Convergence analysis of generalized viscosity implicit rules for a nonexpansive semigroup with gauge functions

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Abstract

In this paper, we introduce an iterative algorithm for finding the set of common fixed points of nonexpansive semigroups by the generalized viscosity implicit rule in certain Banach spaces which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function. We prove strong convergence theorems of proposed algorithm under appropriate conditions. As applications, we apply main result to solving the fixed point problems of countable family of nonexpansive mappings and the problems of zeros of accretive operators. Furthermore, we give some numerical examples for supporting our main results.

Keywords: Nonexpansive semigroup, Banach spaces, strong convergence, fixed point problem, iterative method.

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1. Introduction

In this paper, we assume that E is a real Banach space with dual space E^* and C is a nonempty subset of E. Let $T : C \to C$ be a mapping. We denote the set of all fixed points of T by $F(T) = \{x \in C : x = Tx\}$. A mapping $T : C \to C$ is called *nonexpansive* if for each $x, y \in C$ such that

$$\|Tx - Ty\| \leqslant \|x - y\|.$$

A mapping $f : C \to C$ is called a *contraction*, if there exists a constant $\rho \in (0,1)$ and for each $x, y \in C$

$$\|f(x) - f(y)\| \leqslant \rho \|x - y\|.$$

The viscosity approximation method has been successfully applied to various problems from calculus of variations as minimal surface problems and plasticity theory and phase transition. Various applications can be obtained in optimal control theory, singular perturbations, game theory, and partial differential equations (see [4] and references therein). In recent years, viscosity approximation method for approximating the set of (common) fixed points of nonlinear mappings have been investigated extensively by many authors in Hilbert and Banach spaces (see [10, 11, 13, 19, 20, 23–25, 30] and the references therein).

Very recently, the implicit midpoint rule (IMR) has become a powerful numerical method for numerically solving ordinary differential equations (in particular, the stiff equations) (see [5, 6, 14, 21, 22, 28]) and differential algebraic equations (see [32]).

Xu et al. [31] combined the Moudafi's viscosity method [19] (see also [30]) with IMR for nonexpansive mappings T and proposed the following *viscosity implicit midpoint rule* (VIMR) in Hilbert spaces H as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geqslant 1, \tag{1.1}$$

where $\{\alpha_n\}$ is a real control condition in $(0,1)$. They also proved that VIMR converges strongly to a point $x^* \in F(T)$ which also solves the variational inequality

$$\langle (f - I)x^*, z - x^* \rangle \leqslant 0, \quad \forall z \in F(T), \tag{1.2}$$

where I is the identity on H.

Later, Ke and Ma [17] improved the VIMR (1.1) by replacing the midpoint by any point of interval $[x_n, x_{n+1}]$. They introduced the so-call *generalized viscosity implicit midpoint rules* to approximating the fixed point of nonexpansive mapping T in Hilbert spaces H. They obtained the following result.

Theorem 1.1 (Theorem KM). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let f be a contraction on C with coefficient $\rho \in (0,1)$. Let $x_1 \in C$, and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}), \quad \forall n \geqslant 1, \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{s_n\}$ are sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that the following conditions hold:

(C1) $\lim_{n \to \infty} \gamma_n = 1$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(C3) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(C4) $0 < \kappa \leqslant s_n \leqslant s_{n+1} < 1$ *for all $n \geqslant 1$.*

Then $\{x_n\}$ converges strongly to a point $x^* \in F(T)$, which also solves (1.2).

The above results naturally bring us to the following questions.

Question 1: Can we obtain strong convergence result of Theorem 1.1 to higher spaces other than Hilbert spaces? Such as a real reflexive strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function.

Question 2: Can we remove the control condition (C1) in Theorem 1.1?

Question 3: Can we weaken the control conditions (C2) and (C3) in Theorem 1.1?

Question 4: Can we extend the generalized viscosity implicit midpoint rules (1.3) to finding the set of common fixed points of a family of mappings? Such as one-parameter semigroups of nonexpansive mappings.

The main objective in this paper is to give an affirmative answer to above questions, we introduce an iterative algorithm for finding the set of common fixed points of nonexpansive semigroups by the generalized viscosity implicit rule in a real reflexive strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function. Then, we prove strong convergence theorems of proposed algorithm with different approach on control conditions. As applications, we apply main results to solving the fixed point problems of family of nonexpansive mappings and the problems of zeros of accretive operators. Furthermore, we also give some numerical examples for support our main results.

2. Preliminaries

The continuous and strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ is said to be *gauge function* if $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\| \varphi(\|x\|), \ \|f^*\| = \varphi(\|x\|), \ \forall x \in E\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality paring. In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J is referred to as the normalized duality mapping. In this case $\varphi(t) = t^{q-1}$, $q > 1$, the duality mapping $J_\varphi = J_q$ is called *generalized duality mapping*. It follows from the definition that $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x)$ for each $x \neq 0$, and $J_q(x) = \|x\|^{q-2} J(x)$, $q > 1$ (see [9]).

Remark 2.1. For the gauge function φ, the function $\Phi : [0, \infty) \to [0, \infty)$ defined by $\Phi(t) = \int_0^t \varphi(\tau) d\tau$ is continuous, convex, and strictly increasing function on $[0, \infty)$. Therefore, Φ has a continuous inverse function Φ^{-1}.

Remark 2.2. It is observe that if $k \in [0, 1]$ then $\varphi(ky) \leqslant \varphi(y)$. Then, we have

$$\Phi(kt) = \int_0^{kt} \varphi(\tau) d\tau = k \int_0^t \varphi(ky) dy \leqslant k \int_0^t \varphi(y) dy = k\Phi(t).$$

Remark 2.3. If a Banach space E has a uniformly Gâteaux differentiable, then $J\varphi$ is single-valued and also denoted by j_φ.

Lemma 2.4 ([18]). *Let E be a Banach space. Then for each $x, y \in E$, the following inequality holds:*

$$\Phi(\|x + y\|) \leqslant \Phi(\|x\|) + \langle y, j_\varphi(x + y) \rangle, \ j_\varphi(x + y) \in J_\varphi(x + y).$$

Definition 2.5. A one-parameter family $\mathcal{S} = \{T_t\}_{t \geqslant 0} : C \to C$ is said to be a *nonexpansive semigroup* if it satisfies the following conditions:

(S1) $T_0 x = x$ for $x \in C$;

(S2) $T_{s+t} = T_s T_t$ for $s, t \geqslant 0$;

(S3) $\lim_{t \to 0^+} T(t)x = x$ for $x \in C$;

(S4) for each $t \geqslant 0$, T_t is nonexpansive, i.e.,

$$\|T_t x - T_t y\| \leqslant \|x - y\|, \ \forall x, y \in C.$$

Remark 2.6. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, i.e., $F(\mathcal{S}) = \bigcap_{t \geqslant 0} F(T_t)$.

Now, we give some examples of semigroup operator. The following classical examples were the main sources for the development of semigroup theory (see [15]).

Example 2.7. Let E be a real Banach space and let $\mathcal{L}(E)$ be the space of all bounded linear operators on E. For $A \in \mathcal{L}(E)$, consider the initial value problem for a linear autonomous differential equation on $[0, \infty)$:

$$u'(t) = Au(t), \ u(0) = x. \tag{2.1}$$

Notice that the solution of problem (2.1) is given by

$$u(t) := T_t x \text{ for all } t \geqslant 0.$$

Then, we can show that the operator $T_t x$ is a semigroup on E.

Example 2.8. Let $E := L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$. Consider the initial value problem for the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u, \quad \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\ u(x, 0) &= f(x), \quad \text{for } x \in \mathbb{R}^n, \end{aligned} \tag{2.2}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator on E. By using Fourier transform, we can write the solution $u(x, t)$ in the form of convolution integral as:

$$u(x, t) = \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} e^{\frac{-\|x - \xi\|^2}{4t}} f(\xi) d\xi = (K_t * f)(x),$$

where $t > 0$, $f \in E$, and K_t is the heat kernel given by $K_t(x) = \frac{1}{\sqrt{(4\pi t)^n}} e^{\frac{-\|x\|^2}{4t}}$. Then the solution of problem (2.2) can be written as:

$$T_t f(x) := u(x, t) = (K_t * f)(x).$$

We can show that the operator $T_t f(x)$ is a semigroup on E.

Definition 2.9 ([1, 2, 8]). A continuous operator semigroup $\mathcal{S} = \{T_t\}_{t \geqslant 0} : C \to C$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) if for all $s \geqslant 0$ and any bounded subset B of C,

$$\lim_{t \to \infty} \sup_{x \in B} \|T_t x - T_s T_t x\| = 0.$$

Example 2.10. Let C be a closed convex subset of a uniformly convex Banach space E. Let $\mathcal{S} = \{T_t\}_{t \geqslant 0} : C \to C$ be a nonexpansive semigroup. Let $\{\sigma_t\}_{t > 0}$ defined by $\sigma_t x = \frac{1}{t} \int_0^t T_s x ds$. Then, for each $h > 0$ and any bounded subset B of C, we have

$$\|\sigma_t x - \sigma_h \sigma_t x\| = \left\| \sigma_t x - \frac{1}{h} \int_0^h T_s \sigma_t x ds \right\| = \left\| \frac{1}{h} \int_0^h (\sigma_t x - T_s \sigma_t x) ds \right\| \leqslant \frac{1}{h} \int_0^h \|\sigma_t x - T_s \sigma_t x\| ds.$$

From Lemma 2.7 of [12], we have

$$\lim_{t \to \infty} \sup_{x \in B} \|\sigma_t x - \sigma_h \sigma_t x\| \leqslant \frac{1}{h} \int_0^h \lim_{t \to \infty} \sup_{x \in B} \|\sigma_t x - T_s \sigma_t x\| ds = 0,$$

i.e., $\{\sigma_t\}_{t > 0}$ is u.a.r..

Theorem 2.11 ([13]). *Let C be a nonempty, closed, and convex subset of a real reflexive strictly convex Banach space E, which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $\mathcal{S} = \{T_t\}_{t \geqslant 0} : C \to C$ be a u.a.r. nonexpansive semigroup such that $F(\mathcal{S}) := \bigcap_{t \geqslant 0} F(T_t) \neq \emptyset$ and f be a contraction on C with coefficient $\rho \in (0, 1)$. Suppose that $\{t_n\}$ is a real divergent sequence and $\{\alpha_n\}$ is a real sequence in $(0, 1)$ with $\lim_{n \to \infty} \alpha_n = 0$. Then, the sequence $\{x_n\}$ defined by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n) T_{t_n} x_n, \quad \forall n \geqslant 1,$$

converges strongly to a point $p \in F(\mathcal{S})$, which also solves the variational inequality

$$\langle f(p) - p, j_\varphi(z - p) \rangle \leqslant 0, \quad \forall z \in F(\mathcal{S}).$$

Lemma 2.12 ([26]). *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all integers $n \geqslant 0$ and $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. Then, $\lim_{n\to\infty} \|l_n - x_n\| = 0$.*

Lemma 2.13 ([29]). *Assume that $\{a_n\}$ is a nonnegative real sequence such that*

$$a_{n+1} \leqslant (1 - \theta_n)a_n + \theta_n \sigma_n,$$

where $\{\theta_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a real sequence such that

(i) $\sum_{n=1}^{\infty} \theta_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leqslant 0$ or $\sum_{n=1}^{\infty} |\theta_n \sigma_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a real reflexive strictly convex Banach space E, which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $\mathcal{S} = \{T_t\}_{t\geqslant 0} : C \to C$ be a u.a.r. nonexpansive semigroup such that $F(\mathcal{S}) := \bigcap_{t\geqslant 0} F(T_t) \neq \emptyset$ and f be a contraction on C with coefficient $\rho \in (0,1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{t_n}(s_n x_n + (1 - s_n)x_{n+1}), \quad \forall n \geqslant 1, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \subset (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, and $\{t_n\} \subset (0,\infty)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$ and $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$;
(C3) $t_{n+1} = h + t_n$ for all $h \geqslant 0$ and $\lim_{n\to\infty} t_n = \infty$;
(C4) $0 < \kappa \leqslant s_n \leqslant s_{n+1} < 1$ for all $n \geqslant 1$.

Then, $\{x_n\}$ defined by (3.1) converges strongly to a point $p \in F(\mathcal{S})$, which also solves the variational inequality

$$\langle f(p) - p, j_\varphi(z - p)\rangle \leqslant 0, \quad \forall z \in F(\mathcal{S}). \tag{3.2}$$

Proof. First, we will show that $\{x_n\}$ generated by (3.1) is well defined. For each $x, u \in C$, define the mapping $S_n : C \to C$ by

$$S_n x := \alpha_n f(x) + \beta_n x + \gamma_n T_{t_n}(s_n u + (1 - s_n)x), \quad \forall n \geqslant 1.$$

For each $x, y \in C$, we have

$$
\begin{aligned}
\|S_n x - S_n y\| &= \|\alpha_n(f(x) - f(y)) + \beta_n(x - y) + \gamma_n[T_{t_n}(s_n u + (1 - s_n)x) - T_{t_n}(s_n u + (1 - s_n)y)]\| \\
&\leqslant \alpha_n \|f(x) - f(y)\| + \beta_n \|x - y\| + \gamma_n \|T_{t_n}(s_n u + (1 - s_n)x) - T_{t_n}(s_n u + (1 - s_n)y)\| \\
&\leqslant \alpha_n \rho \|x - y\| + \beta_n \|x - y\| + \gamma_n(1 - s_n)\|x - y\| \\
&= (1 - (1 - \rho)\alpha_n - \gamma_n \kappa)\|x - y\| \leqslant (1 - (1 - \rho)\alpha_n)\|x - y\|,
\end{aligned}
$$

this mean that S_n is a contraction. So S_n has a unique fixed point. Therefore, the sequence $\{x_n\}$ defined by (3.1) is well-defined.

Next, we show that $\{x_n\}$ is bounded. For each $p \in F(\mathcal{S})$, we have

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(T_{t_n}(s_n x_n + (1 - s_n)x_{n+1}) - p)\| \\
&\leqslant \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|T_{t_n}(s_n x_n + (1 - s_n)x_{n+1}) - p\|
\end{aligned}
$$

$$\leqslant \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|s_n(x_n - p) + (1 - s_n)(x_{n+1} - p)\|$$
$$\leqslant \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + r_n(s_n \|x_n - p\| + (1 - s_n)\|x_{n+1} - p\|)$$
$$\leqslant (\alpha_n \rho + \beta_n + \gamma_n s_n)\|x_n - p\| + \alpha_n \|f(p) - p\| + \gamma_n(1 - s_n)\|x_{n+1} - p\|,$$

which implies that

$$\|x_{n+1} - p\| \leqslant \frac{\alpha_n \rho + \beta_n + \gamma_n s_n}{1 - \gamma_n(1 - s_n)}\|x_n - p\| + \frac{\alpha_n}{1 - \gamma_n(1 - s_n)}\|f(p) - p\|$$
$$= \left(1 - \frac{(1 - \rho)\alpha_n}{1 - \gamma_n(1 - s_n)}\right)\|x_n - p\| + \frac{(1 - \rho)\alpha_n}{1 - \gamma_n(1 - s_n)}\frac{\|f(p) - p\|}{1 - \rho}$$
$$\leqslant \max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \rho}\right\}.$$

By induction, we have

$$\|x_n - p\| \leqslant \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1 - \rho}\right\}, \quad \forall n \geqslant 1.$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{f(x_n)\}$ and $\{T_{t_n}(s_n x_n + (1 - s_n)x_{n+1})\}$ are bounded.

Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Let $z_n = s_n x_n + (1 - s_n)x_{n+1}$ for all $n \geqslant 1$. Then, we have

$$\|z_{n+1} - z_n\| = \|s_{n+1}x_{n+1} + (1 - s_{n+1})x_{n+2} - (s_n x_n + (1 - s_n)x_{n+1})\|$$
$$\leqslant (1 - s_{n+1})\|x_{n+2} - x_{n+1}\| + s_n\|x_{n+1} - x_n\|.$$

Let $y_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$ for all $n \geqslant 1$. Then, we drive that

$$y_{n+1} - y_n = \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$$
$$= \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}T_{t_{n+1}}z_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n T_{t_n}z_n}{1 - \beta_n}$$
$$= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}f(x_{n+1}) - \frac{\alpha_n}{1 - \beta_n}f(x_n) + T_{t_{n+1}}z_{n+1} - T_{t_n}z_n + \frac{\alpha_n}{1 - \beta_n}T_{t_n}z_n - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}T_{t_{n+1}}z_{n+1}.$$

It follows that

$$\|y_{n+1} - y_n\| \leqslant \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - T_{t_{n+1}}z_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|T_{t_n}z_n - f(x_n)\| + \|T_{t_{n+1}}z_{n+1} - T_{t_{n+1}}z_n\|$$
$$\quad + \|T_{t_{n+1}}z_n - T_{t_n}z_n\|$$
$$\leqslant \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - T_{t_{n+1}}z_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|T_{t_n}z_n - f(x_n)\| \tag{3.3}$$
$$\quad + (1 - s_{n+1})\|x_{n+2} - x_{n+1}\| + s_n\|x_{n+1} - x_n\| + \|T_{t_{n+1}}z_n - T_{t_n}z_n\|.$$

Now, we estimate $\|x_{n+2} - x_{n+1}\|$. Observe that

$$\|x_{n+2} - x_{n+1}\| = \|\alpha_{n+1}f(x_{n+1}) + \beta_{n+1}x_{n+1} + \gamma_{n+1}T_{t_{n+1}}z_{n+1} - (\alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{t_n}z_n)\|$$
$$= \|\alpha_{n+1}(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)(f(x_n) - T_{t_n}z_n) + \beta_{n+1}(x_{n+1} - x_n)$$
$$\quad + (\beta_{n+1} - \beta_n)(x_n - T_{t_n}z_n) + \gamma_{n+1}(T_{t_{n+1}}z_{n+1} - T_{t_n}z_n)\|$$
$$\leqslant \alpha_{n+}\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|\|f(x_n) - T_{t_n}z_n\| + \beta_{n+1}\|x_{n+1} - x_n\|$$
$$\quad + |\beta_{n+1} - \beta_n|\|x_n - T_{t_n}z_n\| + \gamma_{n+1}\|T_{t_{n+1}}z_{n+1} - T_{t_{n+1}}z_n\| + \gamma_{n+1}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|$$

$$\leqslant (\alpha_{n+1}\rho + \beta_{n+1})\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|f(x_n) - T_{t_n}z_n\|$$
$$+ |\beta_{n+1} - \beta_n|\|x_n - T_{t_n}z_n\|$$
$$+ \gamma_{n+1}((1 - s_{n+1})\|x_{n+2} - x_{n+1}\| + s_n\|x_{n+1} - x_n\|) + \gamma_{n+1}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|$$
$$\leqslant (\alpha_{n+1}\rho + \beta_{n+1} + \gamma_{n+1}s_n)\|x_{n+1} - x_n\| + (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|)M_1$$
$$+ \gamma_{n+1}(1 - s_{n+1})\|x_{n+2} - x_{n+1}\| + \gamma_{n+1}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|,$$

where $M_1 = \sup_{n \geqslant 1}\{\|f(x_n)\| + \|T_{t_n}z_n\|, \|x_n\| + \|T_{t_n}z_n\|\}$. It follows that

$$\|x_{n+2} - x_{n+1}\|$$
$$\leqslant \frac{\alpha_{n+1}\rho + \beta_{n+1} + \gamma_{n+1}s_n}{1 - \gamma_{n+1}(1 - s_{n+1})}\|x_{n+1} - x_n\| + \left(\frac{|\alpha_{n+1} - \alpha_n|}{1 - \gamma_{n+1}(1 - s_{n+1})} + \frac{|\beta_{n+1} - \beta_n|}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)M_1$$
$$+ \frac{\gamma_{n+1}}{1 - \gamma_{n+1}(1 - s_{n+1})}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|$$
$$= \left(1 - \frac{(1 - \rho)\alpha_{n+1}}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)\|x_{n+1} - x_n\| + \left(\frac{|\alpha_{n+1} - \alpha_n|}{1 - \gamma_{n+1}(1 - s_{n+1})} + \frac{|\beta_{n+1} - \beta_n|}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)M_1$$
$$+ \frac{\gamma_{n+1}}{1 - \gamma_{n+1}(1 - s_{n+1})}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|. \tag{3.4}$$

Substituting (3.4) into (3.3), we get that

$$\|y_{n+1} - y_n\| \leqslant \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - T_{t_{n+1}}z_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|T_{t_n}z_n - f(x_n)\|$$
$$+ (1 - s_{n+1})\left\{\left(1 - \frac{(1 - \rho)\alpha_{n+1}}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)\|x_{n+1} - x_n\|\right.$$
$$+ \left(\frac{|\alpha_{n+1} - \alpha_n|}{1 - \gamma_{n+1}(1 - s_{n+1})} + \frac{|\beta_{n+1} - \beta_n|}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)M_1$$
$$+ \left.\frac{\gamma_{n+1}}{1 - \gamma_{n+1}(1 - s_{n+1})}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|\right\}$$
$$+ s_n\|x_{n+1} - x_n\| + \|T_{t_{n+1}}z_n - T_{t_n}z_n\|$$
$$\leqslant \left(1 - \frac{(1 - \rho)\alpha_{n+1}(1 - s_{n+1})}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)\|x_{n+1} - x_n\| + \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - T_{t_{n+1}}z_{n+1}\|$$
$$+ \frac{\alpha_n}{1 - \beta_n}\|T_{t_n}z_n - f(x_n)\| + \left(\frac{|\alpha_{n+1} - \alpha_n|}{1 - \gamma_{n+1}(1 - s_{n+1})} + \frac{|\beta_{n+1} - \beta_n|}{1 - \gamma_{n+1}(1 - s_{n+1})}\right)M_1$$
$$+ \frac{1}{1 - \gamma_{n+1}(1 - s_{n+1})}\|T_{t_{n+1}}z_n - T_{t_n}z_n\|. \tag{3.5}$$

Since $t_{n+1} = h + t_n$ for all $h \geqslant 0$, we have

$$\lim_{n\to\infty}\|T_{t_{n+1}}z_n - T_{t_n}z_n\| = \lim_{n\to\infty}\|T_h T_{t_n}z_n - T_{t_n}z_n\| \leqslant \lim_{n\to\infty}\sup_{x\in\{z_n\}}\|T_h T_{t_n}x - T_{t_n}x\| = 0.$$

Then from (3.5), we have

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0.$$

By Lemma 2.12, we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Consequently, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|y_n - x_n\| = 0$$

and

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}(1 - s_n)\|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Next, we show that $\lim_{n\to\infty}\|x_n - T_h x_n\| = 0$ for all $h \geqslant 0$. Since

$$\|x_{n+1} - T_{t_n}x_n\| \leqslant \alpha_n\|f(x_n) - T_{t_n}z_n\| + \beta_n\|x_n - T_{t_n}z_n\| + \gamma_n\|T_{t_n}z_n - T_{t_n}x_n\|$$
$$\leqslant \alpha_n\|f(x_n) - T_{t_n}x_n\| + \beta_n\|x_n - x_{n+1}\| + \beta_n\|x_{n+1} - T_{t_n}x_n\| + \gamma_n\|z_n - x_n\|,$$

and hence

$$\|x_{n+1} - T_{t_n}x_n\| \leqslant \frac{\alpha_n}{1 - \beta_n}\|f(x_n) - T_{t_n}x_n\| + \frac{\beta_n}{1 - \beta_n}\|x_{n+1} - x_n\| + \frac{\gamma_n}{1 - \beta_n}\|z_n - x_n\| \to 0 \text{ as } n \to \infty.$$

Consequently,

$$\|z_n - T_{t_n}z_n\| \leqslant \|z_n - x_{n+1}\| + \|x_{n+1} - T_{t_n}z_n\|$$
$$\leqslant s_n\|x_n - x_{n+1}\| + \|x_{n+1} - T_{t_n}x_n\| + \|T_{t_n}x_n - T_{t_n}z_n\|$$
$$\leqslant s_n\|x_n - x_{n+1}\| + \|x_{n+1} - T_{t_n}x_n\| + \|x_n - z_n\| \to 0 \text{ as } n \to \infty.$$

Then, for all $h \geqslant 0$, we obtain that

$$\|z_n - T_h z_n\| \leqslant \|z_n - T_{t_n}z_n\| + \|T_{t_n}z_n - T_h T_{t_n}z_n\| + \|T_h T_{t_n}z_n - T_h z_n\|$$
$$\leqslant 2\|z_n - T_{t_n}z_n\| + \sup_{x\in\{z_n\}}\|T_{t_n}x - T_h T_{t_n}x\| \to 0 \text{ as } n \to \infty.$$

From (3.6), we also have

$$\lim_{n\to\infty}\|x_n - T_h x_n\| = 0, \quad \forall h \geqslant 0.$$

Let $u_m = \alpha_m f(u_m) + (1 - \alpha_m)T_{t_m}u_m$, where $\{\alpha_m\}$ and $\{t_m\}$ satisfy the condition of Theorem 2.11. From these, we know that $\{u_m\}$ converges strongly to p, where $p \in F(\mathcal{S})$ is a unique solution of (3.2). Since

$$\|u_m - x_n\|\varphi(\|u_m - x_n\|) = \alpha_n\langle f(u_m) - x_n, j_\varphi(u_m - x_n)\rangle + (1 - \alpha_m)\langle T_{t_m}u_m - x_n, j_\varphi(u_m - x_n)\rangle$$
$$= \alpha_m\langle f(u_m) - f(p) - u_m + p, j_\varphi(u_m - x_n)\rangle + \alpha_m\langle f(p) - p, j_\varphi(u_m - x_n)\rangle$$
$$+ \alpha_m\langle u_m - x_n, j_\varphi(u_m - x_n)\rangle + (1 - \alpha_m)\langle T_{t_m}u_m - T_{t_m}x_n, j_\varphi(u_m - x_n)\rangle$$
$$+ (1 - \alpha_m)\langle T_{t_m}x_n - x_n, j_\varphi(u_m - x_n)\rangle$$
$$\leqslant \|u_m - x_n\|\varphi(\|u_m - x_n\|) + \|T_{t_m}x_n - x_n\|\varphi(\|u_m - x_n\|)$$
$$+ \alpha_m(1 + \rho)\varphi(\|u_m - x_n\|)\|u_m - p\| + \alpha_m\langle f(p) - p, j_\varphi(u_m - x_n)\rangle,$$

which implies that

$$\langle f(p) - p, j_\varphi(x_n - u_m)\rangle \leqslant \left(\frac{\|T_{t_m}x_n - x_n\|}{\alpha_m} + (1 + \rho)\|u_m - p\|\right)M_2, \tag{3.7}$$

where $M_2 = \sup_{n\geqslant 1}\{\varphi(\|u_m - x_n\|)\}$. Now, taking the upper limit as $n \to \infty$ and as $m \to \infty$, respectively in (3.7), we obtain

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle f(p) - p, j_\varphi(x_n - u_m)\rangle \leqslant 0. \tag{3.8}$$

Since j_φ is norm-weak* uniformly continuous on bounded sets, as $m \to \infty$, then

$$\langle f(p) - p, j_\varphi(x_n - u_m)\rangle \to \langle f(p) - p, j_\varphi(x_n - p)\rangle.$$

Hence, for each $\epsilon > 0$, there exists $N \geqslant 1$ such that if $m > N$, for all $n \geqslant 1$ we have

$$\langle f(p) - p, j_\varphi(x_n - p) \rangle < \langle f(p) - p, j_\varphi(x_n - u_m) \rangle + \epsilon. \tag{3.9}$$

Thus taking upper limit as $n \to \infty$ and as $m \to \infty$ in both sides of (3.9), we get from (3.8) that

$$\limsup_{n \to \infty} \langle f(p) - p, j_\varphi(x_n - p) \rangle \leqslant \epsilon.$$

Since $\epsilon > 0$ is arbitrary, then we obtain

$$\limsup_{n \to \infty} \langle f(p) - p, j_\varphi(x_n - p) \rangle \leqslant 0. \tag{3.10}$$

Finally, we show that x_n converges strongly to p. We have from Lemma 2.4 that

$$\begin{aligned}
\Phi(\|x_{n+1} - p\|) &= \Phi(\|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(T_{t_n}(s_n x_n + (1 - s_n)x_{n+1}) - p)\|) \\
&\leqslant \Phi(\|\alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) + \gamma_n(T_{t_n}(s_n x_n + (1 - s_n)x_{n+1}) - p)\|) \\
&\quad + \alpha_n \langle f(p) - p, j_\varphi(x_{n+1} - p) \rangle \\
&\leqslant \alpha_n \Phi(\|f(x_n) - f(p)\|) + \beta_n \Phi(\|x_n - p\|) + \gamma_n \Phi(\|T_{t_n}(s_n x_n + (1 - s_n)x_{n+1}) - p\|) \\
&\quad + \alpha_n \langle f(p) - p, j_\varphi(x_{n+1} - p) \rangle \\
&\leqslant \alpha_n \rho \Phi(\|x_n - p\|) + \beta_n \Phi(\|x_n - p\|) + \gamma_n \Phi(\|s_n(x_n - p) + (1 - s_n)(x_{n+1} - p)\|) \\
&\quad + \alpha_n \langle f(p) - p, j_\varphi(x_{n+1} - p) \rangle \\
&\leqslant \alpha_n \rho \Phi(\|x_n - p\|) + \beta_n \Phi(\|x_n - p\|) + \gamma_n \big(s_n \Phi(\|x_n - p\|) + (1 - s_n)\Phi(\|x_{n+1} - p\|)\big) \\
&\quad + \alpha_n \langle f(p) - p, j_\varphi(x_{n+1} - p) \rangle \\
&= (\alpha_n \rho + \beta_n + \gamma_n s_n)\Phi(\|x_n - p\|) + \gamma_n(1 - s_n)\Phi(\|x_{n+1} - p\|) \\
&\quad + \alpha_n \langle f(p) - p, j_\varphi(x_{n+1} - p) \rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\Phi(\|x_{n+1} - p\|) &\leqslant \frac{\alpha_n \rho + \beta_n + \gamma_n s_n}{1 - \gamma_n(1 - s_n)} \Phi(\|x_n - p\|) + \frac{\alpha_n}{\gamma_n(1 - s_n)} \langle f(p) - p, j_\varphi(x_{n+1} - p) \rangle \\
&= \left(1 - \frac{(1 - \rho)\alpha_n}{1 - \gamma_n(1 - s_n)}\right) \Phi(\|x_n - p\|) + \frac{\alpha_n}{\gamma_n(1 - s_n)} \langle f(p) - p, j_\varphi(x_{n+1} - p) \rangle \\
&= (1 - \theta_n)\Phi(\|x_n - p\|) + \theta_n \sigma_n,
\end{aligned}$$

where $\theta_n = \frac{(1 - \rho)\alpha_n}{1 - \gamma_n(1 - s_n)}$ and $\sigma_n = \frac{1}{1 - \rho} \langle f(p) - p, j_\varphi(x_{n+1} - p) \rangle$. From (C1) and (3.10), we see that $\sum_{n=1}^{\infty} \theta_n = \infty$ and $\limsup_{n \to \infty} \sigma_n \leqslant 0$. We conclude by Lemma 2.13 that $\Phi(\|x_n - p\|) \to 0$ as $n \to \infty$. By the property of Φ, we obtain that $\{x_n\}$ converges strongly to p as $n \to \infty$. This completes the proof. $\square$

Corollary 3.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\mathcal{S} = \{T_t\}_{t \geqslant 0} : C \to C$ be a u.a.r. nonexpansive semigroup such that $F(\mathcal{S}) := \bigcap_{t \geqslant 0} F(T_t) \neq \emptyset$ and f be a contraction on C with coefficient $\rho \in (0, 1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{t_n}\big(s_n x_n + (1 - s_n)x_{n+1}\big), \quad \forall n \geqslant 1. \tag{3.11}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\}$, and $\{t_n\}$ be the same as in Theorem 3.1. Then, $\{x_n\}$ defined by (3.11) converges strongly to a point $p \in F(\mathcal{S})$, which also solves the variational inequality

$$\langle f(p) - x^*, z - p \rangle \leqslant 0, \quad \forall z \in F(\mathcal{S}).$$

4. Some applications

4.1. Convergence theorem for a family of mappings

Definition 4.1. Let C be a subset of a Banach space E. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of mappings such

that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. We say that $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition ([3]) if

$$\sum_{n=1}^{\infty} \sup_{x \in C} \|T_{n+1}x - T_n x\| < \infty. \tag{4.1}$$

Lemma 4.2 ([3]). *Suppose that $\{T_n\}_{n=1}^{\infty}$ satisfy the AKTT-condition. Then, for any $x \in C$, $\{T_n x\}_{n=1}^{\infty}$ converges strongly to some point of C. Further, let $T : C \to C$ defined by $Tx = \lim_{n \to \infty} T_n x$ for all $x \in C$. Then, $\lim_{n \to \infty} \sup_{x \in C} \|Tx - T_n x\| = 0$.*

In the sequel, we say that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition and T is defined by (4.1) with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Theorem 4.3. *Let C be a nonempty, closed, and convex subset of a real reflexive strictly convex Banach space E, which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a sequence of nonexpansive mappings such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and f be a contraction on C with coefficient $\rho \in (0, 1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n \big(s_n x_n + (1 - s_n) x_{n+1}\big), \quad \forall n \geqslant 1. \tag{4.2}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{s_n\}$ be the same as in Theorem 3.1. Suppose in addition, $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition. Then, $\{x_n\}$ defined by (4.2) converges strongly to a point $p \in \bigcap_{n=1}^{\infty} F(T_n)$, which also solves the variational inequality

$$\langle f(p) - p, j_\varphi(z - p) \rangle \leqslant 0, \quad \forall z \in \bigcap_{n=1}^{\infty} F(T_n).$$

Proof. Following the proof line as in Theorem 3.1, we can show that $\{x_n\}$ is bounded and $\lim_{n \to \infty} \|x_n - T_n x_n\| = 0$. Since $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition, we obtain from Lemma 4.2 that

$$\|T_{n+1}z_n - T_n z_n\| = \|T_{n+1}z_n - T z_n\| + \|T z_n - T_n z_n\|$$
$$\leqslant \sup_{x \in \{z_n\}} \|T_{n+1}x - Tx\| + \sup_{x \in \{z_n\}} \|Tx - T_n x\| \to 0 \text{ as } n \to \infty.$$

On the other hand, we need to show that $\lim_{n \to \infty} \|x_n - T x_n\| = 0$. Again, since $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition, then we obtain that

$$\|x_n - T x_n\| \leqslant \|x_n - T_n x_n\| + \|T_n x_n - T x_n\| \leqslant \|x_n - T_n x_n\| + \sup_{x \in \{x_n\}} \|T_n x - Tx\| \to 0 \text{ as } n \to \infty.$$

Some parts of the proof are also the same as the Theorem 3.1. Then, we can obtain the desired conclusion easily. This completes the proof. $\square$

Example 4.4. Let $C = E = \mathbb{R}$ with the usual norm. For each $n \geqslant 1$, define T_n by

$$T_n x = \begin{cases} 0, & x = 0, \\ \sin x + \frac{1}{n^2}, & x \neq 0 \end{cases}$$

for all $x \in C$. It is not hard to show that $\{T_n\}_{n=1}^{\infty}$ is nonexpansive and satisfies the AKTT-condition with $\bigcap_{n=1}^{\infty} F(T_n) = F(T) = \{0\}$, where $Tx = \lim_{n \to \infty} T_n x$ for all $x \in C$.

4.2. The problem of finding zeros of accretive operators

Let $A \subset E \times E$ be an operator. We denote by $D(A)$ and $\overline{D(A)}$ the domain of A and closure of $D(A)$, respectively. We say that A is said to be *accretive* if there exists $j_\varphi(x_1 - x_2) \in J_\varphi(x_1 - x_2)$ such that $\langle y_1 - y_2, j_\varphi \rangle \geqslant 0$, where $(x_i, y_i) \in A$ for $i = 1, 2$. We say that A is said to satisfy the range condition

if $\overline{D(A)} = R(I + \lambda A)$ for all $\lambda > 0$, where $R(I + \lambda A)$ is the range of $I + \lambda A$. It is well known that if A is an accretive operator which satisfies the range condition, then we can defined a single-valued mapping $J_\lambda^A : R(I + \lambda A) \to D(A)$ by $J_\lambda = (I + \lambda A)^{-1}$, which is called the *resolvent* of A. We denote by $A^{-1}0$ the set of zeros of A, i.e., $A^{-1}0 = \{x \in D(A) : 0 \in Ax\}$. It is well known that J_λ is nonexpansive and $F(J_\lambda) = A^{-1}0$ (see [27]). We also know the following [16]: For each $\lambda, \mu > 0$ and $x \in R(I + \lambda A) \cap R(I + \mu A)$, it holds that

$$\|J_\lambda x - J_\mu x\| \leqslant \frac{|\lambda - \mu|}{\lambda} \|x - J_\lambda x\|.$$

Lemma 4.5 ([3]). *Let C be a nonempty, closed, and convex subset of a Banach space E. Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ which satisfies the condition $\overline{D(A)} \subset C \subset \bigcap_{\lambda>0} R(I + \lambda A)$. Suppose that $\{\lambda_n\} \subset (0, \infty)$ such that $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ and $\lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0$. Then $\{J_{\lambda_n}\}$ satisfies the AKTT-condition. Consequently, for each $x \in C$, $\{J_{\lambda_n} x\}$ converges strongly to some point of C. Moreover, let $J_\lambda : C \to C$ defined by $J_\lambda x = \lim_{n \to \infty} J_{\lambda_n} x$ for all $x \in C$ and $F(J_\lambda) = \bigcap_{n=1}^{\infty} F(J_{\lambda_n})$, where $\lambda_n \to \lambda$ as $n \to \infty$. Then, $\lim_{n \to \infty} \sup_{x \in C} \|J_\lambda x - J_{\lambda_n} x\| = 0$.*

Theorem 4.6. *Let C be a nonempty, closed, and convex subset of a real reflexive strictly convex Banach space E, which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ which satisfies the condition $\overline{D(A)} \subset C \subset \bigcap_{\lambda>0} R(I + \lambda A)$ and f be a contraction on C with coefficient $\rho \in (0, 1)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{\lambda_n}(s_n x_n + (1 - s_n)x_{n+1}), \quad \forall n \geqslant 1, \tag{4.3}$$

where $\{\lambda_n\}$ is a real sequence in $(0, \infty)$ with $\inf\{\lambda_n : n \in \mathbb{N}\} > 0$ and $\lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\}$ be the same as in Theorem 3.1. Then $\{x_n\}$ defined by (4.3) converges strongly to a point $p \in A^{-1}0$, which also solves the variational inequality

$$\langle f(p) - p, j_\varphi(z - p) \rangle \leqslant 0, \quad \forall z \in A^{-1}(0).$$

Proof. Since $(\{J_{\lambda_n}\}, J_\lambda)$ satisfies the AKTT-condition, by following the proof line in Theorem 4.3, we can conclude the desired conclusion immediately. $\qquad\square$

5. Numerical examples

In this section, we present two numerical experiments to support the main result.

Example 5.1. Let $E = C = \mathbb{R}^2$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, where $x_i, y_i \in \mathbb{R}$ for $i = 1, 2$. Let $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the inner product defined by $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2$ and let $\| \cdot \| : \mathbb{R}^2 \to \mathbb{R}$ be the usual norm defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(\mathbf{x}) = \frac{1}{4}\mathbf{x}$. For each $t \geqslant 0$, let $T_t : \mathbb{R}^2 \to \mathbb{R}^2$ be a u.a.r. nonexpansive semigroup defined by

$$T_t \mathbf{x} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}.$$

It is not hard to see that $\bigcap_{t \geqslant 0} F(T_t) = p = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$. Let $t_n = \frac{n}{2}$, $s_n = \frac{n}{n+1}$, $\alpha_n = \frac{1}{50n+1}$, $\beta_n = \frac{n}{50n+1}$, we have $\gamma_n = \frac{49n}{50n+1}$. So our algorithm (3.1) has the following form:

$$\begin{pmatrix} x_1^{n+1} \\ x_2^{n+1} \end{pmatrix} = \frac{5n+1}{200n+4} \begin{pmatrix} x_1^n \\ x_2^n \end{pmatrix} + \frac{49n}{50n+1} \begin{pmatrix} e^{-n} & 0 \\ 0 & 1 \end{pmatrix} \left[\frac{n}{n+1} \begin{pmatrix} x_1^n \\ x_2^n \end{pmatrix} + \frac{1}{n+1} \begin{pmatrix} x_1^{n+1} \\ x_2^{n+1} \end{pmatrix} \right], \quad \forall n \geqslant 1.$$

Choose $\mathbf{x_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ be the initial point. Then, we obtain the numerical results shown in Table 1 and Figure 1.

Table 1: Numerical results of Example 5.1.

| n	$\mathbf{x}^n = (x_1^n, x_2^n)^{\mathsf{T}}$	$\|\mathbf{x}_n - p\|$
1	$(2.00000000, 3.00000000)^{\mathsf{T}}$	2.00000000
2	$(0.49707287, 3.00000000)^{\mathsf{T}}$	0.49707287
3	$(0.05745620, 3.00000000)^{\mathsf{T}}$	0.05745620
4	$(0.00337702, 3.00000000)^{\mathsf{T}}$	0.00337702
5	$(0.00012027, 3.00000000)^{\mathsf{T}}$	0.00012027
6	$(3.18033e\text{-}06, 3.00000000)^{\mathsf{T}}$	3.18033e-06
7	$(7.26788e\text{-}08, 3.00000000)^{\mathsf{T}}$	7.26788e-08
8	$(1.55817e\text{-}09, 3.00000000)^{\mathsf{T}}$	1.55817e-09
9	$(3.25133e\text{-}11, 3.00000000)^{\mathsf{T}}$	3.25133e-11
10	$(6.70392e\text{-}13, 3.00000000)^{\mathsf{T}}$	6.70392e-13
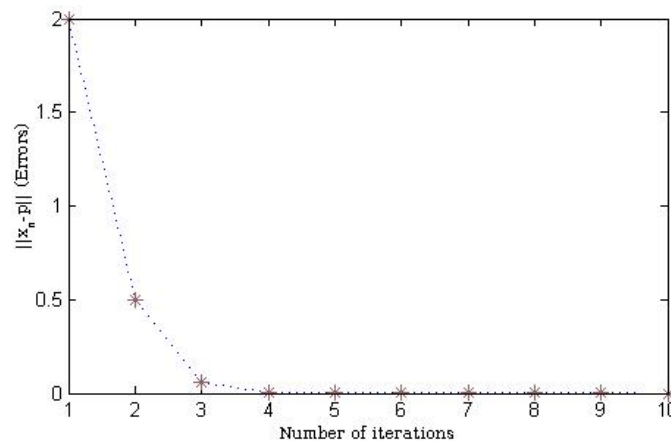
Figure 1: Behavior of convergence error values.

Example 5.2. Let $E = C = \mathbb{R}^3$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$, and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$, where $x_i, y_i \in \mathbb{R}$ for $i = 1, 2, 3$. Let $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the inner product defined by $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$ and let $\| \cdot \| : \mathbb{R}^3 \to \mathbb{R}$ be the usual norm defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}$. For each $t \geqslant 0$, let $T_t : \mathbb{R}^3 \to \mathbb{R}^3$ be a u.a.r. nonexpansive semigroup defined by

$$T_t \mathbf{x} = e^{-t} \begin{pmatrix} \cos \sqrt{2}t & \sin \sqrt{2}t & 0 \\ -\sin \sqrt{2}t & \cos \sqrt{2}t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

It is not hard to see that $\bigcap_{t \geqslant 0} F(T_t) = p = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Let $t_n = 2n$, $s_n = \frac{1}{2}$, $\alpha_n = \frac{1}{2n+1}$, $\beta_n = \frac{n}{2n+1}$, we have $\gamma_n = \frac{n}{2n+1}$. So our algorithm (3.1) has the following form:

$$\begin{pmatrix} x_1^{n+1} \\ x_2^{n+1} \\ x_3^{n+1} \end{pmatrix} = \begin{pmatrix} 0.5x_1^n \\ 0.5x_2^n \\ 0.5x_3^n \end{pmatrix} + \frac{ne^{-2n}}{2n+1} \begin{pmatrix} \cos 2\sqrt{2}n & \sin 2\sqrt{2}n & 0 \\ -\sin 2\sqrt{2}n & \cos 2\sqrt{2}n & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 0.5x_1^n \\ 0.5x_2^n \\ 0.5x_3^n \end{pmatrix} + \begin{pmatrix} 0.5x_1^{n+1} \\ 0.5x_2^{n+1} \\ 0.5x_3^{n+1} \end{pmatrix} \right], \quad \forall n \geqslant 1.$$

Choose $\mathbf{x_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ be the initial point. Then, we obtain the numerical results shown in Table 2 and Figures 2 and 3, respectively.

Table 2: Numerical results of Example 5.2.

| n	$\mathbf{x^n} = (x_1^n, x_2^n, x_3^n)^\mathsf{T}$			$\|\mathbf{x}_n - p\|$
1	(1.00000000,	-1.00000000,	$2.00000000)^\mathsf{T}$	2.44948974
2	(0.45843057,	-0.47840944,	$1.06922917)^\mathsf{T}$	1.25788918
3	(0.23280903,	-0.23985213,	$0.54051127)^\mathsf{T}$	0.63551673
4	(0.11614084,	-0.11996289,	$0.27068651)^\mathsf{T}$	0.31804241
5	(0.05808722,	-0.05997330,	$0.13537353)^\mathsf{T}$	0.15905004
6	(0.02904268,	-0.02998755,	$0.06768886)^\mathsf{T}$	0.07952680
7	(0.01452138,	-0.01499369,	$0.03384457)^\mathsf{T}$	0.03976351
8	(0.00726069,	-0.00749685,	$0.01692230)^\mathsf{T}$	0.01988176
9	(0.00363035,	-0.00374843,	$0.00846115)^\mathsf{T}$	0.00994088
10	(0.00181517,	-0.00187421,	$0.00423057)^\mathsf{T}$	0.00497044
⋮	⋮	⋮	⋮	⋮
20	(1.77263e-06,	-1.83029e-06,	$4.13142e-06)^\mathsf{T}$	4.85395e-06
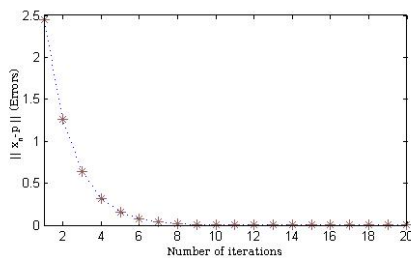
Figure 2: Behavior of convergence error values.
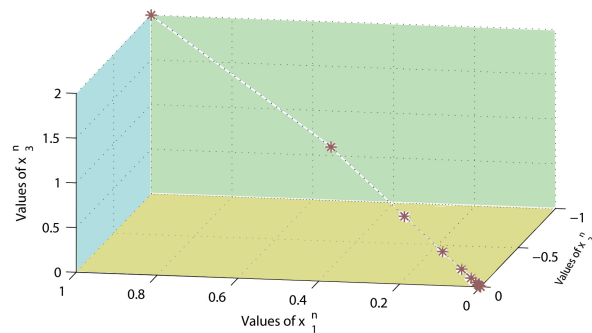
Figure 3: Behavior of convergence values of $\{x^n\}$.

Acknowledgment

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