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Iterative methods for solving the split common fixed point problem of demicontractive mappings in Hilbert spaces

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Abstract

The split common fixed point problem was proposed in recent years which required to find a common fixed point of a family of mappings in one space whose image under a linear transformation is a common fixed point of another family of mappings in the image space. In this paper, we study two iterative algorithms for solving this split common fixed point problem for the class of demicontractive mappings in Hilbert spaces. Under mild assumptions on the parameters, we prove the convergence of both iterative algorithms. As a consequence, we obtain new convergence theorems for solving the split common fixed point problem for the class of directed mappings. We compare the performance of the proposed iterative algorithms with the existing iterative algorithms and conclude from the numerical experiments that our iterative algorithms converge faster than these existing iterative algorithms in terms of iteration numbers.

Keywords: Split common fixed point problem, demicontractive mappings, cyclic iteration method, simultaneous iteration method.

2010 MSC: 47H10, 90C25.

1. Introduction

Linear inverse problems often arise in many real-world applications problems, such as signal and image processing, medical image reconstruction and compressive sensing, etc.. It often reduced to solve a particular optimization problem. The split feasibility problem (SFP) is a general way to characterize many significant optimization problems in the above concerned. The SFP was first introduced by Censor and Elfving [6] in finite-dimensional Hilbert spaces. It has been studied extensively, see for example [3, 4, 7, 25, 26] and references therein. In 2009, Censor and Segal [8] introduced the split common fixed point problem as follows. Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Given operators $\{U_i\}_{i=1}^t : H_1 \to H_1$ and $\{T_j\}_{j=1}^r : H_2 \to H_2$, where $t, r \geqslant 1$ are nonnegative

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integers, respectively. The split common fixed point problem (say SCFPP for short) is formulated as finding a point x^* satisfying the property

$$x^* \in \bigcap_{i=1}^{t} \mathrm{Fix}(U_i), \text{ such that } Ax^* \in \bigcap_{j=1}^{r} \mathrm{Fix}(T_j). \tag{1.1}$$

Here, $\mathrm{Fix}(U)$ denotes the fixed point set of U, i.e., $\mathrm{Fix}(U) = \{x | x = Ux\}$. In particular, if $t = r = 1$, then the SCFPP (1.1) reduces to finding a point x^* with the property

$$x^* \in \mathrm{Fix}(U), \text{ such that } Ax^* \in \mathrm{Fix}(T). \tag{1.2}$$

The above problem (1.2) is usually called the two-operators SCFPP or two-sets SCFPP.

The SCFPP (1.1) is a generalization of the split feasibility problem. In fact, let $U = P_C, T = P_Q$ (here P_C denotes the metric projection onto the corresponding set C), then the two-sets SCFPP (1.2) becomes the SFP below,

$$\text{Finding a point } x^* \in C, \text{ such that } Ax^* \in Q,$$

where C and Q are nonempty closed convex sets in H_1 and H_2, respectively.

The appearance of the SCFPP (1.1) and (1.2) give researchers a more flexible way to deal with several type of optimization problems. A natural question was raised immediately: Under what condition on the mappings U_i and T_j, an iterative algorithm could be defined to solve them. Many interesting works have been done on this issue, see for example [5, 8–24, 27, 28]. In particular, Censor and Segal [8] proposed an iterative algorithm to solve the two-sets SCFPP (1.2) for the class of directed mappings (see Definition 2.1) in finite-dimensional Hilbert spaces,

$$x_{k+1} = U(x_k - \gamma A^t(I - T)Ax_k), k \geqslant 0, \tag{1.3}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the largest eigenvalue of A^tA (t stands for matrix transposition). By using the product space technique, they introduced a parallel iterative algorithm for the general SCFPP (1.1) which was defined by the iterative procedure,

$$x_{k+1} = x_k + \gamma \left[\sum_{i=1}^{t} \alpha_i (U_i(x_k) - x_k) + \sum_{j=1}^{r} \beta_j A^t (T_j - I)Ax_k \right], k \geqslant 0, \tag{1.4}$$

where $\{\alpha_i\}_{i=1}^t, \{\beta_j\}_{j=1}^r$ are nonnegative constants, $0 < \gamma < 2/L$ with $L = \sum_{i=1}^t \alpha_i + \lambda \sum_{j=1}^r \beta_j$. Under some mild assumptions, they proved that the sequence defined by (1.3) and (1.4) converged to a solution of problems (1.2) and (1.1), respectively. Wang and Xu [24] proposed a cyclic iterative algorithm for solving the SCFPP (1.1) of directed mappings.

$$x_{k+1} = U_{[k]_1}\left(x_k + \gamma A^*(T_{[k]_2} - I)Ax_k\right), \ k \geqslant 0, \tag{1.5}$$

where $0 < \gamma < 2/\rho(A^*A)$, $\rho(A^*A)$ being the spectral radius of the operator A^*A, $[k]_1 := (k \bmod t) + 1$ and $[k]_2 := (k \bmod r) + 1$, the mod functions which take values in $\{0, 1, 2, \dots, t-1\}$ and $\{0, 1, 2, \dots, r-1\}$, respectively. They proved that the sequence $\{x_n\}$ generated by (1.5) converges weakly to a solution of the SCFPP (1.1) in a infinite-dimensional Hilbert spaces.

To extend the SCFPP (1.1) for a more general class of nonlinear mappings, Moudafi [17] introduced a relaxed iterative algorithm for the class of quasi-nonexpansive mappings. The relaxed iterative scheme was defined by the following procedure,

$$\begin{aligned} u_k &= x_k + \gamma A^*(T - I)Ax_k, \\ x_{k+1} &= (1 - \alpha_k)u_k + \alpha_k U u_k, \ k \geqslant 0, \end{aligned} \tag{1.6}$$

where $\{\alpha_k\} \subset (0,1)$ and $\gamma \in (0, 1/\lambda)$ with λ being the spectral radius of the operator A^*A. Furthermore, in [16], Moudafi extended the iterative algorithm (1.6) to the class of μ-demicontractive mappings under the parameters assumptions that $\{\alpha_k\} \subset (0,1)$ and $\gamma \in (0, \frac{1-\mu}{\lambda})$. Weak convergence of iterative algorithm (1.6) was obtained in [16] and [17], respectively. Inspired by the work of Wang and Xu [24], Tang et al. [20] introduced a cyclic iterative algorithm for solving the SCFPP (1.1) of demicontractive mappings. Wang and Cui [23] improved the results of [20] by discarding the requirement of $\{U_i\}_{i=1}^t$ and $\{T_j\}_{j=1}^r$ are continuous. Recently, Tang et al. [21] introduced a new simultaneous iterative algorithm for solving the SCFPP (1.1) of demicontractive mappings and proved its convergence under mild assumptions on the parameters. The idea of developing simultaneous iterative algorithm came from parallel iterative method for solving convex feasibility problem.

The purpose of this paper is to continue to study the SCFPP (1.1) for the class of demicontractive mappings. The contribution of this paper is two folds: (i) Based on the technique used in Wang and Cui [23], we present a novel convergence analysis for the iterative algorithm appeared in [21]. The convergence condition required here is weaker than in [21]. (ii) We introduce a new iterative algorithm for solving the SCFPP (1.1) for the class of demicontractive mappings. The convergence of the proposed iterative algorithm is also proved. As applications, we obtain new convergence theorems for solving the SCFPP (1.1) of directed mappings. Numerical experiments compared to the iterative algorithm proposed by [23] and [21] are presented and discussed.

2. Preliminaries

In this section, we collect some important definitions and useful lemmas which will be used in the following section. Throughout this paper, let H be a real Hilbert space, $\langle \cdot, \cdot \rangle$ denotes the inner product, and $\| \cdot \|$ stands for the corresponding norm. In the sequel we shall use the following notations: (i) Ω denotes the solution set of SCFPP (1.1); (ii) $x_k \to x$ and $x_k \rightharpoonup x$ stand for the strong convergence and weak convergence of $\{x_k\}$ to x, respectively; (iii) $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_k\}$.

Recall the orthogonal projection operator P_C. Let C be a nonempty closed convex subset of a Hilbert space H, the orthogonal projection $P_C : H \to C$ is defined by

$$P_C x = \arg\min_{y \in C} \|x - y\|.$$

It is well known that $P_C x$ is characterized by the inequality:

$$\langle x - P_C x, z - P_C x \rangle \leqslant 0, \quad \forall z \in C.$$

Definition 2.1 ([8])**.** Define a mapping $T : H \to H$. Assume that $\mathrm{Fix}(T)$ is nonempty, then T is said to be

(i) nonexpansive, if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in H$;
(ii) quasi-nonexpansive, if $\|Tx - q\| \leqslant \|x - q\|$ for all $x \in H$ and $q \in \mathrm{Fix}(T)$;
(iii) k-strictly pseudocontractive, if $\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$ for all $x, y \in H$, where $k \in [0, 1)$;
(iv) directed, if $\langle Tx - q, Tx - x \rangle \leqslant 0$ for all $x \in H$ and $q \in \mathrm{Fix}(T)$; or equivalently, $\|Tx - q\|^2 \leqslant \|x - q\|^2 - \|Tx - x\|^2$ for all $x \in H$ and $q \in \mathrm{Fix}(T)$;
(v) k-demicontractive, if $\|Tx - q\|^2 \leqslant \|x - q\|^2 + k\|Tx - x\|^2$ for all $x \in H$ and $q \in \mathrm{Fix}(T)$, where $k \in [0, 1)$.

Remark 2.2. The class of demicontractive mappings is fundamental because it includes many types of nonlinear mappings arising in applied mathematics and optimization. We can see from the above definitions that the demicontractive mappings contains these mappings such as the directed mappings, the quasi-nonexpansive mappings, and the strictly pseudocontractive mappings with nonempty fixed point set.

The demiclosedness of T usually plays an important role in dealing with the convergence of fixed point iterative algorithms.

Definition 2.3 ([2])**.** Let $T : H \to H$, $I - T$ is called demiclosed at zero, if for any sequence $\{x_k\} \subset H$ and $x \in H$, we have $x_k \rightharpoonup x$ and $(I-T)x_k \to 0$, then $x \in \mathrm{Fix}(T)$.

Definition 2.4 ([1])**.** Let C be a nonempty closed convex subset of H and $\{x_k\}$ is a sequence in H. The sequence $\{x_k\}$ is called Fejér-monotone with respect to C, if

$$\|x_{k+1} - z\| \leqslant \|x_k - z\| \text{ for all } z \in C \text{ and } k \geqslant 0.$$

The concept of Fejér monotone is basic for many iterative algorithms and the following lemma presents a basic property of Fejér monotone sequence, some other properties can be found in [1, 2].

Lemma 2.5 ([1])**.** *If a sequence $\{x_k\}$ is Fejér-monotone respect to a closed subset of C, then $x_k \rightharpoonup x^* \in C$ if and only if $\omega_w(x_k) \subset C$.*

It is well known that the following equalities hold in a real Hilbert space H, see [2] for more.

Lemma 2.6. *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Then*

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \forall x, y \in H$ *and* $\forall \alpha \in [0,1]$;
(iii) $\|\sum_{i=1}^n \lambda_i x_i\|^2 = \sum_{i=1}^n \lambda_i \|x_i\|^2 - \frac{1}{2}\sum_{i,j=1}^n \lambda_i\lambda_j\|x_i - x_j\|^2$, $n \geqslant 2$,

where $\lambda_i \in [0,1]$ *for all* $i = 1, 2, \ldots, n$ *with* $\sum_{i=1}^n \lambda_i = 1$.

The following lemma presents two equivalent definitions about the demicontractive mappings and a characterization of relaxed demicontractive mappings.

Lemma 2.7 ([16])**.** *Let $T : H \to H$ be a k-demicontractive mapping,*

(i) *Set $T_\alpha = (1-\alpha)I + \alpha T$, $\alpha \in (0,1]$, then T_α is quasi-nonexpansive provided that $\alpha \in [0, 1-k]$ and $\|T_\alpha x - q\|^2 \leqslant \|x-q\|^2 - \alpha(1-k-\alpha)\|Tx - x\|^2$, $x \in H$, $q \in \mathrm{Fix}(T)$.*
(ii) *The following two inequalities are equivalent to the definition of demicontractive mapping,*
   (a) $\langle x - Tx, x - q\rangle \geqslant \frac{1-k}{2}\|x - Tx\|^2$ *for all* $q \in \mathrm{Fix}(T)$, $x \in H$;
   (b) $\langle x - Tx, q - Tx\rangle \leqslant \frac{1+k}{2}\|x - Tx\|^2$ *for all* $q \in \mathrm{Fix}(T)$, $x \in H$.

3. Iterative algorithms for solving the SCFPP (1.1)

In this section, we consider several new iterative algorithms for solving the SCFPP (1.1). In what follows, we consider a family of β_i-demicontractive mappings $\{U_i\}_{i=1}^t : H_1 \to H_1$ and μ_j-demicontractive mappings $\{T_j\}_{j=1}^r : H_2 \to H_2$. Let $\beta = \max\{\beta_1, \beta_2, \ldots, \beta_t\}$ and $\mu = \max\{\mu_1, \mu_2, \ldots, \mu_r\}$, then $\{U_i\}_{i=1}^t$ are β-demicontractive mappings and $\{T_j\}_{j=1}^r$ are μ-demicontractive mappings, respectively.

Algorithm 3.1 (Inner simultaneous and outer cyclic iterative algorithm for solving the SCFPP (1.1))**.** For any initial $x_0 \in H_1$, define the iterative sequence $u_k = x_k + \gamma A^* \sum_{j=1}^r \eta_j (T_j - I)Ax_k$ and the sequence $\{x_k\}$ is given as follows,

$$x_{k+1} = (1-\alpha_k)u_k + \alpha_k U_{[k]}u_k, \; k \geqslant 0, \tag{3.1}$$

where $[k] = (k \bmod t) + 1$, the mod function takes values in $\{0, 1, 2, \ldots, t-1\}$, the constant $\gamma > 0$, $\{\alpha_k\} \subset (0,1)$, and $\{\eta_j\}_{j=1}^r \subset [0,1]$ with $\sum_{j=1}^r \eta_j = 1$.

Now we are in the position to prove the convergence of Algorithm 3.1.

Theorem 3.2. *Let $A : H_1 \to H_2$ be a bounded linear operator, and $\{U_i\}_{i=1}^t : H_1 \to H_1$ be β_i-demicontractive and $\{T_j\}_{j=1}^r : H_2 \to H_2$ be μ_j-demicontractive mappings. Assume that $\{I - U_i\}_{i=1}^t$ and $\{I - T_j\}_{j=1}^r$ are demiclosed at zero. If the solution set Ω of SCFPP (1.1) is nonempty, then the iterative sequence $\{x_k\}$ generated by Algorithm 3.1 converges weakly to a solution of the SCFPP (1.1), where the parameters satisfying that $\gamma \in (0, \frac{1-\mu}{\lambda})$, $\lambda = \rho(A^*A)$ (i.e., λ is equal to the spectral radius of the operator A^*A) and $\alpha_k \in (\delta_1, 1-\beta-\delta_2)$ for some $\delta_1 > 0, \delta_2 > 0$ with $\delta_1 + \delta_2 < 1 - \beta$.*

Proof. First, we prove that the iterative sequence $\{x_k\}$ is Fejér-monotone with respect to the Ω.

Let $p \in \Omega$. By Lemma 2.7 (i), we have

$$\|x_{k+1} - p\|^2 \leqslant \|u_k - p\|^2 - \alpha_k(1-\beta-\alpha_k)\|U_{[k]}u_k - u_k\|^2. \tag{3.2}$$

It follows from the definition of $\{u_k\}$ and Lemma 2.6 that

$$\begin{aligned} \|u_k - p\|^2 &= \left\| x_k + \gamma A^* \sum_{j=1}^r \eta_j (T_j - I)Ax_k - p \right\|^2 \\ &= \left\| \sum_{j=1}^r \eta_j (x_k + \gamma A^*(T_j - I)Ax_k - p) \right\|^2 \\ &\leqslant \sum_{j=1}^r \eta_j \left\| x_k + \gamma A^*(T_j - I)Ax_k - p \right\|^2 \\ &= \|x_k - p\|^2 + \sum_{j=1}^r \eta_j \gamma^2 \left\|A^*(T_j - I)Ax_k\right\|^2 + 2\gamma \sum_{j=1}^r \eta_j \langle x_k - p, A^*(T_j - I)Ax_k\rangle. \end{aligned} \tag{3.3}$$

For the last term of (3.2) and with the help of Lemma 2.7 (ii), we get

$$\begin{aligned} 2\gamma \sum_{j=1}^r \eta_j \left\langle x_k - p, A^*(T_j - I)Ax_k \right\rangle &= 2\gamma \sum_{j=1}^r \eta_j \langle Ax_k - Ap + (T_j - I)Ax_k - (T_j - I)Ax_k, (T_j - I)Ax_k\rangle \\ &= 2\gamma \sum_{j=1}^r \eta_j \left(\langle T_j(Ax_k) - Ap, (T_j - I)Ax_k\rangle - \|(T_j - I)Ax_k\|^2\right) \\ &\leqslant 2\gamma \sum_{j=1}^r \eta_j \left(\frac{1+\mu}{2}\|(T_j - I)Ax_k\|^2 - \|(T_j - I)Ax_k\|^2 \right) \\ &= -\gamma(1-\mu) \sum_{j=1}^r \eta_j \|(T_j - I)Ax_k\|^2. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3), the inequality (3.2) becomes

$$\|x_{k+1} - p\|^2 \leqslant \|x_k - p\|^2 - \alpha_k(1-\beta-\alpha_k) \left\|U_{[k]}u_k - u_k\right\|^2 - \gamma(1-\mu-\gamma\lambda) \sum_{j=1}^r \eta_j \left\|(T_j - I)Ax_k\right\|^2. \tag{3.5}$$

The condition on the parameters γ and $\{\alpha_k\}$ ensure that $\alpha_k(1-\beta-\alpha_k) > 0$ and $\gamma(1-\mu-\gamma\lambda) > 0$, it follows that

$$\|x_{k+1} - p\| \leqslant \|x_k - p\|,$$

which means that $\{x_k\}$ is Fejér-monotone with respect p in Ω. Therefore, $\lim_{k\to\infty} \|x_k - p\|$ exists and $\{x_k\}$ is bounded.

Second, we show that $\omega_w(x_n) \subset \Omega$. Again from (3.5), we know that

$$\delta_1\delta_2 \left\|U_{[k]}u_k - u_k\right\|^2 \leqslant \|x_k - p\|^2 - \|x_{k+1} - p\|^2.$$

Taking the limit on the both side of the above inequality, we obtain

$$\lim_{k\to\infty} \left\|U_{[k]}u_k - u_k\right\| = 0.$$

Similarly, we have that $\lim_{k\to\infty} \|(T_j - I)Ax_k\| = 0$ for any $j = 1, 2, \dots, r$.

Since the sequence $\{x_k\}$ is bounded, so $\omega_w(x_k)$ is nonempty. Let $p^* \in \omega_w(x_k)$. Then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $x_{k_n} \rightharpoonup p^*$. By the demiclosed requirement of $\{I - T_j\}_{j=1}^r$ at zero, we obtain

$$(T_j - I)Ap^* = 0 \text{ for any } j = 1, 2, \dots, r.$$

i.e., $Ap^* \in \bigcap_{j=1}^r \mathrm{Fix}(T_j)$. Recall the definition of $u_k = x_k + \gamma A^* \sum_{j=1}^r (T_j - I)Ax_k$, then $u_{k_n} \rightharpoonup p^*$ which is the same as the iterative sequence $\{x_{k_n}\}$. Since the pool of $\{1, 2, \dots, t\}$ is finite, we can choose a subsequence $\{k_{n_l}\} \subset \{k_n\}$ such that $[k_{n_l}] = i$, $i = 1, 2, \dots, t$, then

$$\left\|U_i(u_{k_{n_l}}) - u_{k_{n_l}}\right\| \to 0, \text{ as } l \to \infty.$$

It follows from $\{I - U_i\}_{i=1}^t$ are demiclosed at zero that $p^* \in \mathrm{Fix}(U_i)$ for any $i = 1, 2, \dots, t$, so $p^* \in \bigcap_{i=1}^t \mathrm{Fix}(U_i)$.

Finally, by Lemma 2.5, we can conclude that the iterative sequence $\{x_k\}$ converges weakly to a solution of the SCFPP (1.1). This completes the proof. □

Remark 3.3. Theorem 3.2 improves Theorem 3.2 of [21] by removing the condition that $\{U_i\}_{i=1}^p$ are continuous.

Corollary 3.4. *Let $A : H_1 \to H_2$ be a bounded linear operator, $\{U_i\}_{i=1}^t : H_1 \to H_1$ be directed mappings, and $\{T_j\}_{j=1}^r : H_2 \to H_2$ be directed mappings. Assume that $\{I - U_i\}_{i=1}^t$ and $\{I - T_j\}_{j=1}^r$ are demiclosed at zero. If the solution set Ω of SCFPP (1.1) is nonempty, then the iterative sequence $\{x_k\}$ generated by Algorithm 3.1 converges weakly to a solution of SCFPP (1.1), where the parameters satisfying that $\gamma \in (0, \frac{1}{\lambda})$, $\lambda = \rho(A^*A)$ and $\alpha_k \in (\delta_1, 1 - \delta_2)$ for some $\delta_1 > 0, \delta_2 > 0$ with $\delta_1 + \delta_2 < 1$.*

Proof. Since the directed mapping is 0-demicontractive mapping, so the constants $\{\beta_i\}_{i=1}^t$ and $\{\mu_j\}_{j=1}^r$ are all equal to zero. Therefore, from Theorem 3.2, we can obtain the result of Corollary 3.4. □

We call the iterative sequence (3.1) that outer cyclic and inner simultaneous iterative algorithm. The algorithmic structure of Algorithm 3.1 inspires us to design a new iterative algorithm for solving the SCFPP (1.1). We exchange the cyclic and outer simultaneous order in Algorithm 3.1 and the new iterative algorithm is presented as follows.

Algorithm 3.5 (Outer simultaneous and inner cyclic iterative algorithm for solving the SCFPP (1.1))**.** For any initial $x_0 \in H_1$, define the iterative sequence $u_k = x_k + \gamma A^* \left(T_{[k]} - I\right) Ax_k$ and the sequence $\{x_k\}$ is defined by

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k \sum_{i=1}^t w_i U_i u_k, k \geqslant 0,$$

where $[k] = (k \bmod r) + 1$, the mod function takes values in $\{0, 1, 2, \dots, r - 1\}$, the constant $\gamma > 0$, $\{\alpha_k\} \subset (0, 1)$, and $\{w_i\}_{i=1}^t \subset [0, 1]$ with $\sum_{i=1}^t w_i = 1$.

The convergence of Algorithm 3.5 is proved in the following theorem.

Theorem 3.6. *Let $A : H_1 \to H_2$ be a bounded linear operator, and $\{U_i\}_{i=1}^t : H_1 \to H_1$ be β_i-demicontractive and $\{T_j\}_{j=1}^r : H_2 \to H_2$ be μ_j-demicontractive mappings. Assume that $\{I - U_i\}_{i=1}^t$ and $\{I - T_j\}_{j=1}^r$ are demiclosed at zero. If the solution set Ω of SCFPP (1.1) is nonempty, then the iterative sequence $\{x_k\}$ generated by Algorithm 3.5 converges weakly to a solution of SCFPP (1.1), where the parameters γ and $\{\alpha_k\}$ are the same as in Theorem 3.2.*

Proof. The proof of Theorem 3.6 is similar to Theorem 3.2. For completeness, we give the detailed steps below.

Let $p \in \Omega$. By Lemma 2.6 and Lemma 2.7 (i), we have

$$\begin{aligned}
\|x_{k+1} - p\|^2 &= \left\|(1-\alpha_k)u_k + \alpha_k \sum_{i=1}^t w_i U_i u_k - p\right\|^2 \\
&= \left\|\sum_{i=1}^t w_i\left((1-\alpha_k)(u_k - p) + \alpha_k(U_i u_k - p)\right)\right\|^2 \\
&\leqslant \sum_{i=1}^t w_i \left\|(1-\alpha_k)(u_k - p) + \alpha_k(U_i u_k - p)\right\|^2 \\
&\leqslant \|u_k - p\|^2 - \alpha_k(1-\beta-\alpha_k)\sum_{i=1}^t w_i\|U_i u_k - u_k\|^2.
\end{aligned} \tag{3.6}$$

It is similar to the deduction of (3.3) and (3.4), we know that

$$\|u_k - p\|^2 \leqslant \|x_k - p\|^2 - \gamma(1-\mu-\gamma\lambda)\left\|(T_{[k]} - I)Ax_k\right\|^2. \tag{3.7}$$

By (3.7) and (3.6), we obtain

$$\|x_{k+1} - p\|^2 \leqslant \|x_k - p\|^2 - \alpha_k(1-\beta-\alpha_k)\sum_{i=1}^t w_i\|U_i u_k - u_k\|^2 - \gamma(1-\mu-\gamma\lambda)\|(T_{[k]} - I)Ax_k\|^2. \tag{3.8}$$

Noting the conditions of γ and $\{\alpha_k\}$, we can conclude that

$$\|x_{k+1} - p\| \leqslant \|x_k - p\|.$$

Then, $\{x_k\}$ is Fejér-monotone with respect to Ω. It also follows from (3.8) that

(i) $\lim_{k\to\infty}\|x_k - p\|$ exists and $\{x_k\}$ is bounded;
(ii) $\lim_{k\to\infty}\|U_i u_k - u_k\| = 0$ and $\lim_{k\to\infty}\left\|(T_{[k]} - I)Ax_k\right\|$ for any $i = 1, 2, \dots, t$.

In the following, we prove that $\omega_w(x_k) \subset \Omega$. In fact, since $\omega_w(x_k)$ is nonempty, let $p^* \in \omega_w(x_k)$. Then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $x_{k_n} \rightharpoonup p^*$. Because of the pool $\{1, 2, \dots, r\}$ is finite, we can choose a subsequence $\{k_{n_l}\} \subset \{k_n\}$ such that $[k_{n_l}] = j$, $j = 1, 2, \dots, r$, then

$$\left\|(T_j - I)Ax_{k_{n_l}}\right\| \to 0, \text{ as } l \to \infty.$$

It follows from the demiclosed at zero of the mappings $\{(I - T_j)\}_{j=1}^r$, we have $(T_j - I)Ap^* = 0$ for any $j = 1, 2, \dots, r$, i.e., $Ap^* \in \bigcap_{j=1}^r \mathrm{Fix}(T_j)$.

Since $u_k = x_k + \gamma A^*(T_{[k]} - I)Ax_k$, $u_{k_n} \rightharpoonup p^*$, and $\{(I - U_i)\}_{i=1}^t$ are also demiclosed at zero. It follows from the fact (ii) that

$$(I - U_i)p^* = 0 \text{ for any } i = 1, 2, \dots, t,$$

i.e., $p^* \in \bigcap_{i=1}^t \mathrm{Fix}(U_i)$.

Therefore, the requirements of Lemma 2.5 are all satisfied, so we can conclude that the iterative sequence $\{x_k\}$ converges weakly to a solution of the SCFPP (1.1). This completes the proof. □

Similar to the proof of Corollary 3.4, we can obtain the following corollary from Theorem 3.6 immediately.

Corollary 3.7. *Let $A : H_1 \to H_2$ be a bounded linear operator, $\{U_i\}_{i=1}^t : H_1 \to H_1$ be directed mappings, and $\{T_j\}_{j=1}^r : H_2 \to H_2$ be directed mappings. Assume that $\{I - U_i\}_{i=1}^t$ and $\{I - T_j\}_{j=1}^r$ are demiclosed at zero. If the solution set Ω of SCFPP (1.1) is nonempty, then the iterative sequence $\{x_k\}$ generated by Algorithm 3.5 converges weakly to a solution of SCFPP (1.1), where the parameters satisfy $\gamma \in (0, \frac{1}{\lambda})$, $\lambda = \rho(A^*A)$, and $\alpha_k \in (\delta_1, 1 - \delta_2)$ for some $\delta_1 > 0, \delta_2 > 0$ with $\delta_1 + \delta_2 < 1$.*

4. Numerical experiments

In this section, we compare the performance of the proposed iterative algorithms with cyclic iterative algorithm [23] and simultaneous iterative algorithm [21]. All the numerical results are completed in a standard Lenovo laptop with Intel(R) Core(TM) i7-4712MQ CPU 2.3GHz and 4 GB memory. The program is implemented in MATLAB 2013a.

Recall that the multiple-set split feasibility problem (MSSFP) as follows,

$$\text{Find a point } x^* \in \bigcap_{i=1}^{t} C_i, \text{ such that } Ax^* \in \bigcap_{j=1}^{r} Q_j, \tag{4.1}$$

where $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ are nonempty closed convex sets in Euclidean spaces R^n and R^m, respectively. The MSSFP (4.1) was introduced by Censor et al. [7] for modeling linear inverse problem of intensity-modulated radiation therapy. Since the MSSFP (4.1) is a special case of the SCFPP (1.1). Based on the results of Corollary 3.4 and Corollary 3.7, we know that the iterative sequences generated by Algorithms 3.1 and 3.5 converge to a solution of the MSSFP (4.1), respectively. We take the example from [19]. For convenience, we denote e_0 be a zero vector and e_1 be a vector with every elements equal to one. The symbol randn is MATLAB function which generates sequences from normal distribution with mean zero and variance one.

Example 4.1. The MSSFP (4.1) with $A = (a_{ij})$ and $a_{ij} \in [0, 1]$ generated from uniformly distribution. $C_i = \{x \in R^n | \|x - d_i\| \leqslant r_i\}$, $i = 1, 2, \ldots, t$, $Q_j = \{y \in R^m | L_j \leqslant y \leqslant U_j\}$, $j = 1, 2, \ldots, r$, where $d_i \in [e_0, 10e_1]$, $r_i \in [40, 60]$, $L_j \in [10e_1, 40e_1]$ and $U_j \in [50e_1, 100e_1]$ are all generated randomly.

We define a function $f(x)$ to measure the distance of a point to all sets of the MSSFP (4.1).

$$f(x) = \frac{1}{2}\Big(\sum_{i=1}^{t} w_i \|x - P_{C_i}(x)\|^2 + \sum_{j=1}^{r} v_j \|Ax - P_{Q_j}(Ax)\|^2\Big),$$

where $w_i > 0$, $v_j > 0$ for all i, j and $\sum_{i=1}^t w_i + \sum_{j=1}^r v_j = 1$. We choose $w_i = v_j = \frac{1}{r+t}$ in the following tests. It is clear that a point x^* is a solution of the MSSFP (4.1) when $f(x^*) = 0$. In practice, if $f(x^*) < \epsilon$, where ϵ is a given small real number. Then, the point x^* is accepted as a solution of the MSSFP (4.1). To ensure a fair comparison, we choose iterative parameter $\gamma = 0.9\frac{1}{\|A\|^2}$ and $\alpha_k = 0.5$ for all the compared iterative algorithms. We report the iteration numbers "Iter" and computation time "T(s)" in CPU . The numerical results are reported in Tables 1 and 2. We can see from Tables 1 and 2 that the proposed Algorithms 3.1 and 3.5 converge to a solution of the MSSFP (4.1) with less iteration numbers than the other two iterative algorithms. All the iterative algorithms converge to the solution in a reasonable amount of the time under given accuracy. It is observed that the cyclic iterative algorithm [23] is the fastest in CPU time among these iterative algorithms.

Table 1: The performance of cyclic iterative algorithm [23], simultaneous iterative algorithm [21], Algorithm 3.1, and Algorithm 3.5 in terms of iteration numbers and computation time when $r = t = 10$, $m = 40, n = 50$.

Initial point	Methods	$\epsilon = 10^{-6}$		$\epsilon = 10^{-9}$		$\epsilon = 10^{-12}$	
		Iter	T(s)	Iter	T(s)	Iter	T(s)
e_0	Algorithm 3.1	263	0.22	410	0.30	558	0.44
	Algorithm 3.5	264	0.39	410	0.57	556	0.83
	Cyclic [23]	2635	0.05	4095	0.07	5555	0.10
	Simultaneous [21]	2624	1.83	4100	2.95	5575	4.12
e_1	Algorithm 3.1	254	0.18	402	0.29	549	0.40
	Algorithm 3.5	255	0.35	401	0.56	547	0.77
	Cyclic [23]	2545	0.04	4005	0.07	5465	0.10
	Simultaneous [21]	2538	1.81	4014	2.86	5490	3.94
$100e_1$	Algorithm 3.1	1278	0.93	1952	1.40	2636	1.90
	Algorithm 3.5	1249	1.77	1915	2.68	2596	3.63
	Cyclic [23]	12738	0.23	19478	0.35	26318	0.47
	Simultaneous [21]	12545	9.10	19187	13.62	25995	18.45
$\mathrm{randn}(n,1)$	Algorithm 3.1	210	0.15	339	0.22	469	0.31
	Algorithm 3.5	206	0.29	334	0.43	462	0.60
	Cyclic [23]	2056	0.03	3336	0.05	4616	0.07
	Simultaneous [21]	2092	1.37	3388	2.24	4684	3.09

Table 2: The performance of cyclic iterative algorithm [23], simultaneous iterative algorithm [21], Algorithm 3.1, and Algorithm 3.5 in terms of iteration numbers and computation time when $r = t = 30$, $m = 50, n = 60$.

Initial point	Methods	$\epsilon = 10^{-6}$		$\epsilon = 10^{-9}$		$\epsilon = 10^{-12}$	
		Iter	T(s)	Iter	T(s)	Iter	T(s)
e_0	Algorithm 3.1	277	1.92	477	3.27	676	5.10
	Algorithm 3.5	278	3.46	476	5.87	673	8.79
	Cyclic [23]	8300	0.14	14240	0.25	20180	0.35
	Simultaneous [21]	8323	55.38	14308	99.97	20292	140.49
e_1	Algorithm 3.1	273	2.08	472	3.37	672	4.93
	Algorithm 3.5	273	3.55	471	5.97	668	8.62
	Cyclic [23]	8180	0.14	14090	0.25	20030	0.34
	Simultaneous [21]	8169	58.11	14153	97.72	20138	140.69
$100e_1$	Algorithm 3.1	1300	9.22	2139	15.77	3019	21.45
	Algorithm 3.5	1617	18.02	2799	31.13	3981	43.24
	Cyclic [23]	38635	0.67	62486	1.10	86932	1.54
	Simultaneous [21]	48895	331.83	84614	591.73	120358	915.29
$\mathrm{randn}(n,1)$	Algorithm 3.1	277	2.31	477	3.95	676	5.58
	Algorithm 3.5	278	3.98	476	6.78	673	9.43
	Cyclic [23]	8300	0.15	14240	0.26	20180	0.36
	Simultaneous [21]	8323	68.70	14308	118.04	20292	167.59

5. Conclusion

In this paper, we studied iterative methods related to the split common fixed point problem for the class of demicontractive mappings. We presented a new convergence analysis of an iterative algorithm proposed by [21] and also have weaken the condition on the given demicontractive mappings. Furthermore, a new iterative algorithm was proposed and its convergence was also proved under some mild conditions. As applications, we applied the proposed iterative algorithms to solve the multiple-set split feasibility problem (4.1). Numerical results showed that our iterative algorithms perform better than the existing iterative algorithms in terms of iteration numbers. We also demonstrated the efficiency of the optimization algorithm and its scalability with the size of the problem.

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