By multiplying (4.38) by $\mu(z_0)$ and using that $\mu(z_0\sigma(z_0))=1$ and subtracting the resulting inequality from the result of (4.39) we get (4.8).

Equation (4.2): Replacing x by $\sigma(x)$ in (4.1) and multiplying the result by $\mu(x)$ we get

$$|\mu(xy)f(\sigma(y)\sigma(x)z_0)-\mu(x)f(\sigma(x)yz_0)-2\mu(x)f(\sigma(x))f(y)|\leqslant\delta. \tag{4.40}$$

Now, we will discuss two cases.

Case 1. If σ is an involutive automorphism of S. By replacing x by yx in (4.8) we obtain

$$|f(yxz_0)-\mu(yx)f(\sigma(y)\sigma(x)z_0)|\leqslant \frac{2\delta+M}{|f(z_0)|}+\frac{M(2\delta+M)}{|f(z_0)|}. \tag{4.41}$$

Adding the result of (4.40) to the result of (4.41) and using the triangle inequality we get

$$|f(yxz_0)-\mu(x)f(\sigma(x)yz_0)-2\mu(x)f(\sigma(x))f(y)|\leqslant \frac{2\delta+M}{|f(z_0)|}+\frac{M(2\delta+M)}{|f(z_0)|}+\delta. \tag{4.42}$$

By interchanging x and y in (4.1) we get

$$|\mu(x)f(\sigma(x)yz_0)-f(yxz_0)-2f(x)f(y)|\leqslant\delta. \tag{4.43}$$

By adding the result of (4.43) and the result of (4.42) and using the triangle inequality we obtain

$$|2f(y)[\mu(x)f(\sigma(x))+f(x)]|\leqslant \frac{2\delta+M}{|f(z_0)|}+\frac{M(2\delta+M)}{|f(z_0)|}+2\delta.$$

Since f is assumed to be unbounded then we obtain (4.2).

Case 2. If σ is an involutive anti-automorphism of S. By replacing x by yx in (4.8) we have

$$|f(yxz_0)-\mu(yx)f(\sigma(x)\sigma(y)z_0)|\leqslant \frac{2\delta+M}{|f(z_0)|}+\frac{M(2\delta+M)}{|f(z_0)|}. \tag{4.44}$$

If we replace y by x and after y by $\sigma(y)$ in (4.1) and we multiply the result by $\mu(y)$ we get

$$|\mu(xy)f(\sigma(x)\sigma(y)z_0)-\mu(y)f(\sigma(y)xz_0)-2\mu(y)f(\sigma(y))f(x)|\leqslant M\delta. \tag{4.45}$$

By adding the results of (4.44) and (4.45) and using the triangle inequality we get

$$|f(yxz_0)-\mu(y)f(\sigma(y)xz_0)-2\mu(y)f(\sigma(y))f(x)|\leqslant M\delta+\frac{2\delta+M}{|f(z_0)|}+\frac{M(2\delta+M)}{|f(z_0)|}. \tag{4.46}$$

By interchanging x by y in (4.1) we get

$$|\mu(x)f(\sigma(x)yz_0) - f(yxz_0) - 2f(y)f(x)| \leqslant \delta. \tag{4.47}$$

By replacing x by $\sigma(x)y$ in (4.8) and multiplying the result obtained by $\mu(x)$ and using that $\mu(x\sigma(x)) = 1$ we get

$$|\mu(x)f(\sigma(x)yz_0) - \mu(y)f(\sigma(y)xz_0)| \leqslant \frac{2\delta + M}{|f(z_0)|} + \frac{M(2\delta + M)}{|f(z_0)|}. \tag{4.48}$$

By subtracting the results of (4.47) from (4.48) and using the triangle inequality we get

$$|\mu(y)f(\sigma(y)xz_0) - f(yxz_0) - 2f(y)f(x)| \leqslant \delta + \frac{2\delta + M}{|f(z_0)|} + \frac{M(2\delta + M)}{|f(z_0)|}. \tag{4.49}$$

By adding the results of (4.49) and (4.46)

$$|2f(x)(f(y) + \mu(y)f(\sigma(y)))| \leqslant 2\frac{2\delta + M}{|f(z_0)|} + \frac{M(2\delta + M)}{|f(z_0)|} + M\delta + \delta$$

for all $x, y \in S$. Since f is unbounded then we we get (4.2).

Equation (4.4): If we replace x by $\sigma(z_0)$ in (4.1) and using (4.2) and using that $\mu(x\sigma(x)) = 1$ for all $x \in S$, we get

$$| - f(yz_0\sigma(z_0)) - f(y\sigma(z_0)z_0) + 2\mu(\sigma(z_0))f(y)f(z_0)| \leqslant \delta,$$

which proves (4.4).

Equation (4.5): By replacing y by z_0 in (4.1) we get

$$|\mu(z_0)f(x\sigma(z_0)z_0) - f(xz_0^2) - 2f(x)f(z_0)| \leqslant \delta. \tag{4.50}$$

Multiplying (4.4) by $\mu(z_0)$ and subtracting the result obtained from the result of (4.50) and using the triangle inequality we deduce (4.5).

Equation (4.10): Let g be the function defined by (4.9). Then we have

$$\begin{aligned} f(z_0)^2[g(xy) + \mu(y)g(\sigma(y)x) - 2g(x)g(y)] &= f(z_0)f(xyz_0) + \mu(y)f(z_0)f(\sigma(y)xz_0) - 2f(xz_0)f(yz_0) \\ &= f(xyz_0)f(z_0) + f(xyz_0^3) + \mu(yz_0)(\mu(\sigma(z_0))f(\sigma(y)xz_0)f(z_0) - f(\sigma(y)x\sigma(z_0)z_0^2)) \\ &\quad + \mu(yz_0)f(\sigma(yz_0)xz_0^2) - f(xyz_0^3) - 2f(xz_0)f(yz_0). \end{aligned}$$

So, from (4.4), (4.5), and (4.1) we get (4.10). Now, since f is unbounded then g is unbounded and satisfies (4.1). So, by using same computations used in [14] g satisfies the variant of d'Alembert's functional equation (4.11).

Finally, from (4.4), (4.5), and the triangle inequality we have

$$|\mu(z_0)f(x\sigma(z_0)z_0) + f(xz_0^2)| \leqslant (1 + M)\delta \tag{4.51}$$

for all $x, y \in S$. By using the definition of g, the inequality (4.51) can be written as follows

$$|f(z_0)[\mu(z_0)g(x\sigma(z_0)) + g(xz_0)]| \leqslant 2\delta.$$

On the other hand g is a solution of μ-d'Alembert's functional equation (4.11) then g is central [25] and we get $|2g(x)g(z_0)| \leqslant \frac{(1+M)\delta}{|f(z_0)|}$ for all $x \in S$. Since g is unbounded then we deduce that $g(z_0) = 0$. That is $f(z_0^2) = 0$. □

Theorem 4.2. *Let S be a semigroup, let σ be an involutive morphism of S. Let μ be a bounded multiplicative function, such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. Let $\delta > 0$ be fixed. If $f : S \longrightarrow \mathbb{C}$ is a function which satisfies the inequality*

$$|\mu(y)f(\sigma(y)xz_0) - f(xyz_0) - 2f(x)f(y)| \leqslant \delta \tag{4.52}$$

for all $x, y \in S$, then, either f is bounded on S or f is a solution of the variant of Van Vleck's functional equation (1.4).

Proof. Assume that f is an unbounded solution of (4.52). Replacing y by yz_0 in (4.1) we get

$$|\mu(yz_0)f(\sigma(y)x\sigma(z_0)z_0) - f(xyz_0^2) - 2f(x)f(yz_0)| \leqslant \delta$$

for all $x, y \in S$. By using (4.1), (4.4), and (4.5) and the triangle inequality we get

$$|f(z_0)f(xy) + \mu(y)f(z_0)f(\sigma(y)x) - 2f(x)f(yz_0)| \leqslant 2\delta + \frac{M\delta}{2} + \frac{M^2\delta}{2}$$

for all $x, y \in S$. Equation which can be written as follows

$$|f(xy) + \mu(y)f(\sigma(y)x) - 2f(x)g(y)| \leqslant \frac{2\delta}{|f(z_0)|} + \frac{M\delta}{2|f(z_0)|} + \frac{M^2\delta}{2|f(z_0)|} \tag{4.53}$$

for all $x, y \in S$ and where g is the function defined by (4.9). Replacing x by xz_0 in (4.1) we get

$$|\mu(y)f(\sigma(y)xz_0^2) - f(xyz_0^2) - 2f(y)f(xz_0)| \leqslant \delta$$

for all $x, y \in S$. By using (4.1), (4.5), and the triangle inequality we get

$$|f(xy) - \mu(y)f(\sigma(y)x) - 2f(y)g(x)| \leqslant \frac{2\delta}{|f(z_0)|} + \frac{3M\delta}{2|f(z_0)|} + \frac{M^2\delta}{2|f(z_0)|} \tag{4.54}$$

for all $x, y \in S$. By adding the result of (4.54) and (4.53) we get

$$|f(xy) - f(x)g(y) - f(x)g(y)| \leqslant \frac{2\delta}{|f(z_0)|} + \frac{M\delta}{|f(z_0)|} + \frac{M^2\delta}{2|f(z_0)|}$$

for all $x, y \in S$. Now, we will show that if $\alpha, \beta \in \mathbb{C}$ and $\alpha f + \beta g$ is a bounded function on S, then $\alpha = \beta = 0$. Assume that there exits N such that

$$|\alpha f(x) + \beta f(xz_0)| \leqslant N \tag{4.55}$$

for all $x \in S$. Then by replacing x by $\sigma(x)$ and multiplying the result by $\mu(x)$ we get $|\alpha\mu(x)f(\sigma(x)) + \beta\mu(x)f(\sigma(x)z_0)| \leqslant NM$. Using (4.8), the triangle inequality, and that $\mu(x)f(\sigma(x)) = -f(x)$ we get

$$|-\alpha f(x) + \beta f(xz_0)| \leqslant M + |\beta|(\frac{2\delta + M}{|f(z_0)|} + \frac{M\delta}{|f(z_0)|} + \frac{M^2}{2|f(z_0)|}). \tag{4.56}$$

By adding the result of (4.55) and (4.56) we get $2\beta f(xz_0)$ is a bounded function. Since g is unbounded then $\beta = 0$ and consequently $\alpha = 0$. Now, from [30, Lemma 2.1] we conclude that the pair f, g is a solution of the sine addition law

$$f(xy) = f(x)g(y) + f(y)g(x) \; x, y \in S.$$

Since $\mu(x)f(\sigma(x)) = -f(x)$ and $\mu(x)g(\sigma(x)) = g(x)$ for all $x \in S$ then the pair f, g satisfies the variant μ-Wilson's functional equation

$$f(xy) + \mu(y)f(\sigma(y)x) = 2f(x)g(y) \; x, y \in S. \tag{4.57}$$

By taking $y = z_0$ in (4.57) and using $g(z_0) = 0$ (see Lemma 4.1) we get

$$f(xz_0) + \mu(z_0)f(\sigma(z_0)x) = 0. \tag{4.58}$$

By replacing y by $z_0\sigma(z_0)$ in (4.57) and using that $\mu(x\sigma(x)) = 1$ for all $x \in S$ we get

$$f(xz_0\sigma(z_0)) + f(xz_0\sigma(z_0)) = 2f(x)g(z_0\sigma(z_0)).$$

That is

$$f(xz_0\sigma(z_0)) = f(x)g(z_0\sigma(z_0)). \tag{4.59}$$

Now from (4.4) and (4.59) we get

$$|f(x)(g(z_0\sigma(z_0)) - \mu(\sigma(z_0))f(z_0))| \leqslant \frac{\delta}{2}$$

for all $x \in S$. Since f is assumed to be unbounded then we get

$$g(z_0\sigma(z_0)) = \mu(\sigma(z_0))f(z_0). \tag{4.60}$$

The function g is a solution of (4.11) and $g(z_0) = 0$, then we get $g(yz_0) = -\mu(z_0)g(y\sigma(z_0))$. So, by using the definition of g, and equations (4.59), (4.2), (4.60) we obtain

$$g(yz_0) = -\mu(z_0)g(y\sigma(z_0)) = \frac{-\mu(z_0)f(y\sigma(z_0)z_0)}{f(z_0)} = \frac{-\mu(z_0)f(y)g(\sigma(z_0)z_0)}{f(z_0)} = \frac{-f(y)f(z_0)}{f(z_0)} = -f(y). \tag{4.61}$$

Finally, from (4.58), (4.57), and (4.61) we have

$$\begin{aligned}\mu(y)f(\sigma(y)xz_0) - f(xyz_0) &= -\mu(yz_0)f(\sigma(y)x\sigma(z_0)) - f(xyz_0)\\ &= -[\mu(yz_0)f(\sigma(yz_0)x) + f(xyz_0)] = -2f(x)g(yz_0) = 2f(x)f(y)\end{aligned}$$

for all $x, y \in S$. This completes the proof. □

5. The superstability of a variant of Kannappan's functional equation (1.5)

In this section we obtain the superstability result of equation (1.5) on semigroups not necessarily abelian. Later, we need the following Lemma.

Lemma 5.1. *Let S be a semigroup, let σ be an involutive morphism of S. Let μ be a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. Let $\delta > 0$ be fixed. If $f : S \longrightarrow \mathbb{C}$ is an unbounded function which satisfies the following inequality*

$$|f(xyz_0) + \mu(y)f(\sigma(y)xz_0) - 2f(x)f(y)| \leqslant \delta \tag{5.1}$$

for all $x, y \in S$, then, for all $x \in S$

$$f(x) = \mu(x)f(\sigma(x)), \tag{5.2}$$

$$|f(x\sigma(z_0)z_0) - \mu(\sigma(z_0))f(x)f(z_0)| \leqslant \frac{\delta + \alpha}{2}, \tag{5.3}$$

where $\alpha = (M+1)\sqrt{\frac{\delta}{2}}$.

$$|f(xz_0^2) - f(x)f(z_0)| \leqslant \frac{M(\delta + \alpha)}{2} + \delta, \tag{5.4}$$

$$f(z_0) \neq 0, \text{ and } f(\sigma(z_0)) \neq 0, \tag{5.5}$$

$$f(x\sigma(z_0)) = \mu(x)f(\sigma(x)\sigma(z_0)), \tag{5.6}$$

$$|f(xz_0) - \mu(x)f(\sigma(x)z_0)| \leqslant (M+1)\alpha. \tag{5.7}$$

The function g *defined by*

$$g(x) = \frac{f(xz_0)}{f(z_0)} \text{ for } x \in S \tag{5.8}$$

is unbounded on S *and satisfies the following inequalities*

$$|g(xy) + \mu(y)g(\sigma(y)x) - 2g(x)g(y)| \leqslant \frac{M(\delta+\alpha)+2\delta}{|f(z_0)|^2}, \tag{5.9}$$

$$|g(xz_0) - g(x)g(z_0)| \leqslant \frac{2\delta + M(\delta+\alpha)}{|f(z_0)|^2} + \frac{\delta + M(\delta+\alpha)}{|f(z_0)|} \tag{5.10}$$

for all $x, y \in S$*. Furthermore,* g *is a non-zero solution of a variant of* μ*-d'Alembert's functional equation*

$$g(xy) + \mu(y)g(\sigma(y)x) = 2g(x)g(y), \ x, y \in S, \tag{5.11}$$

and satisfies the condition

$$g(xz_0) = g(z_0)g(x) \text{ for all } x \in S. \tag{5.12}$$

Proof. First we prove that $x \longrightarrow f(x) - \mu(x)f(\sigma(x))$ is a bounded function on S. Interchanging x and y in (5.1) and multiplying the result by $\mu(\sigma(y))$ we get

$$|\mu(x\sigma(y))f(\sigma(x)yz_0) + \mu(\sigma(y))f(yxz_0) - 2f(x)\mu(\sigma(y))f(y)| \leqslant M\delta. \tag{5.13}$$

Replacing y by $\sigma(y)$ in (5.1) we obtain

$$|\mu(\sigma(y))f(yxz_0) + f(x\sigma(y)z_0) - 2f(x)f(\sigma(y))| \leqslant \delta. \tag{5.14}$$

By subtracting (5.14) from (5.13) and using the triangle inequality we get

$$|\mu(x\sigma(y))f(\sigma(x)yz_0) - f(x\sigma(y)z_0) - 2f(x)[\mu(\sigma(y))f(y) - f(\sigma(y))]| \leqslant M\delta + \delta. \tag{5.15}$$

By replacing x by $\sigma(x)$ in (5.15) we have

$$|\mu(\sigma(x)\sigma(y))f(xyz_0) - f(\sigma(x)\sigma(y)z_0) - 2f(\sigma(x))[\mu(\sigma(y))f(y) - f(\sigma(y))]| \leqslant M\delta + \delta. \tag{5.16}$$

Replacing y by $\sigma(y)$ in (5.15) and multiplying the result by $\mu(\sigma(y)\sigma(x))$ we obtain

$$|f(\sigma(x)\sigma(y)z_0) - \mu(\sigma(x)\sigma(y))f(xyz_0) - 2f(x)\mu(\sigma(x))[f(\sigma(y)) - \mu(\sigma(y))f(y)]| \leqslant M^2\delta + M\delta. \tag{5.17}$$

Now, by adding (5.16) and (5.17) and using the triangle inequality we get

$$|[f(\sigma(x)) - \mu(\sigma(x))f(x)][f(\sigma(y)) - \mu(\sigma(y))f(y)]| \leqslant \frac{\delta}{2}(M^2 + 2M + 1). \tag{5.18}$$

Replacing y by x in (5.18) we deduce that

$$|f(\sigma(x)) - \mu(\sigma(x))f(x)| \leqslant (M+1)\sqrt{\frac{\delta}{2}} = \alpha,$$

which gives by replacing x by $\sigma(x)$,

$$|f(x) - \mu(x)f(\sigma(x))| \leqslant \alpha \tag{5.19}$$

for all $x \in S$.

Equation (5.3): First we prove that $f(z_0) = \mu(z_0)f(\sigma(z_0))$. Replacing y by z_0 in (5.1) we get

$$|f(xz_0^2) + \mu(z_0)f(x\sigma(z_0)z_0) - 2f(x)f(z_0)| \leqslant \delta. \tag{5.20}$$

Replacing y by $\sigma(z_0)$ in (5.1) we get

$$|f(x\sigma(z_0)z_0) + \mu(\sigma(z_0))f(xz_0^2) - 2f(x)f(\sigma(z_0))| \leqslant \delta. \tag{5.21}$$

Multiplying (5.21) by $\mu(z_0)$ and using that $\mu(x\sigma(x)) = 1$ we get

$$|\mu(z_0)f(x\sigma(z_0)z_0) + f(xz_0^2) - 2\mu(z_0)f(\sigma(z_0))f(x)| \leqslant M\delta. \tag{5.22}$$

Subtracting (5.20) from (5.22) we get

$$|2f(x)[f(z_0) - \mu(z_0)f(\sigma(z_0))]| \leqslant M\delta + \delta.$$

Since f is assumed to be unbounded, then

$$f(z_0) = \mu(z_0)f(\sigma(z_0)). \tag{5.23}$$

Now, if we replace x by $\sigma(z_0)$ in (5.1) we get

$$|f(y\sigma(z_0)z_0) + \mu(y)f(\sigma(y)\sigma(z_0)z_0) - 2f(\sigma(z_0))f(y)| \leqslant \delta. \tag{5.24}$$

Replacing x by $y\sigma(z_0)z_0$ in (5.19) and using that $\mu(x\sigma(x)) = 1$ we get

$$|f(y\sigma(z_0)z_0) - \mu(y)f(\sigma(y)\sigma(z_0)z_0)| \leqslant \alpha. \tag{5.25}$$

Adding (5.24) and (5.25) we get

$$|f(y\sigma(z_0)z_0) - f(\sigma(z_0))f(y)| \leqslant \frac{\delta + \alpha}{2}. \tag{5.26}$$

From (5.23) and (5.26) we deduce (5.3).

Equation (5.4): Multiplying (5.3) by $\mu(z_0)$ we get

$$|\mu(z_0)f(x\sigma(z_0)z_0) - f(z_0)f(x)| \leqslant \frac{M(\delta + \alpha)}{2}. \tag{5.27}$$

Subtracting the result of (5.27) from (5.20) and using the triangle inequality we deduce (5.4).

Equation (5.5): Assume that f is an unbounded function which satisfies the inequality (5.1) and that $f(z_0) = 0$. Replacing x by xz_0, y by yz_0 in (5.1) we get

$$|f(xyz_0^3) + \mu(yz_0)f(\sigma(yz_0)xz_0z_0) - 2f(xz_0)f(yz_0)| \leqslant \delta. \tag{5.28}$$

In view of (5.3) and (5.4) we have

$$|f(z_0\sigma(y)x\sigma(z_0)z_0) - \mu(\sigma(z_0))f(z_0)f(z_0\sigma(y)x)| \leqslant \frac{\delta + \alpha}{2},$$
$$|f(xyz_0^3) - f(z_0)f(xyz_0)| \leqslant \frac{M(\delta + \alpha)}{2} + \delta.$$

Since $f(z_0) = 0$, then we get

$$|f(z_0\sigma(z_0)\sigma(y)xz_0)| \leqslant \frac{\delta + \alpha}{2}, \quad |f(xyz_0^3)| \leqslant \frac{M(\delta + \alpha)}{2} + \delta.$$

From (5.28) we conclude that the function $h(x) = f(xz_0)$ is a bounded function on S, in particular the functions $(x, y) \longrightarrow f(xyz_0)$; $(x, y) \longrightarrow f(\sigma(y)xz_0)$ are bounded on $S \times S$. So, from (5.1) we deduce that f is a bounded function, which contradicts the assumption that f is an unbounded function on S. So, we deduce that $f(z_0) \neq 0$, and that $f(\sigma(z_0)) \neq 0$ because $f(\sigma(z_0)) = \mu(\sigma(z_0))f(z_0)$ and $\mu(x) \neq 0$ for all $x \in S$. This proves (5.5).

Equation (5.6): Replacing x by $x\sigma(z_0)$ in (5.1) we get

$$|\mu(y)f(\sigma(y)x\sigma(z_0)z_0) + f(xy\sigma(z_0)z_0) - 2f(y)f(x\sigma(z_0))| \leqslant \delta. \tag{5.29}$$

Now we will discuss two cases.

Case 1. If σ is an involutive anti-automorphism of S. By replacing x by $\sigma(x)y\sigma(z_0)z_0$ in (5.19) we get

$$|f(\sigma(x)y\sigma(z_0)z_0) - \mu(\sigma(x)y)f(\sigma(y)x\sigma(z_0)z_0)| \leqslant \alpha. \tag{5.30}$$

Multiplying (5.30) by $\mu(x)$ and using that $\mu(x\sigma(x)) = 1$ we obtain

$$|\mu(x)f(\sigma(x)y\sigma(z_0)z_0) - \mu(y)f(\sigma(y)x\sigma(z_0)z_0)| \leqslant M\alpha. \tag{5.31}$$

By adding the result of (5.29) to the result of (5.31) and using the triangle inequality we get

$$|\mu(x)f(\sigma(x)y\sigma(z_0)z_0) + f(xy\sigma(z_0)z_0) - 2f(y)f(x\sigma(z_0))| \leqslant M\alpha + \delta. \tag{5.32}$$

Replacing x by $\sigma(x)$ in (5.32) and multiplying the result by $\mu(x)$ and using that $\mu(x\sigma(x)) = 1$ we get

$$|f(xy\sigma(z_0)z_0) + \mu(x)f(\sigma(x)y\sigma(z_0)z_0) - 2\mu(x)f(y)f(\sigma(x)\sigma(z_0))| \leqslant M^2\alpha + M\delta. \tag{5.33}$$

Subtracting the result of (5.33) from the result of (5.32) and using the triangle inequality we get

$$|2f(y)[f(x\sigma(z_0)) - \mu(x)f(\sigma(x)\sigma(z_0))] \leqslant M^2\alpha + M(\alpha+\delta) + \delta.$$

Since f is assumed to be unbounded, then we have (5.6).

Case 2. If σ is an involutive automorphism of S. Replacing y by $\sigma(x)\sigma(z_0)$, and x by y in (5.1) we get

$$|f(y\sigma(x)\sigma(z_0)z_0) + \mu(\sigma(x)\sigma(z_0))f(xyz_0^2) - 2f(y)f(\sigma(x)\sigma(z_0))| \leqslant \delta. \tag{5.34}$$

Multiplying (5.34) by $\mu(x)$ and using that $\mu(x\sigma(x)) = 1$ we obtain

$$|\mu(x)f(y\sigma(x)\sigma(z_0)z_0) + \mu(\sigma(z_0))f(xyz_0^2) - 2\mu(x)f(\sigma(x)\sigma(z_0))f(y)| \leqslant M\delta. \tag{5.35}$$

Subtracting (5.35) from (5.29) we get

$$\begin{aligned}|f(xy\sigma(z_0)z_0) - \mu(\sigma(z_0))f(xyz_0^2) + [\mu(y)f(\sigma(y)x\sigma(z_0)z_0) - \mu(x)f(y\sigma(x)\sigma(z_0)z_0)]\\ - 2f(y)[f(x\sigma(z_0)) - \mu(x)f(\sigma(x)\sigma(z_0))]| \leqslant \delta + M\delta.\end{aligned} \tag{5.36}$$

On the other hand, if we multiply (5.4) by $\mu(\sigma(z_0))$ we get

$$|\mu(\sigma(z_0))f(xz_0^2) - \mu(\sigma(z_0))f(x)f(z_0)| \leqslant \frac{M^2(\alpha+\delta)}{2} + M\delta. \tag{5.37}$$

Subtracting (5.37) from (5.3) we get

$$|f(x\sigma(z_0)z_0) - \mu(\sigma(z_0))f(xz_0^2)| \leqslant \frac{M^2(\alpha+\delta)}{2} + M\delta + \frac{\delta+\alpha}{2}. \tag{5.38}$$

Replacing x in (5.38) by xy we get

$$|f(xy\sigma(z_0)z_0) - \mu(\sigma(z_0))f(xyz_0^2)| \leqslant \frac{M^2(\alpha+\delta)}{2} + M\delta + \frac{\delta+\alpha}{2}. \tag{5.39}$$

Replacing x by $\sigma(y)x\sigma(z_0)z_0$ in (5.19) we get

$$|f(\sigma(y)x\sigma(z_0)z_0) - \mu(\sigma(y)x)f(y\sigma(x)\sigma(z_0)z_0))| \leqslant \alpha. \tag{5.40}$$

Multiplying (5.40) by $\mu(y)$ and using that $\mu(y\sigma(y))=1$ we obtain

$$|\mu(y)f(\sigma(y)x\sigma(z_0)z_0)-\mu(x)f(y\sigma(x)\sigma(z_0)z_0)|\leqslant M\alpha. \tag{5.41}$$

From (5.39), (5.41), (5.36), and the triangle inequality we deduce that

$$|f(y)[f(x\sigma(z_0))-\mu(x)f(\sigma(x)\sigma(z_0))]|\leqslant \frac{\alpha+\delta}{4}M^2+\frac{2\delta+\alpha}{2}M+\frac{3\delta+\alpha}{4}.$$

Since f is assumed to be unbounded we deduce (5.6).

Equation(5.7): If we replace x by xz_0 in (5.19) we obtain

$$|f(xz_0)-\mu(xz_0)f(\sigma(x)\sigma(z_0))|\leqslant \alpha. \tag{5.42}$$

In view of (5.6), the inequality (5.42) can be written as follows

$$|f(xz_0)-\mu(z_0)f(x\sigma(z_0))|\leqslant \alpha. \tag{5.43}$$

Replacing x by $x\sigma(z_0)$ in (5.19) we get

$$|f(x\sigma(z_0))-\mu(x\sigma(z_0))f(\sigma(x)z_0)|\leqslant \alpha. \tag{5.44}$$

Multiplying (5.44) by $\mu(z_0)$ and using that $\mu(z_0\sigma(z_0))=1$ we get

$$|\mu(z_0)f(x\sigma(z_0))-\mu(x)f(\sigma(x)z_0)|\leqslant M\alpha. \tag{5.45}$$

By adding the results of (5.43) and (5.45) we deduce (5.7).

Equation (5.2): Replacing x by $\sigma(x)$ in (5.1) and multiplying the result by $\mu(x)$ we get

$$|\mu(xy)f(\sigma(y)\sigma(x)z_0)+\mu(x)f(\sigma(x)yz_0)-2\mu(x)f(\sigma(x))f(y)|\leqslant M\delta. \tag{5.46}$$

Now, we will discuss two cases.

Case 1. If σ is an involutive automorphism of S. By replacing x by yx in (5.7) we obtain

$$|f(yxz_0)-\mu(yx)f(\sigma(y)\sigma(x)z_0)|\leqslant (M+1)\alpha. \tag{5.47}$$

Adding the result of (5.46) to the result of (5.47) and using the triangle inequality we get

$$|f(yxz_0)+\mu(x)f(\sigma(x)yz_0)-2\mu(x)f(\sigma(x))f(y)|\leqslant M(\alpha+\delta)+\alpha. \tag{5.48}$$

By interchanging x and y in (5.1) we get

$$|\mu(x)f(\sigma(x)yz_0)+f(yxz_0)-2f(x)f(y)|\leqslant \delta. \tag{5.49}$$

By subtracting the result of (5.48) from the result of (5.49) and using the triangle inequality we obtain

$$|2f(y)[\mu(x)f(\sigma(x))-f(x)]|\leqslant M(\delta+\alpha)+\alpha+\delta.$$

Since f is assumed to be unbounded then we obtain (5.2).

Case 2. If σ is an involutive anti-automorphism of S. By replacing x by yx in (5.7) we have

$$|f(yxz_0)-\mu(yx)f(\sigma(x)\sigma(y)z_0)|\leqslant (M+1)\alpha. \tag{5.50}$$

If we replace y by x and x by $\sigma(y)$ in (5.1) and we multiply the result by $\mu(y)$ we get

$$|\mu(xy)f(\sigma(x)\sigma(y)z_0)+\mu(y)f(\sigma(y)xz_0)-2\mu(y)f(\sigma(y))f(x)|\leqslant M\delta. \tag{5.51}$$

By adding the results of (5.50) and (5.51) and using the triangle inequality we get

$$|f(yxz_0)+\mu(y)f(\sigma(y)xz_0)-2\mu(y)f(\sigma(y))f(x)| \leqslant M(\alpha+\delta)+\alpha. \tag{5.52}$$

By interchanging x by y in (5.1) we get

$$|\mu(x)f(\sigma(x)yz_0)+f(yxz_0)-2f(y)f(x)| \leqslant \delta. \tag{5.53}$$

By replacing x by $\sigma(x)y$ in (5.7) and multiplying the result obtained by $\mu(x)$ and using that $\mu(x\sigma(x)) = 1$ we get

$$|\mu(x)f(\sigma(x)yz_0)-\mu(y)f(\sigma(y)xz_0)| \leqslant M^2\alpha+M\alpha. \tag{5.54}$$

By subtracting the results of (5.54) from (5.53) and using the triangle inequality we get

$$|\mu(y)f(\sigma(y)xz_0)+f(yxz_0)-2f(y)f(x)| \leqslant M^2\alpha+M\alpha+\delta. \tag{5.55}$$

By subtracting the results of (5.55) from the result of (5.52) and using the triangle inequality we get

$$|2f(x)(f(y)-\mu(y)f(\sigma(y)))| \leqslant M^2\alpha+M(2\alpha+\delta)+\alpha+\delta$$

for all $x, y \in S$. Since f is unbounded then we we get (5.2).

Equation (5.9): In the following we will show that the function g defined by (5.8) is unbounded. If g is bounded, then the function $x \longrightarrow f(xz_0)$ is also bounded. From (5.1) and the triangle inequality we get that the function $(x,y) \longrightarrow f(x)f(y)$ is bounded on $S \times S$ and this implies that f is bounded. This contradicts the assumption that f is assumed to be unbounded on S. From the inequalities (5.1), (5.3), (5.4), and that $\mu(x\sigma(x)) = 1$ for all $x \in S$ we get

$$\begin{aligned}
&(f(z_0))^2[g(xy)+\mu(y)g(\sigma(y)x)-2g(x)g(y)]\\
&\quad= f(z_0)f(xyz_0)+\mu(y)f(z_0)f(\sigma(y)xz_0)-2f(xz_0)f(yz_0)\\
&\quad= f(z_0)f(xyz_0)-f(xyz_0^3)+\mu(yz_0)[\mu(\sigma(z_0))f(z_0)f(\sigma(y)xz_0)-f(\sigma(y)xz_0\sigma(z_0)z_0)]\\
&\qquad+\mu(yz_0)f(\sigma(yz_0)xz_0z_0)+f(xyz_0^3)-2f(xz_0)f(yz_0) \leqslant M(\delta+\alpha)+2\delta,
\end{aligned}$$

which gives (5.9).

Equation (5.10): For all $x \in S$, we have

$$g(xz_0)-g(x)g(z_0) = \frac{f(xz_0^2)}{f(z_0)}-\frac{f(z_0^2)f(xz_0)}{(f(z_0))^2} = \frac{f(xz_0^2)f(z_0)-f(z_0^2)f(xz_0)}{(f(z_0))^2}.$$

Replacing x by xz_0^2 and y by z_0 in (5.1) we get

$$|f(xz_0^4)+\mu(z_0)f(xz_0^2\sigma(z_0)z_0)-2f(xz_0^2)f(z_0)| \leqslant \delta.$$

By replacing x by xz_0 and y by z_0^2 in (5.1) we get

$$|f(xz_0^4)+\mu(z_0^2)f(\sigma(z_0^2)xz_0^2)-2f(z_0^2)f(xz_0)| \leqslant \delta.$$

By using the fact that $\mu(y\sigma(y)) = 1$ we get

$$\begin{aligned}
&2f(z_0^2)f(xz_0)-2f(xz_0^2)f(z_0)\\
&\quad= [2f(z_0^2)f(xz_0)-f(xz_0^4)-\mu(z_0^2)f(xz_0^2\sigma(z_0^2))]\\
&\qquad-[2f(xz_0^2)f(z_0)-f(xz_0^4)-\mu(z_0)f(xz_0^3\sigma(z_0))]+\mu(z_0^2)[f(xz_0^2\sigma(z_0^2))-\mu(\sigma(z_0))f(x\sigma(z_0)z_0)f(z_0)]\\
&\qquad+\mu(z_0)[f(x\sigma(z_0)z_0)f(z_0)-\mu(\sigma(z_0))f(x)(f(z_0))^2]-\mu(z_0)[f(xz_0^3\sigma(z_0))-\mu(\sigma(z_0))f(xz_0^2)f(z_0)]\\
&\qquad-[f(xz_0^2)f(z_0)-f(x)(f(z_0))^2].
\end{aligned}$$

From inequalities (5.1), (5.2), (5.3), and the above relations we get

$$2f(z_0^2)f(xz_0) - 2f(xz_0^2)f(z_0) \leqslant 2\delta + M(\delta+\alpha) + f(z_0)(\delta + M(\delta+\alpha)),$$

which implies that

$$|g(xz_0) - g(x)g(z_0)| \leqslant \frac{2\delta + M(\delta+\alpha)}{|f(z_0)^2|} + \frac{\delta + M(\delta+\alpha)}{|f(z_0)|}$$

and this proves (5.10). Now, since g is unbounded and satisfies the inequality (5.9) then, from [14], we deduce that g satisfies μ-d'Alembert's functional equation (5.11). We will show that $g(xz_0) = g(x)g(z_0)$ for all $x \in S$,

$$\begin{aligned}
2|g(y)||g(xz_0) - g(x)g(z_0)| &= |2g(y)g(xz_0) - 2g(x)g(y)g(z_0)| \\
&= |[g(xyz_0) + \mu(y)g(\sigma(y)xz_0)] - g(z_0)[g(xy) + \mu(y)g(\sigma(y)x)]| \\
&= |g(xyz_0) - g(z_0)g(xy) + \mu(y)[g(\sigma(y)xz_0) - g(z_0)g(\sigma(y)x)]| \\
&\leqslant |g(xyz_0) - g(xy)g(z_0)| + |g(\sigma(y)xz_0) - g(\sigma(y)x)g(z_0)|.
\end{aligned}$$

In view of inequality (5.10) we obtain

$$2|g(y)||g(xz_0) - g(x)g(z_0)| \leqslant 2\left(\frac{2\delta + M(\delta+\alpha)}{|f(z_0)|^2} + \frac{\delta + M(\delta+\alpha)}{|f(z_0)|}\right).$$

Since g is an unbounded function on S then we get $g(xz_0) = g(x)g(z_0)$ for all $x \in S$. This completes the proof. □

Now, we are ready to prove the main result of the present section.

Theorem 5.2. *Let S be a semigroup, and σ be an involutive morphism of S. Let μ be a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$, and z_0 is contained in the center of S. Let $\delta > 0$ be fixed. If $f : S \longrightarrow \mathbb{C}$ satisfies the inequality*

$$|f(xyz_0) + \mu(y)f(\sigma(y)xz_0) - 2f(x)f(y)| \leqslant \delta \tag{5.56}$$

for all $x, y \in S$, then either f is bounded or f is a solution of the variant of Kannappan's functional equation (1.5).

Proof. Assume that f is an unbounded solution of (5.56). Replacing y by yz_0 in (5.56) we get

$$|f(xyz_0^2) + \mu(yz_0)f(\sigma(z_0)\sigma(y)xz_0) - 2f(x)f(yz_0)| \leqslant \delta$$

for all $y \in S$. From (5.3), (5.4), $\mu(x\sigma(x)) = 1$ for all $x \in S$, and the triangle inequality we get

$$|f(z_0)f(xy) + \mu(y)f(z_0)f(\sigma(y)x) - 2f(x)f(yz_0)| \leqslant 2\delta + M(\delta+\alpha) \tag{5.57}$$

for all $x, y \in S$. Since from (5.5) we have $f(z_0) \neq 0$, then the inequality (5.57) can be written as follows

$$|f(xy) + \mu(y)f(\sigma(y)x) - 2f(x)g(y)| \leqslant \frac{2\delta + M(\delta+\alpha)}{|f(z_0)|}$$

for all $x, y \in S$, where g is the function defined in Lemma 5.1. Now, by using the same computation used in [14, Theorem 3.7 (iv)] we conclude that f, g are solutions of the variant of μ-Wilson's functional equation

$$f(xy) + \mu(y)f(\sigma(y)x) = 2f(x)g(y) \tag{5.58}$$

for all $x, y \in S$. By replacing x by z_0 in (5.58) we get $f(z_0y)+\mu(y)f(\sigma(y)z_0)=2g(y)f(z_0)$. Since $\mu(y)f(\sigma(y)) = f(y)$ and $\mu(y\sigma(y)) = 1$, then we get

$$f(z_0y) + \mu(y)f(\sigma(y)z_0) = f(yz_0) + \mu(z_0)f(\sigma(z_0)y) = 2f(y)g(z_0).$$

Then we have $f(y)g(z_0) = g(y)f(z_0)$. So, $g = \frac{g(z_0)}{f(z_0)}f$.

For all $x, y \in S$ we have

$$f(xyz_0) + \mu(y)f(\sigma(y)xz_0) = [f(xz_0y) + \mu(y)f(\sigma(y)xz_0)] = 2f(xz_0)g(y) = 2\beta f(x)f(y), \quad (5.59)$$

where $\beta = \frac{(g(z_0))^2}{f(z_0)}$. Substituting this into (5.1) we obtain $|2(\beta - 1)f(y)f(x)| \leqslant \delta$ for all $x, y \in S$. Since f is assumed to be unbounded, then we deduce that $\beta = 1$ and then from (5.59) we deduce that f is a solution of (1.5). This completes the proof. □

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