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Ulam-Hyers stability of fractional impulsive differential equations

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Abstract

In this paper, we first prove the existence and uniqueness for a fractional differential equation with time delay and finite impulses on a compact interval. Secondly, Ulam-Hyers stability of the equation is established by Picard operator and abstract Gronwall's inequality.

Keywords: Ulam-Hyers stability, fractional order impulsive equation, delay differential equation.

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1. Introduction and Preliminaries

In the past decades, the fractional order differential systems have been extensively studied due to its wide applications to science and engineering. The Ulam-Hyers stability of fractional differential equations has become one of most active areas, and has attracted many researchers, see [2, 3, 5, 6, 10, 11, 13–17]. For the stability theory of impulsive dynamical systems and its applications, Wang et al. [9] considered Ulam type stability of impulsive ordinary differential equation. And many impulsive differential equations are also studied by mathematicians, see [8, 12] and the references therein. In [1], S. Abbas and M. Benchohra reported the Ulam stability result for partial fractional differential equations with not instantaneous impulses. However, there are few results about the Ulam stability of impulsive fractional equation with finite delay.

Recently, I. A. Rus [7] proposed a unified framework for studying the Ulam-Hyers stability problems, using Picard and weakly Picard operators. In the present paper, we generalize the results of [18] to fractional impulsive delay differential equations. Motivated by [4, 7, 14], we will investigate existence, uniqueness, Ulam-Hyers stability results for some problems associated with impulsive delay differential equations

$$\begin{cases} {}^{C}D_t^{\alpha} z(t) = F(t, z(t), z(g(t))), & t \in J = [0, t_f] \setminus \{t_1, t_2, \cdots, t_m\}, \\ z(t) = h(t), & t \in [-\lambda, 0], \\ \triangle z(t_k) = z(t_k^+) - z(t_k^-) = I_k(z(t_k^-)), & k = 1, 2, \cdots, m, \end{cases} \tag{1.1}$$

where $\lambda > 0, t_f > 0, F : [0, t_f] \times \mathbb{R}^2 \to \mathbb{R}, I_k : \mathbb{R} \to \mathbb{R}$, and $h : [-\lambda, 0] \to \mathbb{R}$ are continuous. $z(t_k^+) =$

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$\lim_{\tau\to 0^+} z(t_k+\tau)$ and $z(t_k^-) = \lim_{\tau\to 0^+} z(t_k-\tau)$ are, respectively, the right and left side limits of $z(t)$ at t_k, where t_k satisfy $0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = t_f < +\infty$. Moreover, $g : [0, t_f] \to [-\lambda, t_f]$ satisfies $g(t) \leqslant t$.

Let $PC^1([-\lambda, t_f], \mathbb{R})$ denotes the Banach space of all piecewise right continuous derivative functions from $[-\lambda, t_f]$ into $\mathbb{R}$ which have left continuous derivative on $[-\lambda, t_f]$ with the norm

$$\|x\| = \sup\{|x(t)| : t \in [-\lambda, t_f]\}.$$

Definition 1.1. Let (X, d) be a metric space. An $\Lambda : X \to X$ is a Picard operator if there exists $x^* \in X$ such that (i) $F_\Lambda = x^*$ where $F_\Lambda = \{x \in X : \Lambda(x) = x\}$ is the fixed point set of Λ; (ii) the sequence $(\Lambda^n(x))_{n\in N}$ converges to x^* for all $x \in X$.

Lemma 1.2. *Let $(X, d, \leqslant)$ be an ordered metric space and let $\Lambda : X \to X$ be an increasing Picard operator with fixed point x^*. Then for any $x \in X, x \leqslant \Lambda(x)$ implies $x \leqslant x^*$ and $x \geqslant \Lambda(x)$ implies $x \geqslant x^*$.*

Lemma 1.3 ([7]). *If for $t \geqslant t_0 \geqslant 0$ we have*

$$x(t) \leqslant a(t) + \sum_{t_0<t_k<t} \xi_k x(t_k^-) + \int_{t_0}^{t} b(s)x(s)ds,$$

where $x, a, b \in PC([t_0, \infty), \mathbb{R})$, a is nondecreasing and $b(t), \xi_k > 0$. Then for $t \geqslant t_0$ the following inequality works:

$$x(t) \leqslant a(t) \prod_{t_0<t_k<t} (1 + \xi_k) \exp\left(\int_{t_0}^{t} b(s)ds\right).$$

For problem (1.1), we focus on the following inequalities

$$\begin{cases} \left|{}^C D_t^\alpha y(t) - F(t, y(t), y(g(t)))\right| \leqslant \epsilon, & t \in J, \\ \left|\triangle y(t_k) - I_k(y(t_k^-))\right| \leqslant \epsilon, & k = 1, 2, \cdots, m. \end{cases} \tag{1.2}$$

Remark 1.4. A function $y \in PC^1([0, t_f], \mathbb{R})$ satisfies (1.2) if and only if there is a function $f \in PC([-\lambda, t_f], \mathbb{R})$ and a sequence f_k such that $|f(t)| \leqslant \epsilon$ for all $t \in [-\lambda, t_f]$, $|f_k| \leqslant \epsilon$ for all $k = 1, 2, \cdots, m$, and

$$\begin{cases} {}^C D_t^\alpha y(t) = F(t, y(t), y(g(t))) + f(t), & t \in J, \\ \triangle y(t_k) = I_k(y(t_k^-)) + f_k, & k = 1, 2, \cdots, m. \end{cases}$$

Lemma 1.5. *Every $y \in PC^1([0, t_f], \mathbb{R})$ that satisfies (1.2) also comes out perfect on the following inequality:*

$$\left| y(t) - y(0) - \sum_{j=1}^{k} I(y(t_j^-)) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s), y(g(s)))ds \right| \leqslant \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + m\right)\epsilon$$

for $t \in (t_k, t_{k+1}] \subset [0, t_f]$.

Proof. If $y \in PC^1([0, t_f], \mathbb{R})$ satisfies (1.2), then by Remark 1.4 we have

$$\begin{cases} {}^C D_t^\alpha y(t) = F(t, y(t), y(g(t))) + f(t), & t \in J, \\ \triangle y(t_k) = I_k(y(t_k^-)) + f_k, & k = 1, 2, \cdots, m. \end{cases}$$

Then

$$\begin{aligned} y(t) =& y(0) + \sum_{j=1}^{k} I(y(t_j^-)) + \sum_{i=1}^{k} f_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s), y(g(s)))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, t \in (t_k, t_{k+1}]. \end{aligned}$$

From this, it follows that

$$\begin{aligned}
&\left|y(t) - y(0) - \sum_{j=1}^{k} I(y(t_j^-)) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, y(s), y(g(s)))ds\right| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s)\right| ds + \sum_{i=1}^{k} |f_i| \\
&\leqslant \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + k\right)\epsilon \leqslant \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + m\right)\epsilon, t \in (t_k, t_{k+1}]. \qquad \square
\end{aligned}$$

2. Main Results

In this section, we will prove existence, uniqueness and Ulam-Hyers stability for Eq. (1.1).

Theorem 2.1. *If*

(a) $F : [0, t_f] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous with the Lipschitz condition:*

$$|F(t, x_1, x_2) - F(t, y_1, y_2)| \leqslant \sum_{i=1}^{2} L\left|x_i - y_i\right|,$$

where $L > 0$*, for all* $t \in [0, t_f]$ *and* $x_i, y_i \in \mathbb{R}, i \in \{1, 2\}$*;*

(b) $I_k : \mathbb{R} \to \mathbb{R}$ *satisfies* $|I_k(x_1) - I_k(x_2)| \leqslant M_k|x_1 - x_2|, M_k > 0$*, for all* $k \in \{1, 2, \cdots, m\}$ *and* $x_1, x_2 \in \mathbb{R}$*;*

(c) $\sum_{j=1}^{k} M_j + \frac{2Lt_f^\alpha}{\Gamma(\alpha+1)} < 1$*, then the Eq.* (1.1) *has*

(i) *a unique solution on* $PC([-\lambda, t_f], \mathbb{R}_+) \cap PC^1([0, t_f], \mathbb{R}_+)$*;*

(ii) *Ulam-Hyers stability on* $[-\lambda, t_f]$*.*

Proof.

(i) Define an operator Λ by

$$(\Lambda z)(t) = \begin{cases} h(t), & t \in [-\lambda, 0], \\ h(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, z(s), z(g(s)))ds, & t \in [0, t_1], \\ h(0) + I_1(z(t_1^-)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, z(s), z(g(s)))ds, & t \in [t_1, t_2], \\ h(0) + \sum_{j=1}^{2} I_j(z(t_j^-)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, z(s), z(g(s)))ds, & t \in [t_2, t_3], \\ \vdots & \\ h(0) + \sum_{j=1}^{m} I_j(z(t_j^-)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, z(s), z(g(s)))ds, & t \in [t_m, t_f]. \end{cases} \tag{2.1}$$

We see that for any $z_1, z_2 \in PC([-\lambda, t_f], \mathbb{R})$ and for all $t \in [-\lambda, 0]$ we have $|(\Lambda z_1)(t) - (\Lambda z_2)(t)| = 0$. For $t \in [t_k, t_{k+1}]$, we consider

$$\begin{aligned}
|(\Lambda z_1)(t) - (\Lambda z_2)(t)| &\leqslant \sum_{j=1}^{k}\left|I_j(z_1(t_j^-)) - I_j(z_2(t_j^-))\right| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\quad |F(s, z_1(s), z_1(g(s))) - F(s, z_2(s), z_2(g(s)))|\, ds \\
&\leqslant \sum_{j=1}^{k} M_j\left|z_1(t_j^-) - z_2(t_j^-)\right| + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|z_1(s) - z_2(s)\right| ds \\
&\quad + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|z_1(g(s)) - z_2(g(s))\right| ds
\end{aligned}$$

$$\leqslant \left(\sum_{j=1}^{k} M_j + \frac{2L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \right) \sup_{t\in[-\lambda,t_f]} |z_1(t) - z_2(t)|$$
$$\leqslant \left(\sum_{j=1}^{k} M_j + \frac{2L t_f^{\alpha}}{\Gamma(\alpha+1)} \right) \|z_1(t) - z_2(t)\|.$$

Following (c), the operator is strictly contractive on $(t_k, t_{k+1}], k = 0, 1, 2, \cdots, m$, and hence Λ is a Picard operator on $PC([-\lambda, t_f], \mathbb{R}_+)$. From (2.1), it follows that the unique fixed point of this operator is in fact the unique solution of (1.1) on $PC([-\lambda, t_f], \mathbb{R}_+) \cap PC^1([0, t_f], \mathbb{R}_+)$.

(ii) Let $y \in PC([-\lambda, t_f], \mathbb{R}_+) \cap PC^1([0, t_f], \mathbb{R}_+)$ be a solution to (1.2). The unique solution

$$z \in PC([-\lambda, t_f], \mathbb{R}_+) \cap PC^1([0, t_f], \mathbb{R}_+)$$

of the differential equation

$$\begin{cases} {}^C D_t^{\alpha} z(t) = F(t, z(t), z(g(t))), & t \in J, \\ z(t) = y(t), & t \in [-\lambda, 0], \\ \triangle z(t_k) = z(t_k^+) - z(t_k^-) = I_k(z(t_k^-)), & k = 1, 2, \cdots, m. \end{cases}$$

is given by

$$z(t) = \begin{cases} y(t), & t \in [-\lambda, 0], \\ y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, z(s), z(g(s))) ds, & t \in [0, t_1], \\ y(0) + I_1(z(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, z(s), z(g(s))) ds, & t \in [t_1, t_2], \\ y(0) + \sum_{j=1}^{2} I_j(z(t_j^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, z(s), z(g(s))) ds, & t \in [t_2, t_3], \\ \vdots \\ y(0) + \sum_{j=1}^{m} I_j(z(t_j^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, z(s), z(g(s))) ds, & t \in [t_m, t_{m+1}]. \end{cases}$$

We observe that for all $t \in [-\lambda, 0]$, $z(t) = y(t)$, so we have $|y(t) - z(t)| = 0$. For $t \in (t_k, t_{k+1}]$, using Lemma 1.5, we have

$$\begin{aligned} |y(t) - z(t)| \leqslant & \left| y(t) - y(0) - \sum_{j=1}^{k} I_j(y(t_j^-)) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s), y(g(s)) ds \right| \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(s, y(s), y(g(s)) ds - F(s, z(s), z(g(s)) ds| \\ & + \sum_{j=1}^{k} \left| I_j(y(t_j^-)) - I_j(z(t_j^-)) \right| \\ \leqslant & \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + m \right) \epsilon + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y(s) - z(s)| \, ds \\ & + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y(g(s)) - z(g(s))| \, ds + \sum_{j=1}^{k} M_j \left| y(t_j^-) - z(t_j^-) \right|. \end{aligned}$$

Next, we show that the operator $\Psi : PC([-\lambda, t_f], \mathbb{R}_+) \to PC([-\lambda, t_f], \mathbb{R}_+)$ given below is an increasing

Picard operator on $PC([-\lambda, t_f], \mathbb{R}_+)$:

$$(\Psi u)(t) = \begin{cases} 0, & t \in [-\lambda, 0], \\ \frac{t^\alpha}{\Gamma(\alpha+1)}\epsilon \\ +\frac{L}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1}u(s)ds + \int_0^t (t-s)^{\alpha-1}u(g(s))\right) ds, & t \in [0, t_1], \\ (\frac{t^\alpha}{\Gamma(\alpha+1)}+1)\epsilon + I_1(u(t_1^-)) \\ +\frac{L}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1}u(s)ds + \int_0^t (t-s)^{\alpha-1}u(g(s))\right) ds, & t \in [t_1, t_2], \\ (\frac{t^\alpha}{\Gamma(\alpha+1)}+2)\epsilon + \sum_{j=1}^2 I_j(u(t_j^-)) \\ +\frac{L}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1}u(s)ds + \int_0^t (t-s)^{\alpha-1}u(g(s))\right) ds, & t \in [t_2, t_3], \\ \vdots \\ (\frac{t^\alpha}{\Gamma(\alpha+1)}+m)\epsilon + \sum_{j=1}^m I_j(u(t_j^-)) \\ +\frac{L}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1}u(s)ds + \int_0^t (t-s)^{\alpha-1}u(g(s))\right) ds, & t \in [t_m, t_{m+1}]. \end{cases} \tag{2.2}$$

For any $u_1, u_2 \in PC([-\lambda, 0], \mathbb{R}_+)$, $|(\Psi u_1)(t) - (\Psi u_2)(t)| = 0$ for all $t \in [-\lambda, 0]$. For $t \in (t_k, t_{k+1}]$ consider

$$\begin{aligned} |(\Psi u_1)(t) - (\Psi u_2)(t)| &\leqslant \sum_{j=1}^k M_j \left| u_1(t_j^-) - u_2(t_j^-) \right| + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad [|u_1(s) - u_2(s)|\, ds + |u_1(g(s)) - u_2(g(s))|]\, ds \\ &\leqslant \sum_{j=1}^m M_j \sup_{t \in [-\lambda, t_f]} |u_1(t) - u_2(t)| + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \left[\sup_{t \in [-\lambda, t_f]} |u_1(t) - u_2(t)| + \sup_{t \in [-\lambda, t_f]} |u_1(g(t)) - u_2(g(t))| \right] ds \\ &\leqslant \left(\sum_{j=1}^m M_j + \frac{2L(t_f)^\alpha}{\Gamma(\alpha+1)} \right) \|u_1 - u_2\|. \end{aligned}$$

Since $\left(\sum_{j=1}^m M_j + \frac{2L(t_f)^\alpha}{\Gamma(\alpha+1)}\right) < 1$, the operator is contractive on $PC([-\lambda, t_f], \mathbb{R}_+)$ for $t \in (t_k, t_k + 1]$, where $k = 0, 1, \cdots, m$. Applying Banach contraction principle to Ψ, we derive that Ψ is a Picard operator with a unique fixed point $u^* \in PC([-\lambda, t_f], \mathbb{R}_+)$, that is,

$$\begin{aligned} u^*(t) = \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + k \right) \epsilon + \sum_{j=1}^k M_j(u^*(t_j^-)) + \frac{L}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} u^*(s) ds \right. \\ \left. + \int_0^t (t-s)^{\alpha-1} u^*(g(s)) ds \right), t \in (t_k, t_{k+1}]. \end{aligned}$$

It remains to verify that the solution u^* is increasing. Indeed, for $0 \leqslant t_1 < t_2 \leqslant t_f$ and denote $p := \min_{s \in J} [u^*(s) + u^*(g(s))] \in \mathbb{R}_+$, we have

$$\begin{aligned} u^*(t_2) - u^*(t_1) &= \left(\frac{t_2^\alpha}{\Gamma(\alpha+1)} - \frac{t_1^\alpha}{\Gamma(\alpha+1)} \right) \epsilon \\ &\quad + \frac{L}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] [u^*(s) + u^*(g(s))]\, ds \\ &\quad + \frac{L}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} [u^*(s) + u^*(g(s))]\, ds \\ &\geqslant \left(\frac{t_2^\alpha}{\Gamma(\alpha+1)} - \frac{t_1^\alpha}{\Gamma(\alpha+1)} \right) \epsilon + \frac{pL}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{pL}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\
&=\left(\frac{t_2^{\alpha}}{\Gamma(\alpha+1)}-\frac{t_1^{\alpha}}{\Gamma(\alpha+1)}\right)\epsilon+\frac{pL}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha})\\
&>0.
\end{aligned}
$$

Then, we obtain that $u^*(t)$ is increasing. So, $u^*(g(t)) \leqslant u^*(t)$ due to $g(t) \leqslant t$ and

$$u^*(t) \leqslant \left(\frac{t_f^{\alpha}}{\Gamma(\alpha+1)}+k\right)\epsilon+\sum_{j=1}^{k}M_j(u^*(t_j^-))+\frac{2L}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u^*(s)ds.$$

Using Lemma 1.3, we get

$$u^*(t) \leqslant \left(\frac{t_f^{\alpha}}{\Gamma(\alpha+1)}+k\right)\epsilon\prod_{0<t_k<t}(1+M_k)exp\left(\frac{2Lt_f^{\alpha}}{\Gamma(\alpha+1)}\right).$$

If we set $u(t)=|y(t)-z(t)|$, for $t\in[-\lambda,t_f]$, then from (2.2), $u(t)\leqslant(\Psi u)(t)$ and applying Lemma 1.2 we obtain $u(t)\leqslant u^*(t)$, thus

$$|y(t)-z(t)| \leqslant \left(\frac{t_f^{\alpha}}{\Gamma(\alpha+1)}+m\right)\epsilon\prod_{0<t_m<t}(1+M_m)exp\left(\frac{2Lt_f^{\alpha}}{\Gamma(\alpha+1)}\right). \qquad \square$$

3. Example

Without loss of generality, we only consider the following impulsive fractional differential equation with time delay

$$\begin{cases} {}^{C}D_t^{\frac{1}{2}}z(t)=\frac{1}{20(t+10)^2}\left(\frac{|z(t)|}{(1+|z(t)|)}+\frac{|z(g(t))|}{(1+|z(g(t))|)}\right), & t\in[0,1]\setminus\{\frac{1}{2}\},\\ z(t)=0, & t\in[-1,0],\\ \triangle z(\frac{1}{2})=\frac{|z(\frac{1}{2}^-)|}{100(1+|z(\frac{1}{2}^-)|)}. \end{cases} \tag{3.1}$$

Here $\alpha=\frac{1}{2}, t_f=1, k=1, \lambda=1, t_1=\frac{1}{2}, h(t)=0$. Also

$$F(t,z(t),z(g(t)))=\frac{1}{20(t+10)^2}\left(\frac{|z(t)|}{(1+|z(t)|)}+\frac{|z(g(t))|}{(1+|z(g(t))|)}\right),$$

obviously, one has $|F(t,x_1,x_2)-F(t,y_1,y_2)|\leqslant\sum_{i=1}^{2}\frac{1}{2000}|x_i-y_i|$ with $L=\frac{1}{2000}$, $|I_1(x_1(\frac{1}{2}^-))-I_1(x_2(\frac{1}{2}^-))|\leqslant\frac{1}{100}|x_1(\frac{1}{2}^-)-x_2(\frac{1}{2}^-)|$ with $M_1=\frac{1}{100}$, and $M_1+\frac{2Lt_f^{\alpha}}{\Gamma(\alpha+1)}=\frac{1}{100}+\frac{1}{500\Gamma(\frac{1}{2})}<1$.

Thus all the assumptions in Theorem 2.1 are satisfied, Eq. (3.1) is Ulam-Hyers stable.

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