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On the oscillation for nth-order nonlinear neutral delay dynamic equations on time scales

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Abstract

In this paper, we investigate the solution's oscillation of nth-order nonlinear dynamic equation

$$[a_n(t)((a_{n-1}(t)(\cdots(a_1(t)(x(t)-p(t)x(\tau(t)))^\Delta)^{\alpha_1})^\Delta\cdots)^\Delta)^{\alpha_n}]^\Delta+f(t,x(\delta(t)))=0$$

on a time scale $\mathbb{T}$ with $n\geqslant 2$. We give some conditions for the oscillation of the above equation.

Keywords: Oscillation, dynamic equation, time scale.

2010 MSC: 34N05, 34K11, 39A21.

1. Introduction

In this paper, we study the nth-order nonlinear neutral delay dynamic equation

$$[a_n(t)((a_{n-1}(t)(\cdots(a_1(t)(x(t)-p(t)x(\tau(t)))^\Delta)^{\alpha_1})^\Delta\cdots)^\Delta)^{\alpha_n}]^\Delta+f(t,x(\delta(t)))=0 \tag{1.1}$$

on a time scale $\mathbb{T}$ satisfying $\inf\mathbb{T}=t_0$ and $\sup\mathbb{T}=\infty$, where $n\geqslant 2$ and $\alpha_k(1\leqslant k\leqslant n)$ are quotients of odd positive integers. Throughout this paper, we assume the following conditions are satisfied:

(H1) $a_k(t)\in C_{rd}(\mathbb{T},(0,\infty)), p(t)\in C_{rd}(\mathbb{T},\mathbb{R}), \lim_{t\to\infty}p(t)=p_0$, where $|p_0|<1$, and

$$\int_{t_0}^{\infty}\left(\frac{1}{a_k(t)}\right)^{\frac{1}{\alpha_k}}\Delta t=\infty\ (1\leqslant k\leqslant n);$$

(H2) $\tau,\delta\in C_{rd}(\mathbb{T},\mathbb{T}), \tau(t)\leqslant t$, and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$;

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(H3) $f \in C(\mathbb{T}\times\mathbb{R},\mathbb{R}), uf(t,u) > 0$ and there exists $q(t) \in C_{rd}(\mathbb{T},(0,\infty))$ such that $|f(t,u)| \geqslant q(t)|u|$ for all $u \neq 0$ and $t \in \mathbb{T}$.

We write

$$S_k(t,x(t)) = \begin{cases} x(t) - p(t)x(\tau(t)), & \text{if } k = 0, \\ a_k(t)\big(S^{\Delta}_{k-1}(t,x(t))\big)^{\alpha_k}, & \text{if } 1 \leqslant k \leqslant n. \end{cases}$$

Then (1.1) reduces to the equation

$$S^{\Delta}_n(t,x(t)) + f(t,x(\delta(t))) = 0. \tag{1.2}$$

In last few decades, there are lots of research concerning oscillation of second and third order delay dynamic equations and we can find in [3–7, 10]. Recently, the number of papers, such as [1, 2, 8, 9, 11], are concerned with the oscillation of higher order dynamic equations. Sun et al. [8] studied the oscillation for higher order dynamic equation

$$\big\{a_n(t)\big[\big(a_{n-1}(t)\big(\cdots(a_1(t)x^{\Delta}(t))^{\Delta}\cdots\big)^{\Delta}\big)^{\Delta}\big]^{\alpha}\big\}^{\Delta} + p(t)x^{\beta}(t) = 0.$$

Zhang and Wang [11] considered the asymptotic and oscillation of nth-order nonlinear dynamic equation

$$(r(t)\Phi_{\gamma}(x^{\Delta^{n-1}}(t)))^{\Delta} + \sum_{i=0}^{k} q_i(t)\Phi_{\alpha_i}(x(\delta_i(t))) = 0.$$

The purpose of this paper is to extend the existing results to more general nth-order dynamic equations, and give some oscillation criteria.

2. Auxiliary results

Lemma 2.1. *Let $x(t)$ be an eventually positive solution of (1.1). If there exists a constant $l \geqslant 0$ such that $\lim_{t\to\infty} S_0(t) = l$, then $\lim_{t\to\infty} x(t) = \frac{l}{1-p_0}$.*

Proof. Suppose that $x(t)$ is an eventually positive solution of (1.1). According to (H1) and (H2), there exist $T_1 \in [t_0,\infty)_{\mathbb{T}}$ and $|p_0| < p_1 < 1$ such that $x(t), x(\tau(t)) > 0$ and $|p(t)| \leqslant p_1$ for $t \in [T_1,\infty)_{\mathbb{T}}$. We claim that $x(t)$ is bounded on $[t_0,\infty)_{\mathbb{T}}$. If not, then there exists $\{t_n\}_{n=1}^{\infty} \subset [T_1,\infty)_{\mathbb{T}}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$x(t_n) = \max_{t_0\leqslant t\leqslant t_n} x(t), \quad \lim_{t\to\infty} x(t_n) = \infty.$$

Noting that $\tau(t) \leqslant t$, so $x(\tau(t_n)) \leqslant x(t_n)$. Then we have

$$S_0(t_n) = x(t_n) - p(t_n)x(\tau(t_n)) \geqslant (1-p_1)x(t_n) \to \infty$$

as $n \to \infty$, which contradicts the fact $\lim_{t\to\infty} S_0(t) = l$. Therefore, $x(t)$ is bounded. Let $\limsup_{t\to\infty} x(t) = x_1$ and $\liminf_{t\to\infty} x(t) = x_2$. If $0 \leqslant p_0 < 1$, we have

$$x_1 - p_0x_2 \leqslant l \leqslant x_2 - p_0x_1,$$

which implies that $x_1 \leqslant x_2$. If $-1 \leqslant p_0 < 0$, we have

$$x_1 - p_0x_1 \leqslant l \leqslant x_2 - p_0x_2,$$

which also implies that $x_1 \leqslant x_2$. Therefore, $\lim_{t\to\infty} x(t)$ exists and $\lim_{t\to\infty} x(t) = \frac{l}{1-p_0}$. □

Lemma 2.2. *If $S^{\Delta}_n(t,x(t)) < 0$ and $x(t) > 0$ for $t \geqslant t_0$, then there exists an integer $m \in [0,n]$ with $m+n$ even such that*

$$(-1)^{m+i}S_i(t,x(t)) > 0 \text{ for } t \geqslant t_0 \text{ and } m \leqslant i \leqslant n, \tag{2.1}$$

and if $m > 1$, then there exists $T \geqslant t_0$ such that

$$S_i(t,x(t)) > 0 \text{ for } t \geqslant T \text{ and } 1 \leqslant i \leqslant m-1. \tag{2.2}$$

Proof. First, we show that $S_n(t,x(t)) > 0$ for $t \geqslant t_0$. If not, then there exists some $T_1 \geqslant t_0$ such that $S_n(T_1,x(T_1)) < 0$. Noting that $S_n^{\Delta}(t,x(t)) < 0$, it follows $S_n(t,x(t))$ is strictly decreasing on $[t_0,\infty)_{\mathbb{T}}$. Therefore, $S_n(t,x(t)) < S_n(T_1,x(T_1)) < 0$ for $t \geqslant T_1$. Then, from (H1), we have

$$S_{n-1}(t,x(t)) = S_{n-1}(T_1,x(T_1)) + \int_{T_1}^{t}\left(\frac{S_n(s,x(s))}{a_n(s)}\right)^{\frac{1}{\alpha_n}}\Delta s \leqslant S_{n-1}(T_1,x(T_1)) + \int_{T_1}^{t}\left(\frac{S_n(T_1,x(T_1))}{a_n(s)}\right)^{\frac{1}{\alpha_n}}\Delta s.$$

Thus $\lim_{t\to\infty} S_{n-1}(t,x(t)) = -\infty$. By induction we can obtain $\lim_{t\to\infty} S_0(t) = -\infty$, which is a contradiction to $S_0(t) > 0$. Thus $S_n(t,x(t)) > 0$. Then we have the following two cases:

(i) $S_i(t) > 0$ for any $0 \leqslant i \leqslant n-1$;
(ii) $S_j(t) < 0$ for some $0 < j < n-1$.

From case (ii), there exists a smallest integer $m \in [0,n]$ with $m+n$ even such that $(-1)^{m+i}S_i(t,x(t)) > 0$ for $t \geqslant t_0$ and $m \leqslant i \leqslant n$.

If $m > 1$, then $S_{m-1}^{\Delta}(t,x(t)) = \left(\frac{S_m(t,x(t))}{a_m(t)}\right)^{\frac{1}{\alpha_m}} > 0$ for $t \geqslant t_0$. So we have two cases: either $S_{m-1}(t,x(t)) \geqslant S_{m-1}(t_1,x(t_1)) > 0$, $t \geqslant t_1$ for some $t_1 \geqslant t_0$ or $S_{m-1}(t,x(t)) < 0$ for all $t \geqslant t_0$. For the first case, similar to the case of $S_n(t,x(t)) < S_n(T_1,x(T_1)) < 0$ for $t \geqslant T_1$, we can show that $\lim_{t\to\infty} S_i(t,x(t)) = \infty$ for $0 \leqslant i \leqslant m-1$. For the second case, using arguments similar to the case of $S_n(t,x(t)) < 0$, we can show that $S_{m-2}(t,x(t)) > 0$ for $t \geqslant t_0$, which contradicts to the definition of m. The proof is completed. □

Lemma 2.3. *Let $x(t)$ be an eventually positive solution of (1.1). If*

$$\int_{t_0}^{\infty} A_{n-1}(s)\Delta s = \infty, \tag{2.3}$$

where

$$A_i(t) = \begin{cases} \left[\frac{1}{a_n(t)}\int_t^{\infty} q(s)\Delta s\right]^{\frac{1}{\alpha_n}}, & \text{if } i = n, \\ \left[\frac{1}{a_i(t)}\int_t^{\infty} A_{i+1}(t)\Delta s\right]^{\frac{1}{\alpha_i}}, & \text{if } 1 \leqslant i \leqslant n-1, \end{cases}$$

then there exists $T \in [t_0,\infty)_{\mathbb{T}}$ sufficiently large such that $S_n^{\Delta}(t,x(t)) < 0$ for $t \geqslant T$. Moreover,

(1) *the following statement holds when n is odd,*

$$S_j(t,x(t))) > 0, \quad j = 1,2,\ldots,n; \tag{2.4}$$

(2) *either (2.4) holds or*

$$(-1)^j S_j(t,x(t))) > 0, \quad j = 1,2,\ldots,n,$$

and $\lim_{t\to\infty} x(t) = 0$, when n is even.

Proof. According to (H1) and (H2), there exist $T \in [t_0,\infty)_{\mathbb{T}}$ and $|p_0| < p_1 < 1$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ and $|p(t)| \leqslant p_1$ for $t \in [T,\infty)_{\mathbb{T}}$. From (H3) and (1.2), we obtain

$$S_n^{\Delta}(t,x(t)) \leqslant -q(t)x(\delta(t)) < 0. \tag{2.5}$$

When n is odd, by Lemma 2.2, m must be an odd number. By (2.1), we can get

$$S_0^{\Delta}(t) = \left(\frac{S_1(t,(x(t))}{a_1(t)}\right)^{\frac{1}{\alpha_1}} > 0.$$

Hence, $\lim_{t\to\infty} S_0(t)$ exists and is positive, or $\lim_{t\to\infty} S_0(t) = \infty$. It follows that there are $T_1 \geqslant T$ and a positive real number b such that $S_0(t) \geqslant b$ for $t \geqslant T_1$. We claim that $m = n$. If not, then, by Lemma 2.1, we have

$$S_{n-1}(t,x(t)) < 0 \quad \text{and} \quad S_{n-2}(t,x(t)) > 0 \text{ for } t \geqslant T. \tag{2.6}$$

Integrating both sides of (2.5) from t to ∞, we get

$$S_n(t,x(t)) \geqslant \int_t^\infty q(s)x(\delta(s))\Delta s,$$

which yields that

$$S^\Delta_{n-1}(t,x(t)) \geqslant \left[\frac{1}{a_n(t)}\int_T^\infty q(s)x(\delta(s))\Delta s\right]^{\frac{1}{\alpha_n}} =: \beta_n(t).$$

Integrating above from T to ∞, we have

$$-S_{n-1}(t,x(t)) \geqslant \int_t^\infty \beta_n(s)\Delta s,$$

which yields that

$$-S^\Delta_{n-2}(t,x(t)) \geqslant \left[\frac{1}{a_{n-1}(t)}\int_t^\infty \beta_n(s)\Delta s\right]^{\frac{1}{\alpha_{n-1}}} =: \beta_{n-1}(t).$$

Again, integrating above from t_0 to ∞, by Lemma 2.1, we obtain

$$\infty > S_{n-2}(t_0,x(t_0)) \geqslant \int_{t_0}^\infty \beta_{n-1}(s)\Delta s \geqslant \frac{b}{1-p_0}\int_{t_0}^\infty A_{n-1}(s)\Delta s,$$

which contradicts (2.1). Hence, $m = n$ and (2.4) holds.

When n is even, by Lemma 2.2, m must be an even integer. By (2.1) and (2.2), we have either $S^\Delta_0(t) > 0$ or $S^\Delta_0(t) < 0$. It means that $\lim_{t\to\infty} S_0(t) = l \geqslant 0$. We claim that $l \neq 0$ implies that $m = n$. Otherwise, (2.6) holds. By a similar arguments as above, we can reach a contradiction to (2.3). This completes the proof. □

Lemma 2.4. *Suppose that $x(t)$ is an eventually positive solution of (1.1) which satisfies (2.4) eventually. Then there exists $T \in [t_0,\infty)_{\mathbb{T}}$ such that, for $t \geqslant T$ and $0 \leqslant j \leqslant n$, we have*

$$S_j(t,x(t)) \geqslant S_n^{\prod_{k=j+1}^n \frac{1}{\alpha_k}}(t,x(t))B_{j+1}(t,T), \tag{2.7}$$

and

$$S^\Delta_0(t) \geqslant S_n^{\prod_{k=1}^n \frac{1}{\alpha_k}}(\sigma(t),x(\sigma(t)))\left(\frac{B_2(t,T)}{a_1(t)}\right)^{\frac{1}{\alpha_1}}, \tag{2.8}$$

and there exist $T_1 > T$ and a constant $c > 0$ such that

$$S_0(t) \leqslant cB_1(t,T) \;\; \textit{for} \;\; t \geqslant T_1, \tag{2.9}$$

where

$$B_j(t,T) = \begin{cases} \int_T^t \left(\frac{1}{a_n(s)}\right)^{\frac{1}{\alpha_n}}\Delta s, & \textit{if} \;\; j = n,\\ \int_T^t \left(\frac{B_{j+1}(s,T)}{a_j(s)}\right)^{\frac{1}{\alpha_j}}\Delta s, & \textit{if} \;\; 1 \leqslant j \leqslant n-1.\end{cases}$$

Proof. According to the hypothesis, there exists $T \in [t_0,\infty)_{\mathbb{T}}$ such that for any $t \geqslant T$ and $0 \leqslant j \leqslant n$, $S_j(t,x(t)) > 0$. So $S_n(t,x(t))$ is decreasing on $[T,\infty)_{\mathbb{T}}$. For $t \geqslant T$, we have

$$\begin{aligned} S_{n-1}(t,x(t)) &= S_{n-1}(T,x(T)) + \int_T^t \left(\frac{S_n(s,x(s))}{a_n(s)}\right)^{\frac{1}{\alpha_n}}\Delta s\\ &\geqslant S_n^{\frac{1}{\alpha_n}}(t,x(t))\int_T^t \left(\frac{1}{a_n(s)}\right)^{\frac{1}{\alpha_n}}\Delta s = S_n^{\frac{1}{\alpha_n}}(t,x(t))B_n(t,T),\end{aligned}$$

$$\begin{aligned} S_{n-2}(t,x(t)) &= S_{n-2}(T,x(T)) + \int_T^t \left(\frac{S_{n-1}(s,x(s))}{a_{n-1}(s)}\right)^{\frac{1}{\alpha_{n-1}}}\Delta s \\ &\geqslant \int_T^t \left(\frac{S_{n-1}(s,x(s))}{a_{n-1}(s)}\right)^{\frac{1}{\alpha_{n-1}}}\Delta s \\ &\geqslant \int_T^t \left(\frac{S_n^{\frac{1}{\alpha_n}}(s,x(s))B_n(s,T)}{a_{n-1}(s)}\right)^{\frac{1}{\alpha_{n-1}}}\Delta s \\ &\geqslant S_n^{\frac{1}{\alpha_n\alpha_{n-1}}}(t,x(t))\int_T^t \left(\frac{B_n(s,T)}{a_{n-1}(s)}\right)^{\frac{1}{\alpha_{n-1}}}\Delta s = S_n^{\frac{1}{\alpha_n\alpha_{n-1}}}(t,x(t))B_{n-1}(t,T). \end{aligned}$$

By induction, it is easy to see that

$$S_1(t,x(t)) \geqslant S_n^{\prod_{k=2}^n \frac{1}{\alpha_k}}(t,x(t))B_2(t,T), \quad S_0(t,x(t)) \geqslant S_n^{\prod_{k=1}^n \frac{1}{\alpha_k}}(t,x(t))B_1(t,T).$$

Then we have

$$S_0^\Delta(t) = \left(\frac{S_1(t,x(t))}{a_1(t)}\right)^{\frac{1}{\alpha_1}} \geqslant S_n^{\prod_{k=1}^n \frac{1}{\alpha_k}}(t,x(t))\left(\frac{B_2(t,T)}{a_1(t)}\right)^{\frac{1}{\alpha_1}}.$$

Since $S_n(t,x(t))$ is decreasing on $[T,\infty)_{\mathbb{T}}$,

$$S_0^\Delta(t) \geqslant S_n^{\prod_{k=1}^n \frac{1}{\alpha_k}}(\sigma(t),x(\sigma(t)))\left(\frac{B_2(t,T)}{a_1(t)}\right)^{\frac{1}{\alpha_1}}.$$

On the other hand, for $t > T$,

$$S_{n-1}(t,x(t)) = S_{n-1}(T,x(T)) + \int_T^t \left(\frac{S_n(s,x(s))}{a_n(s)}\right)^{\frac{1}{\alpha_n}}\Delta s \leqslant S_{n-1}(T,x(T)) + S_n^{\frac{1}{\alpha_n}}(T,x(T))B_n(t,T).$$

Thus, there exist $T_1 > T$ and $b_1 > 0$ such that

$$S_{n-1}(t,x(t)) \leqslant b_1B_n(t,T) \quad \text{for} \quad t \geqslant T_1.$$

Similarly, we have

$$S_{n-2}(t,x(t)) = S_{n-2}(T_1,x(T_1)) + \int_{T_1}^t \left(\frac{S_{n-1}(s,x(s))}{a_{n-1}(s)}\right)^{\frac{1}{\alpha_{n-1}}}\Delta s \leqslant S_{n-1}(T_1,x(T_1)) + b_1\int_T^t \left(\frac{B_n(s,T)}{a_{n-1}(s)}\right)^{\frac{1}{\alpha_{n-1}}}\Delta s.$$

Thus, there exists a constant $b_2 > 0$ such that

$$S_{n-2}(t,x(t)) \leqslant b_2B_{n-1}(t,T) \quad \text{for} \quad t \geqslant T_1.$$

By induction, it is easy to see that there exist $T_1 > T$ and $b_n > 0$ such that

$$S_0(t) \leqslant b_nB_1(t,T) \quad \text{for} \quad t \geqslant T_1.$$

This completes the proof. □

3. Main results

Theorem 3.1. *Suppose that (2.3) holds, $p_0 \in (0,1)$, $\delta(t) > t$, and $\prod_{k=1}^n \alpha_k \geqslant 1$. If there exists $z \in C_{rd}(\mathbb{T},(0,\infty))$ such that for all sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty}\int_T^t \left[z(s)q(s) - \frac{(z^\Delta(s))^2}{4Mz(s)\delta^\Delta(s)}\left(\frac{a_1(\delta(s))}{B_2(\delta(s),T)}\right)^{\frac{1}{\alpha_1}}\right]\Delta s = \infty, \tag{3.1}$$

where M is a positive constant, then

(1) *every solution of* (1.1) *is either oscillatory or tends to zero when* n *is even;*
(2) *every solution of* (1.1) *is oscillatory when* n *is odd.*

Proof. Assume that $x(t)$ is a non-oscillatory solution of (1.1). Then there is a $T \geqslant t_0$ sufficiently large such that $x(t), x(\tau(t)), x(\delta(t)) > 0$ and $p(t) > 0$ for $t \geqslant T$. From Lemma 2.3, we see that (2.4) holds when n is odd, and either (2.4) holds or $\lim_{t\to\infty} x(t) = 0$ when n is even.

Assume that n is odd. Define w by

$$w(t) = \frac{z(t)S_n(t, x(t))}{S_0(\delta(t))} \quad \text{for} \quad t \geqslant T. \tag{3.2}$$

Then $w(t) > 0$. Using the product rule, we have

$$w^{\Delta}(t) = (S_n(t, x(t)))^{\sigma}\left(\frac{z(t)}{S_0(\delta(t))}\right)^{\Delta} + (S_n(t, x(t)))^{\Delta}\frac{z(t)}{S_0(\delta(t))}.$$

By the definition of $S_0(t)$, we obtain $x(t) \geqslant S_0(t)$ for $t \geqslant T$. By the quotient rule and applying (2.5), we get

$$w^{\Delta}(t) \leqslant (S_n(t, x(t)))^{\sigma}\frac{z^{\Delta}(t)S_0(\delta(t)) - z(t)(S_0(\delta(t)))^{\Delta}}{S_0(\delta(t))S_0(\delta^{\sigma}(t))} - z(t)q(t)\frac{S_0(\delta(t))}{S_0(\delta(t))}.$$

From (3.2), it follows that

$$w^{\Delta}(t) \leqslant -z(t)q(t) + \frac{z^{\Delta}(t)}{z(\sigma(t))}w(\sigma(t)) - (S_n(t, x(t)))^{\sigma}\frac{z(t)S_0^{\Delta}(\delta(t))\delta^{\Delta}(t)}{S_0(\delta(t))S_0(\delta^{\sigma}(t))}. \tag{3.3}$$

Since $S_n(t, x(t))$ is decreasing on $[t_1, \infty)_{\mathbb{T}}$, there exists a constant $d > 0$ such that

$$(S_n(t, x(t)))^{\sigma} \leqslant S_n(t, x(t)) \leqslant d \quad \text{for} \quad t \geqslant T.$$

Applying (3.3) to (2.8) and noting that $\prod_{k=1}^{n} \alpha_k \geqslant 1$, we have

$$S_0^{\Delta}(t) \geqslant d^{(\prod_{k=1}^{n}\frac{1}{\alpha_k})-1} S_n(\sigma(t), x(\sigma(t)))\left(\frac{B_2(t,T)}{a_1(t)}\right)^{\frac{1}{\alpha_1}}. \tag{3.4}$$

Let $M = d^{(\prod_{k=1}^{n}\frac{1}{\alpha_k})-1}$. From (3.2), (3.3), (3.4), and noting that $S_0^{\Delta}(t) > 0$, we get

$$w^{\Delta}(t) \leqslant -z(t)q(t) + \frac{z^{\Delta}(t)}{z(\sigma(t))}w(\sigma(t)) - \frac{Mz(t)B_2^{\frac{1}{\alpha_1}}(\delta(t),T)\delta^{\Delta}(t)}{z^2(\sigma(t))a_1^{\frac{1}{\alpha_1}}(\delta(t))}w^2(\sigma(t)). \tag{3.5}$$

By completing the square for $w(\sigma(t))$ on the right-hand side of (3.5), we have

$$w^{\Delta}(t) \leqslant -z(t)q(t) + \frac{(z^{\Delta}(t))^2}{4Mz(t)\delta^{\Delta}(t)}\left(\frac{a_1(\delta(t))}{B_2(\delta(t),T)}\right)^{\frac{1}{\alpha_1}}.$$

Integrating the above inequality from T to t for $t \geqslant T$, we get

$$\int_T^t \left[z(s)q(s) - \frac{(z^{\Delta}(s))^2}{4Mz(s)\delta^{\Delta}(s)}\left(\frac{a_1(\delta(s))}{B_2(\delta(s),T)}\right)^{\frac{1}{\alpha_1}}\right]\Delta s \leqslant w(T) - w(t) < w(T).$$

Taking the lim sup on both sides of the above inequality as $t \to \infty$, we obtain a contradiction to (3.1).

In similar fashion, we can show that either every solution of (1.1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$, when n is even. This completes the proof. □

Theorem 3.2. *Suppose that* (2.3) *holds,* $p_0 \in (0,1)$, $\delta(t) > t$ *and* $\prod_{k=1}^{n} \alpha_k \geqslant 1$. *If there exist positive functions* $H, C \in C_{rd}(\mathbb{D}, (0,\infty))$, *where* $\mathbb{D} = \{(t,s) \in \mathbb{T}^2 : t \geqslant s \geqslant t_0\}$, *such that*

$$H(t,t) = 0, H(t,s) > 0 \ \text{and} \ H_s^{\Delta}(t,s) \leqslant 0 \ \text{for} \ t > s \geqslant t_0, \quad C(t,s) = H_s^{\Delta}(t,s) + H(t,s)\frac{z^{\Delta}(s)}{z^{\sigma}(s)},$$

and for sufficiently large T,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)z(s)q(s) - \frac{C^2(t,s)z^2(\sigma(s))a_1^{\frac{1}{\alpha_1}}(\delta(s)}{4Mz(s)\delta^{\Delta}(s)B_2^{\frac{1}{\alpha_1}}(\delta(s),T)H(t,s)} \right] \Delta s = \infty, \tag{3.6}$$

where z, M *are defined as in Theorem* 3.1. *Then*

(1) *every solution of* (1.1) *is either oscillatory or tends to zero when* n *is even;*
(2) *every solution of* (1.1) *is oscillatory when* n *is odd.*

Proof. Assume that $x(t)$ is a non-oscillatory solution of (1.1). Then there is a $T \geqslant t_0$ sufficiently large such that $x(t), x(\tau(t)), x(\delta(t)) > 0$ and $p(t) > 0$ for $t \geqslant T$. From Lemma 2.3, we see that (2.4) holds when n is odd, and either (2.4) holds or $\lim_{t\to\infty} x(t) = 0$ when n is even. Assume that n is odd. We define $w(t)$ by (3.2) and proceed as the proof of Theorem 3.1 to get (3.5). Multiplying (3.5) by $H(t,s)$ and integrating from T to t, we have

$$\begin{aligned}\int_T^t H(t,s)z(s)q(s)\Delta s \leqslant &-\int_T^t H(t,s)w^{\Delta}(s)\Delta s + \int_T^t H(t,s)\frac{z^{\Delta}(s)}{z(\sigma(s))}w(\sigma(s))\Delta s \\ &- \int_T^t H(t,s)\frac{Mz(s)B_2^{\frac{1}{\alpha_1}}(\delta(s),T)\delta^{\Delta}(s)}{z^2(\sigma(s))a_1^{\frac{1}{\alpha_1}}(\delta(s))}w^2(\sigma(s))\Delta s.\end{aligned}$$

By integration by parts we obtain

$$-\int_T^t H(t,s)w^{\Delta}(s)\Delta s = H(t,T)w(T) + \int_T^t H_s^{\Delta}(t,s)w(\sigma(s))\Delta s.$$

It follows that

$$\begin{aligned}\int_T^t H(t,s)z(s)q(s)\Delta s &\leqslant H(t,T)w(T) + \int_T^t \left[H_s^{\Delta}(t,s) + H(t,s)\frac{z^{\Delta}(s)}{z(\sigma(s))}\right]w(\sigma(s))\Delta s \\ &\quad - \int_T^t H(t,s)\frac{Mz(s)B_2^{\frac{1}{\alpha_1}}(\delta(s),T)\delta^{\Delta}(s)}{z^2(\sigma(s))a_1^{\frac{1}{\alpha_1}}(\delta(s))}w^2(\sigma(s))\Delta s \\ &= H(t,T)w(T) + \int_T^t C(t,s)w(\sigma(s))\Delta s - \int_T^t H(t,s)\frac{Mz(s)B_2^{\frac{1}{\alpha_1}}(\delta(s),T)\delta^{\Delta}(s)}{z^2(\sigma(s))a_1^{\frac{1}{\alpha_1}}(\delta(s))}w^2(\sigma(s))\Delta s.\end{aligned}$$

By completing the square for $w(\sigma(t))$ on the right-hand side, we get

$$\int_T^t H(t,s)z(s)q(s)\Delta s \leqslant H(t,T)w(T) + \int_T^t \left[\frac{C^2(t,s)z^2(\sigma(s))}{4Mz(s)\delta^{\Delta}(s)H(t,s)}\left(\frac{a_1(\delta(s))}{B_2(\delta(s),T)}\right)^{\frac{1}{\alpha_1}}\right]\Delta s,$$

and this implies that

$$\frac{1}{H(t,T)}\int_T^t \left[H(t,s)z(s)q(s) - \frac{C^2(t,s)z^2(\sigma(s))}{4Mz(s)\delta^{\Delta}(s)H(t,s)}\left(\frac{a_1(\delta(s))}{B_2(\delta(s),T)}\right)^{\frac{1}{\alpha_1}}\right]\Delta s \leqslant w(T),$$

which contradicts (3.6).

In similar fashion, we can show that either every solution of (1.1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$ when n is even. This completes the proof. □

Theorem 3.3. *Suppose that (2.3) holds and $p_0 \in (0,1)$, $\delta(t) > t$. If for all sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$, there exist positive constants d_1, d_2 such that*

$$\limsup_{t\to\infty} B_1^{\prod_{k=1}^n \alpha_k}(\delta(t),T)\gamma(\delta(t),T,d_1,d_2)\int_t^\infty q(s)\Delta s > 1, \tag{3.7}$$

where

$$\gamma(\delta(t),T,d_1,d_2) = \begin{cases} 1, & \text{if } \prod_{k=1}^n \alpha_k = 1, \\ d_1, & \text{if } \prod_{k=1}^n \alpha_k < 1, \\ d_2 B_1^{1-\prod_{k=1}^n \alpha_k}(\delta(t),T), & \text{if } \prod_{k=1}^n \alpha_k > 1. \end{cases}$$

Then

(1) *every solution of (1.1) is either oscillatory or tends to zero when n is even;*
(2) *every solution of (1.1) is oscillatory when n is odd.*

Proof. Assume that $x(t)$ is a non-oscillatory solution of (1.1). Then, without loss of generality, there is a $T \geqslant t_0$ sufficiently large such that $x(t), x(\tau(t)), x(\delta(t)) > 0$ for $t \geqslant T$. From Lemma 2.3, we see that (2.4) holds when n is odd, and either (2.4) holds or $\lim_{t\to\infty} x(t) = 0$ when n is even.

Assume that n is odd. From (2.5) and (2.7), we get for $t > T$,

$$\int_t^\infty q(s)S_0(\delta(s))\Delta s \leqslant S_n(t,x(t)) \leqslant \left[\frac{S_0(t)}{B_1(t,T)}\right]^{\prod_{k=1}^n \alpha_k}.$$

Noting that $S_0^\Delta(t) > 0$ and $\delta(t) > t$, we obtain

$$S_0(\delta(t))\int_t^\infty q(s)\Delta s \leqslant S_n(t,x(t)) \leqslant \left[\frac{S_0(\delta(t))}{B_1(t,T)}\right]^{\prod_{k=1}^n \alpha_k}.$$

Thus

$$B_1^{\prod_{k=1}^n \alpha_k}(t,T)S_0^{1-\prod_{k=1}^n \alpha_k}(\delta(t))\int_t^\infty q(s)\Delta s \leqslant 1.$$

The rest of the proof is separated into three cases:

Case 1. If $\prod_{k=1}^n \alpha_k = 1$, then

$$S_0^{1-\prod_{k=1}^n \alpha_k}(\delta(t)) = 1 \quad \text{for} \quad t \geqslant T. \tag{3.8}$$

Case 2. If $\prod_{k=1}^n \alpha_k < 1$, then

$$S_0(\delta(t)) \geqslant S_0(\delta(T)) \quad \text{for} \quad t \geqslant T. \tag{3.9}$$

Thus

$$S_0^{1-\prod_{k=1}^n \alpha_k}(\delta(t)) \geqslant d_1 S_0^{1-\prod_{k=1}^n \alpha_k}(\delta(T)). \tag{3.10}$$

Case 3. If $\prod_{k=1}^n \alpha_k > 1$, then from (2.9), there exists a $T_1 > T$ and a constant c such that

$$S_0(\delta(t)) \leqslant cB_1(\delta(t),T) \quad \text{for} \quad t \geqslant T_1.$$

Thus

$$S_0^{1-\prod_{k=1}^n \alpha_k}(\delta(t)) \geqslant c^{1-\prod_{k=1}^n \alpha_k} B_1^{1-\prod_{k=1}^n \alpha_k}(\delta(t),T). \tag{3.11}$$

Let $d_2 = c^{1-\prod_{k=1}^n \alpha_k}$, we have

$$S_0^{1-\prod_{k=1}^n \alpha_k}(\delta(t)) \geqslant d_2 B_1^{1-\prod_{k=1}^n \alpha_k}(\delta(t),T). \tag{3.12}$$

According to (3.8)-(3.12), we obtain that for $t \geqslant T_1$,

$$B_1^{\prod_{k=1}^n \alpha_k}(\delta(t),T)\gamma(\delta(t),T,d_1,d_2)\int_t^\infty q(s)\Delta s \leqslant 1,$$

which is a contradiction to (3.7).

In similar fashion, we can show that either every solution of (1.1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$ when n is even. The proof is completed. □

4. Examples

Example 4.1. Consider the equation

$$\left[\frac{1}{t}((\frac{1}{t}(\cdots(\frac{1}{t}(x(t)-\frac{1}{2}x(\tau(t)))^{\Delta})^{n})^{\Delta}\cdots)^{\Delta})^{\frac{1}{n}}\right]^{\Delta}+t^{n}x(t^{n})=0, \tag{4.1}$$

where n is odd and $n \geqslant 2$, $\mathbb{T}=[1,\infty)$. Here we have $a_k(t)=\frac{1}{t}$ $(1\leqslant k\leqslant n)$, $\alpha_1=n$, $\alpha_k=1$ $(2\leqslant k\leqslant n-1)$, $\alpha_n=\frac{1}{n}$, $p(t)=\frac{1}{2}$, and $q(t)=\delta(t)=t^n$. Clearly,

$$\begin{aligned}
&\int_{t_0}^{\infty}\left(\frac{1}{a_1(t)}\right)^{\frac{1}{\alpha_1}}\Delta t=\int_1^{\infty}t^{\frac{1}{n}}\Delta t=\infty,\\
&\int_{t_0}^{\infty}\left(\frac{1}{a_n(t)}\right)^{\frac{1}{\alpha_n}}\Delta t=\int_1^{\infty}t^{n}\Delta t=\infty,\\
&\int_{t_0}^{\infty}\left(\frac{1}{a_k(t)}\right)^{\frac{1}{\alpha_k}}\Delta t=\int_1^{\infty}t\Delta t=\infty\ (2\leqslant k\leqslant n-1),\\
&A_n(t)=\left[\frac{1}{a_n(t)}\int_t^{\infty}q(s)\Delta s\right]^{\frac{1}{\alpha_n}}=\left[t\int_t^{\infty}s^n\Delta s\right]^{n}=t^n\left[\int_t^{\infty}s^n\Delta s\right]^{n}=\infty,\\
&A_{n-1}(t)=\left[\frac{1}{a_{n-1}(t)}\int_t^{\infty}A_n(s)\Delta s\right]^{\frac{1}{\alpha_{n-1}}}=t\int_t^{\infty}A_n(s)\Delta s=\infty,\\
&\int_{t_0}^{\infty}A_{n-1}(s)\Delta s=\int_1^{\infty}A_{n-1}(s)\Delta s=\infty.
\end{aligned}$$

Let $z(t)=1$, we see that for all sufficiently large $T\in[t_0,\infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty}\int_T^t\left[z(s)q(s)-\frac{(z^{\Delta}(s))^2}{4Mz(s)\delta^{\Delta}(s)}\left(\frac{a_1(\delta(s))}{B_2(\delta(s),T)}\right)^{\frac{1}{\alpha_1}}\right]\Delta s=\limsup_{t\to\infty}\int_T^t s^n\Delta s=\infty.$$

Hence the conditions of Theorem 3.1 are satisfied. By Theorem 3.1, every solution $x(t)$ of (4.1) is oscillatory.

Example 4.2. Consider the equation

$$\left[\frac{1}{t}((\frac{1}{t}(\cdots(\frac{1}{t}(x(t)-\frac{1}{3}x(\tau(t)))^{\Delta})^{n+1})^{\Delta}\cdots)^{\Delta})^{\frac{1}{n-1}}\right]^{\Delta}+t^{n}x(t^{n})=0, \tag{4.2}$$

where n is even and $n\geqslant 2$, $\mathbb{T}=[1,\infty)$. Here we have $a_k(t)=\frac{1}{t}$ $(1\leqslant k\leqslant n)$, $\alpha_1=n+1$, $\alpha_k=1$ $(2\leqslant k\leqslant n-1)$, $\alpha_n=\frac{1}{n-1}$, $p(t)=\frac{1}{3}$, and $q(t)=\delta(t)=t^n$. Clearly,

$$\begin{aligned}
&\int_{t_0}^{\infty}\left(\frac{1}{a_1(t)}\right)^{\frac{1}{\alpha_1}}\Delta t=\int_1^{\infty}t^{\frac{1}{n+1}}\Delta t=\infty,\\
&\int_{t_0}^{\infty}\left(\frac{1}{a_n(t)}\right)^{\frac{1}{\alpha_n}}\Delta t=\int_1^{\infty}t^{n-1}\Delta t=\infty,\\
&\int_{t_0}^{\infty}\left(\frac{1}{a_k(t)}\right)^{\frac{1}{\alpha_k}}\Delta t=\int_1^{\infty}t\Delta t=\infty\ (2\leqslant k\leqslant n-1),\\
&A_n(t)=\left[\frac{1}{a_n(t)}\int_t^{\infty}q(s)\Delta s\right]^{\frac{1}{\alpha_n}}=\left[t\int_t^{\infty}s^n\Delta s\right]^{n-1}=t^{n-1}\left(\int_t^{\infty}s^n\Delta s\right)^{n-1}=\infty,\\
&A_{n-1}(t)=\left[\frac{1}{a_{n-1}(t)}\int_t^{\infty}A_n(s)\Delta s\right]^{\frac{1}{\alpha_{n-1}}}=t\int_t^{\infty}A_n(s)\Delta s=\infty,\\
&\int_{t_0}^{\infty}A_{n-1}(s)\Delta s=\int_1^{\infty}A_{n-1}(s)\Delta s=\infty.
\end{aligned}$$

Let $z(t) = 1$, we see that for all sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty} \int_T^t \left[z(s)q(s) - \frac{(z^{\Delta}(s))^2}{4Mz(s)\delta^{\Delta}(s)} \left(\frac{a_1(\delta(s))}{B_2(\delta(s), T)} \right)^{\frac{1}{\alpha_1}} \right] \Delta s = \limsup_{t\to\infty} \int_T^t s^n \Delta s = \infty.$$

Hence the conditions of Theorem 3.1 are satisfied.

By Theorem 3.1, every solution $x(t)$ of (4.2) is either oscillatory or tends to zero.

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