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On the local convergence of Gargantini-Farmer-Loizou method for simultaneous approximation of multiple polynomial zeros

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Abstract

The paper deals with a well known iterative method for simultaneous computation of all zeros (of known multiplicities) of a polynomial with coefficients in a valued field. This method was independently introduced by Farmer and Loizou [M. R. Farmer, G. Loizou, Math. Proc. Cambridge Philos. Soc., **82** (1977), 427–437] and Gargantini [I. Gargantini, SIAM J. Numer. Anal., **15** (1978), 497–510]. If all zeros of the polynomial are simple, the method coincides with the famous Ehrlich's method [L. W. Ehrlich, Commun. ACM, **10** (1967), 107–108]. We provide two types of local convergence results for the Gargantini-Farmer-Loizou method. The first main result improves the results of [N. V. Kyurkchiev, A. Andreev, V. Popov, Ann. Univ. Sofia Fac. Math. Mech., **78** (1984), 178–185] and [A. I. Iliev, C. R. Acad. Bulg. Sci., **49** (1996), 23–26] for this method. Both main results of the paper generalize the results of Proinov [P. D. Proinov, Calcolo, **53** (2016), 413–426] for Ehrlich's method. The results in the present paper are obtained by applying a new approach for convergence analysis of Picard type iterative methods in finite-dimensional vector spaces.

Keywords: Iterative methods, simultaneous methods, Ehrlich method, multiple polynomial zeros, Gargantini-Farmer-Loizou method, local convergence, error estimates.

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1. Introduction

Throughout this paper $(\mathbb{K}, |\cdot|)$ denotes a valued field with an absolute value $|\cdot|$ (see, e.g. [2]), and $\mathbb{K}[z]$ denotes the ring of polynomials (in one variable) over $\mathbb{K}$. Let $f \in \mathbb{K}[z]$ be an arbitrary polynomial of degree $n \geqslant 2$ which splits in $\mathbb{K}$, and let $\xi_1, \ldots, \xi_s$ be all pairwise distinct zeros of f of known multiplicities $m_1, \ldots, m_s$ $(m_1 + \ldots + m_s = n)$, respectively. We consider the zeros of f as a vector $\xi = (\xi_1, \ldots, \xi_s)$ in the vector space $\mathbb{K}^s$. We equip the space $\mathbb{K}^s$ with the norm $\|x\|_p = (\sum_{i=1}^{s} |x_i|^p)^{1/p}$ for some $1 \leqslant p \leqslant \infty$.

The most famous iterative method for simultaneous computation of all zeros of f with known multiplicities is defined in $\mathbb{K}^s$ by the following fixed-point iteration:

$$x^{(k+1)} = \Phi(x^{(k)}), \qquad k = 0, 1, 2, \ldots, \tag{1.1}$$

where $x^{(0)} \in \mathbb{K}^s$ is an initial approximation of ξ, and the operator $\Phi\colon \mathscr{D} \subset \mathbb{K}^s \to \mathbb{K}^s$ is defined by

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$\Phi(x) = (\Phi_1(x), \ldots, \Phi_s(x))$ with

$$\Phi_i(x) = \begin{cases} x_i - \dfrac{m_i}{\dfrac{f'(x_i)}{f(x_i)} - \displaystyle\sum_{j \neq i} \dfrac{m_j}{x_i - x_j}}, & \text{if } f(x_i) \neq 0, \\ x_i, & \text{if } f(x_i) = 0, \end{cases} \qquad (i = 1, \ldots, s), \tag{1.2}$$

and the domain $\mathscr{D}$ of Φ is the set

$$\mathscr{D} = \left\{ x \in \mathcal{D} \, : \, \frac{f'(x_i)}{f(x_i)} - \sum_{j \neq i} \frac{m_j}{x_i - x_j} \neq 0 \ \text{ whenever } \ f(x_i) \neq 0 \right\}. \tag{1.3}$$

Here and throughout the paper, $\mathcal{D}$ denotes the set of all vectors in $\mathbb{K}^s$ with pairwise distinct components, that is

$$\mathcal{D} = \left\{ x \in \mathbb{K}^s \, : \, x_i \neq x_j \ \text{ whenever } \ i \neq j \right\}.$$

The iterative method (1.1) was independently introduced by Farmer and Loizou [3] in 1977 and Gargantini [4] in 1978, and we call it the *Gargantini-Farmer-Loizou method*. In the case when all the zeros of f are simple ($s = n$ and $m_1 = \cdots = m_s = 1$), the method (1.1) coincides with the famous *Ehrlich's method* [1]. A detailed convergence analysis (local and semilocal) of Ehrlich's method was given in [18, 19]. Some other generalizations of Ehrlich's method, as well as their convergence analysis, can be found in [7, 8, 12–14, 22]. In [9] a family of iterative methods that includes the method (1.1) as particular one was constructed and studied.

Local convergence results for the method (1.1) with error estimates at each iteration were established by Kyurkchiev et al. [9] in 1984 and Iliev [5, 6] in 1996. In these works, the authors find an expression $\mathcal{R}_h$ (in an implicit form) that depends on $n, m_1, \ldots, m_s$ and a variable h. They prove that if an initial approximation $x^{(0)}$ satisfies a condition of the type

$$\|x^{(0)} - \xi\|_\infty \leqslant c\, h \tag{1.4}$$

for some $h \in (0, 1)$ and $c \in [0, \mathcal{R}_h \operatorname{sep}(f)]$, then

$$\|x^{(k)} - \xi\|_\infty \leqslant c\, h^{3^k} \quad \text{for all } \ k \geqslant 0. \tag{1.5}$$

Here and throughout the paper, we denote by $\operatorname{sep}(f)$ the *separation number* of f, which is defined to be the minimum distance between two distinct zeros of f, that is

$$\operatorname{sep}(f) = \min_{i \neq j} |\xi_i - \xi_j|. \tag{1.6}$$

It follows from the above result that the Gargantini-Farmer-Loizou method (1.1) has the order of convergence *three*, and that the open ball $U(\xi, r)$ with center ξ and radius $r = R \operatorname{sep}(f)$ is a convergence ball of this method.

This paper deals with construction of initial conditions that guarantee the convergence of the iterative method (1.1) and provide error bounds at each iteration. For some results on the computational efficiency of the method (1.1), we refer the interested reader to [11] and [15, Section 7.4].

Very recently, Proinov [20] have presented a new approach for convergence analysis (local and semilocal) of the Picard iteration in $\mathbb{K}^s$. In this paper, we apply this approach to the Gargantini-Farmer-Loizou method (1.1). We provide two types of local convergence results for this method. The first main result (Theorem 3.4) improves the results [5, 6, 9] mentioned above as well as some other existing results [10, 23–25]. Both main results of the paper (Theorems 3.4 and 4.3) generalize the local convergence results for Ehrlich's method, which were established in [19].

2. Preliminaries

In this section, we recall two general convergence results of [20] for iterative processes of the type

$$x^{(k+1)} = T(x^{(k)}), \qquad k = 0, 1, 2, \ldots, \tag{2.1}$$

where $T\colon D \subset \mathbb{K}^s \to \mathbb{K}^s$ is an iteration function. These results play a central role in the proofs of the main results.

2.1. Notations

First we introduce some more notations and definitions that we use in the following sections. Let $\mathbb{K}^s$ be equipped with coordinate-wise ordering $\preceq$ defined by

$$x \preceq y \quad \text{if and only if} \quad x_i \leqslant y_i \quad \text{for each} \quad i = 1, \ldots, s,$$

and let $\mathbb{K}^s$ be endowed with the vector-valued norm (cone norm) $\|\cdot\|\colon \mathbb{K}^s \to \mathbb{R}^s$ defined by

$$\|x\| = (\,|x_1|, \ldots, |x_s|\,).$$

Furthermore, for two vectors $x \in \mathbb{K}^s$ and $y \in \mathbb{R}^s$ we denote by $\dfrac{x}{y}$ a vector in $\mathbb{R}^s$ defined by

$$\frac{x}{y} = \left(\frac{|x_1|}{y_1}, \ldots, \frac{|x_s|}{y_s}\right),$$

provided that y has no zero components. Also, we use the function $d\colon \mathbb{K}^s \to \mathbb{R}^s$ defined by

$$d(x) = (d_1(x), \ldots, d_s(x)) \quad \text{with} \quad d_i(x) = \min_{j \neq i} |x_i - x_j| \quad (i = 1, \ldots, s).$$

For a nonnegative integer k and $r \geqslant 1$, $S_k(r)$ stands for the sum

$$S_k(r) = \sum_{0 \leqslant j < k} r^j.$$

Here and throughout the paper, we assume by definition that $0^0 = 1$.

Definition 2.1 ([16])**.** A function $\varphi\colon J \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *quasi-homogeneous* of degree $r \geqslant 0$ if $\varphi(\lambda t) \leqslant \lambda^r \varphi(t)$ for all $\lambda \in [0, 1]$ and $t \in J$.

2.2. General local convergence of the first type

Let $T\colon D \subset \mathbb{K}^s \to \mathbb{K}^s$ be an iteration function, and $\xi \in \mathbb{K}^s$ be a vector with pairwise distinct components. The following theorem study the local convergence of the Picard iteration (2.1) with respect to the function of initial conditions $E\colon \mathbb{K}^s \to \mathbb{R}_+$ defined by

$$E(x) = \left\| \frac{x - \xi}{d(\xi)} \right\|_p \qquad (1 \leqslant p \leqslant \infty). \tag{2.2}$$

Theorem 2.2 ([20])**.** *Let $T\colon D \subset \mathbb{K}^s \to \mathbb{K}^s$ be an iteration function, $\xi \in \mathbb{K}^s$ be a vector with pairwise distinct components, and $E\colon \mathbb{K}^s \to \mathbb{R}_+$ be defined by (2.2). Suppose there exists a quasi-homogeneous function $\phi\colon J \to \mathbb{R}_+$ of degree $m \geqslant 0$ such that for each vector $x \in \mathbb{K}^s$ with $E(x) \in J$, the following two conditions are satisfied:*

(a) $x \in D$;

(b) $\|T(x) - \xi\| \preceq \phi(E(x))\, \|x - \xi\|$.

Let $x^{(0)} \in \mathbb{K}^s$ *be an initial approximation such that*

$$E(x^{(0)}) \in J \quad \text{and} \quad \phi(E(x^{(0)})) < 1. \tag{2.3}$$

Then the Picard iteration (2.1) *is well defined and converges to* ξ *with order* $r = m + 1$ *and with error estimates*

$$\|x^{(k+1)} - \xi\| \preceq \lambda^{r^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \lambda^{S_k(r)} \|x^{(0)} - \xi\| \quad \text{for all } k \geqslant 0,$$

where $\lambda = \phi(E(x^{(0)}))$.

Remark 2.3. We note that if $\phi(R) \leqslant 1$ for some $R \in J$, then the initial conditions (2.3) of Theorem 2.2 can be rewritten in the following equivalent form:

$$E(x^{(0)}) < R, \tag{2.4}$$

where the function E is defined by (2.2).

2.3. General local convergence of the second type

Let $T\colon D \subset \mathbb{K}^s \to \mathbb{K}^s$ be an iteration function, and let $\xi \in \mathbb{K}^s$ be a vector. The next theorem study the local convergence of the Picard iteration (2.1) with respect to the function of initial conditions $E\colon \mathcal{D} \to \mathbb{R}_+$ defined by

$$E(x) = \left\| \frac{x - \xi}{d(x)} \right\|_p \qquad (1 \leqslant p \leqslant \infty). \tag{2.5}$$

In what follows, for a nondecreasing function $\beta\colon J \to \mathbb{R}_+$, we define the functions $\psi, \Psi\colon J \to \mathbb{R}$ by

$$\psi(t) = 1 - bt(1 + \beta(t)), \tag{2.6}$$

$$\Psi(t) = 1 - bt - \beta(t)(1 + bt), \tag{2.7}$$

where $b = 2^{1/q}$ and $1 \leqslant q \leqslant \infty$ is defined by the condition $1/p + 1/q = 1$. It is easy to see that $\Psi = \psi - \beta$. If ψ is positive, we can define the function $\phi\colon J \to \mathbb{R}_+$ by

$$\phi(t) = \frac{\beta(t)}{\psi(t)}. \tag{2.8}$$

Theorem 2.4 ([20])**.** *Let* $T\colon D \subset \mathbb{K}^s \to \mathbb{K}^s$ *be an iteration function,* $\xi \in \mathbb{K}^s$ *be a vector, and* $E\colon \mathcal{D} \subset \mathbb{K}^s \to \mathbb{R}_+$ *be defined by* (2.5)*. Suppose there exists a nonzero quasi-homogeneous function* $\beta\colon J \to \mathbb{R}_+$ *of degree* $m \geqslant 0$ *such that for any* $x \in \mathcal{D}$ *with* $E(x) \in J$*, the following two conditions are satisfied:*

(a) $x \in D$;

(b) $\|T(x) - \xi\| \preceq \beta(E(x)) \|x - \xi\|$.

Let $x^{(0)} \in \mathcal{D}$ *be an initial approximation such that*

$$E(x^{(0)}) \in J \quad \text{and} \quad \Psi(E(x^{(0)})) \geqslant 0, \tag{2.9}$$

where the function Ψ *is defined by* (2.7)*. Then the Picard iteration* (2.1) *is well defined and converges to* ξ *with error estimates*

$$\|x^{(k+1)} - \xi\| \preceq \theta\lambda^{r^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \theta^k \lambda^{S_k(r)} \|x^{(0)} - \xi\| \quad \text{for all } k \geqslant 0,$$

where $r = m + 1$, $\lambda = \phi(E(x^{(0)}))$, $\theta = \psi(E(x^{(0)}))$ *and the functions* ψ *and* ϕ *are defined by* (2.6) *and* (2.8)*, respectively. Besides, if the inequality in* (2.9) *is strict, then the order of convergence is at least* r.

Remark 2.5. We note that if $\Psi(R) \geqslant 0$ for some $R \in J$, then the initial conditions (2.9) of Theorem 2.4 can be rewritten in the following equivalent form:

$$E(x^{(0)}) \leqslant R, \tag{2.10}$$

where the function E is defined by (2.5).

3. Local convergence theorem of the first type

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geqslant 2$ which splits over $\mathbb{K}$ and $\xi_1, \ldots, \xi_s$ be all distinct zeros of f with multiplicity $m_1, \ldots, m_s$, respectively.

In this section, we study the local convergence of the Gargantini-Farmer-Loizou iteration (1.1) with respect to the function of initial conditions $E\colon \mathbb{K}^s \to \mathbb{R}_+$ defined by (2.2) for some $1 \leqslant p \leqslant \infty$. The derivation of our first convergence theorem for this method is based on Theorem 2.2 applied to the operator $\Phi\colon \mathscr{D} \subset \mathbb{K}^s \to \mathbb{K}^s$. In Lemma 3.3 below, we find a quasi-homogeneous function $\phi\colon [0,\tau) \to \mathbb{R}_+$ of the second degree, which satisfies the conditions (a) and (b) of Theorem 2.2.

For the sake of brevity, we define the quantities $a = a(p, m_1, \ldots, m_s)$ and $b = b(p)$ as follows

$$a = \max_{1 \leqslant i \leqslant s} \frac{1}{m_i} \left(\sum_{j \neq i} m_j^q \right)^{1/q} \qquad \text{and} \qquad b = 2^{1/q}, \tag{3.1}$$

where $1 \leqslant q \leqslant \infty$ is defined by $1/p + 1/q = 1$. In Table 1, we give the values of a and b for $p = \infty$ and $p = 1$, where we use the denotations

$$m = \min_{1 \leqslant i \leqslant s} m_i \quad \text{and} \quad M = \max_{1 \leqslant i \leqslant s} m_i\,. \tag{3.2}$$

We note that the quantity a defined in (3.1) was introduced by Proinov and Cholakov [21].

Table 1: Values of a and b for $p = \infty$ and $p = 1$.

$p = \infty$	$q = 1$	$a = n/m - 1$	$b = 2$
$p = 1$	$q = \infty$	$a = M/m$	$b = 1$

In the following lemma, we give lower and upper bounds for the quantities a and b defined by (3.1).

Lemma 3.1. *Let $n \geqslant 2$, $1 \leqslant p \leqslant \infty$, and $m_1, \ldots, m_s$ be positive integers such that $m_1 + \ldots + m_s = n$. Then*

$$(s-1)^{1/q} \leqslant a \leqslant n/m - 1 \quad \textit{and} \quad 1 \leqslant b \leqslant 2, \tag{3.3}$$

where a, b are defined by (3.1) and m is defined in (3.2).

Proof. We prove only the first part of the claim (3.3), since the second is obvious. Let us define the quantities $a_1, \ldots, a_s$ by

$$a_i = \frac{1}{m_i} \left(\sum_{j \neq i} m_j^q \right)^{1/q}, \qquad i = 1, \ldots, s.$$

By the inequality of arithmetic and geometric means, we obtain

$$\left(\sum_{j \neq i} m_j^q \right)^{1/q} \geqslant (s-1)^{1/q} \left(\prod_{j \neq i} m_j \right)^{1/(s-1)}.$$

From this, we get

$$\prod_{i=1}^{s} a_i \geqslant (s-1)^{s/q} \prod_{i=1}^{s} \frac{1}{m_i} \left(\prod_{j \neq i} m_j \right)^{1/(s-1)} = (s-1)^{s/q} \frac{\left(\prod_{i=1}^{s} \prod_{j \neq i} m_j \right)^{1/(s-1)}}{\prod_{i=1}^{s} m_i} = (s-1)^{s/q}.$$

It follows from this that $a_i \geqslant (s-1)^{1/q}$ for some i. Therefore, $a \geqslant (s-1)^{1/q}$ since $a = \max\{a_1, \dots, a_s\}$. The estimate $a \leqslant n/m - 1$ follows from the inequality

$$a_i = \frac{1}{m_i} \left(\sum_{j \neq i} m_j^q \right)^{1/q} \leqslant \frac{1}{m_i} \sum_{j \neq i} m_j = \frac{n}{m_i} - 1 \leqslant \frac{n}{m} - 1.$$

This completes the proof. □

Lemma 3.2 ([19, Lemma 6.1])**.** *Let $x, \xi \in \mathbb{K}^s$ and $1 \leqslant p \leqslant \infty$ $(s \geqslant 2)$. If ξ has pairwise distinct components, then for $i \neq j$ we have*

$$|x_i - x_j| \geqslant (1 - bE(x))\, d_i(\xi) \quad \textit{and} \quad |x_i - \xi_j| \geqslant (1 - E(x))\, d_j(\xi),$$

where $E\colon \mathbb{K}^s \to \mathbb{R}_+$ is defined by (2.2), and b is defined in (3.1).

Lemma 3.3. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geqslant 2$ which splits over $\mathbb{K}$, $\xi_1, \dots, \xi_s$ be all pairwise distinct zeros of f with multiplicity $m_1, \dots, m_s$ and $1 \leqslant p \leqslant \infty$. Suppose $x \in \mathbb{K}^s$ is a vector satisfying the following condition*

$$E(x) < \tau = \frac{2}{b + 1 + \sqrt{(b-1)^2 + 4a}}, \tag{3.4}$$

where the function $E\colon \mathbb{K}^s \to \mathbb{R}_+$ is defined by (2.2) and a, b are defined by (3.1). Then $x \in \mathscr{D}$ and

$$\|\Phi(x) - \xi\| \preceq \phi(E(x))\, \|x - \xi\|, \tag{3.5}$$

where the function $\phi\colon [0, \tau) \to \mathbb{R}_+$ is defined by

$$\phi(t) = \frac{a t^2}{(1-t)(1-bt) - a t^2}. \tag{3.6}$$

Proof. First, we shall prove that $x \in \mathscr{D}$. It follows from Lemma 3.2 that $x \in \mathcal{D}$. Let $f(x_i) \neq 0$ for some i. According to (1.3), it remains to prove that

$$\frac{f'(x_i)}{f(x_i)} - \sum_{j \neq i} \frac{m_j}{x_i - x_j} \neq 0. \tag{3.7}$$

It is well known that if $z \in \mathbb{K}$ is not a zero of f, then

$$\frac{f'(z)}{f(z)} = \sum_{j \neq i} \frac{m_j}{z - \xi_j}.$$

Applying this identity to $z = x_i$, we obtain

$$\frac{f'(x_i)}{f(x_i)} - \sum_{j \neq i} \frac{m_j}{x_i - x_j} = \frac{m_i}{x_i - \xi_i} + \sum_{j \neq i} \left(\frac{m_j}{x_i - \xi_j} - \frac{m_j}{x_i - x_j} \right) = \frac{m_i (1 - \sigma_i)}{x_i - \xi_i}, \tag{3.8}$$

where $\sigma_i \in \mathbb{K}$ is defined by

$$\sigma_i = \frac{x_i - \xi_i}{m_i} \sum_{j \neq i} \frac{m_j (x_j - \xi_j)}{(x_i - \xi_j)(x_i - x_j)}. \tag{3.9}$$

According to Lemma 3.1, we have $a \geqslant 0$ and $b \geqslant 1$. Hence, $R \leqslant 1/b$. Then it follows from (3.1) that $E(x) < 1/b$. From the triangle inequality in $\mathbb{K}$, Lemma 3.2, and the definition (2.2) of the function E, we obtain

$$|\sigma_i| \leqslant \frac{|x_i - \xi_i|}{m_i} \sum_{j \neq i} \frac{m_j\, |x_j - \xi_j|}{|x_i - \xi_j|\, |x_i - x_j|} \leqslant \frac{E(x)}{(1 - E(x))(1 - bE(x))} \frac{1}{m_i} \sum_{j \neq i} \frac{m_j\, |x_j - \xi_j|}{d_j(\xi)}. \tag{3.10}$$

By Hölder's inequality and the definition (3.1) of the quantity a, we get

$$\frac{1}{m_i}\sum_{j\neq i}\frac{m_j\,|x_j-\xi_j|}{d_j(\xi)} \leqslant \frac{1}{m_i}\left(\sum_{j\neq i} m_j^q\right)^{\frac{1}{q}}\left(\sum_{j\neq i}\left(\frac{|x_j-\xi_j|}{d_j(\xi)}\right)^p\right)^{\frac{1}{p}} \leqslant a\,E(x). \tag{3.11}$$

From (3.10) and (3.11), we obtain for σ_i the following estimate

$$|\sigma_i| \leqslant \frac{aE(x)^2}{(1-E(x))(1-bE(x))}. \tag{3.12}$$

Let us note that τ, defined in (3.4), is a positive solution of the equation $(1-t)(1-bt)=at^2$. Then from (3.12) and (3.4), we conclude that $|\sigma_i|<1$. This means that $\sigma_i\neq 1$. Then it follows from (3.8) that the inequality (3.7) is satisfied. Hence, $x\in\mathscr{D}$.

Second, we shall prove (3.5). In other words, we have to prove that for every $i=1,\dots,s$, we have

$$|\Phi_i(x)-\xi_i| \leqslant \phi(E(x))\,|x_i-\xi_i|. \tag{3.13}$$

If $x_i=\xi_i$ for some i, then $\Phi_i(x)=\xi_i$ and so (3.13) becomes an equality. Suppose $x_i\neq\xi_i$. In this case, it follows from Lemma 3.2 that $f(x_i)\neq 0$. Then from (1.2) and (3.8), we obtain

$$\Phi_i(x)-\xi_i = x_i-\xi_i-m_i\left(\frac{f'(x_i)}{f(x_i)}-\sum_{j\neq i}\frac{m_j}{x_i-x_j}\right)^{-1} = -\frac{\sigma_i}{1-\sigma_i}(x_i-\xi_i). \tag{3.14}$$

From this and (3.12), we get

$$|\Phi_i(x)-\xi_i| = \frac{|\sigma_i|}{|1-\sigma_i|}|x_i-\xi_i| \leqslant \frac{|\sigma_i|}{1-|\sigma_i|}|x_i-\xi_i| \leqslant \phi(E(x))\,|x_i-\xi_i|,$$

which proves (3.13). This completes the proof. □

Now we are able to state the main result of this section. This result improves the results of Kyurkchiev et al. [9] and Iliev [5, 6] for the Gargantini-Farmer-Loizou iteration (1.1). It also generalizes and improves the results for Ehrlich's method due to Milovanović and Petković [10], Kyurkchiev and Tashev [23], Wang and Zhao [25], and Proinov [19].

Theorem 3.4. *Let $f\in\mathbb{K}[z]$ be a polynomial of degree $n\geqslant 2$ which splits over $\mathbb{K}$, $\xi_1,\dots,\xi_s$ be all distinct zeros of f with multiplicities $m_1,\dots,m_s$ $(m_1+\dots+m_s=n)$, and $1\leqslant p\leqslant\infty$. Let $x^{(0)}\in\mathbb{K}^s$ be an initial approximation satisfying*

$$E(x^{(0)}) = \left\|\frac{x^{(0)}-\xi}{d(\xi)}\right\|_p < R = \frac{2}{b+1+\sqrt{(b-1)^2+8a}}, \tag{3.15}$$

where $\xi=(\xi_1,\dots,\xi_s)$, the function $E\colon\mathbb{K}^s\to\mathbb{R}_+$ is defined by (2.2), and a and b are defined by (3.1). Then the Gargantini-Farmer-Loizou iteration (1.1) is well defined and converges cubically to ξ with error estimates

$$\|x^{(k+1)}-\xi\| \preceq \lambda^{3^k}\|x^{(k)}-\xi\| \quad \textit{and} \quad \|x^{(k)}-\xi\| \preceq \lambda^{(3^k-1)/2}\|x^{(0)}-\xi\| \quad \textit{for all } k\geqslant 0, \tag{3.16}$$

where $\lambda=\phi(E(x^{(0)}))$ and the real function ϕ is defined by (3.6).

Proof. If $s=1$, then the statements hold true trivially because $x^{(k)}=\xi$ for every $k\geqslant 0$. In the nontrivial case $s\geqslant 2$, we shall apply Theorem 2.2 to the iteration function $\Phi\colon\mathscr{D}\subset\mathbb{K}^s\to\mathbb{K}^s$ defined by (1.2). Let $J=[0,\tau)$ and $\phi\colon J\to\mathbb{R}_+$ be defined by (3.6). It is easy to show that ϕ is a quasi-homogeneous function of degree $m=2$ and $\phi(R)=1$. According to Lemma 3.3, ϕ satisfies conditions (a) and (b) of Theorem 2.2. On the other hand, it follows from (3.15) that the initial approximation $x^{(0)}$ satisfies condition (2.4). Now it follows from Theorem 2.2 that the iteration (1.1) is well defined and converges to ξ with order $r=3$ and with error estimates (3.16). This completes the proof. □

It follows from Theorem 3.4 that if an initial condition $x^{(0)} \in \mathbb{K}^s$ satisfies the condition

$$\|x^{(0)} - \xi\| \leqslant R \operatorname{sep}(f),$$

where $\operatorname{sep}(f)$ is defined by (1.6), then the iteration (1.1) converges cubically to ξ. Therefore, the open ball $U(\xi, r)$ with center ξ and radius $r = R \operatorname{sep}(f)$ is a convergence ball of the Gargantini-Farmer-Loizou iteration (1.1).

The main role in Theorem 3.4 is played by the quantity $R = R(p, m_1, \dots, m_s)$ defined in (3.15). In Table 2, we give the values of R for $p = \infty$ and $p = 1$, where m and M are defined by (3.2).

Table 2: Values of R for $p = \infty$ and $p = 1$.

$p = \infty$	$R = 2/(3 + \sqrt{8n/m - 7}\,)$
$p = 1$	$R = 1/(3 + \sqrt{2M/m}\,)$

Remark 3.5. If all zeros of the polynomial f are simple, then Theorem 3.4 coincides with Theorem 2.1 of Proinov [19] for Ehrlich's method. Hence, it generalizes and improves also the previous convergence result of the first type for Ehrlich's method, which belongs to Milovanović and Petković [10], Kyurkchiev and Tashev [23], and Wang and Zhao [25].

Theorem 3.4 can be reformulated in the following equivalent form.

Theorem 3.6. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geqslant 2$ which splits over $\mathbb{K}$, $\xi_1, \dots, \xi_s$ be all distinct zeros of f with multiplicity $m_1, \dots, m_s$ $(m_1 + \dots + m_s = n)$, $1 \leqslant p \leqslant \infty$, and $0 < h < 1$. Suppose $x^{(0)} \in \mathbb{K}^s$ is an initial approximation which satisfies the following condition*

$$E(x^{(0)}) = \left\| \frac{x^{(0)} - \xi}{d(\xi)} \right\|_p \leqslant R_h = \frac{2}{b + 1 + \sqrt{(b-1)^2 + 4a(1 + 1/h)}}\,. \tag{3.17}$$

Then the Gargantini-Farmer-Loizou method (1.1) is well defined and converges cubically to ξ with error estimates

$$\|x^{(k+1)} - \xi\| \preceq h^{3^k} \|x^{(k)} - \xi\| \quad \textit{and} \quad \|x^{(k)} - \xi\| \preceq h^{(3^k-1)/2} \|x^{(0)} - \xi\| \quad \textit{for all } k \geqslant 0. \tag{3.18}$$

Proof. We consider only the nontrivial case $s \geqslant 2$. It follows from the initial condition (3.17), the obvious inequality $R_h < R$ and Theorem 3.4 that the iteration (1.1) is well defined and converges cubically to ξ with error estimates (3.16). The estimates (3.18) follow from (3.16) and $\lambda = \phi(E(x^{(0)})) \leqslant \phi(R_h) = h$. □

Corollary 3.7. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geqslant 2$ which splits over $\mathbb{K}$, $\xi_1, \dots, \xi_s$ be all distinct zeros of f with multiplicity $m_1, \dots, m_s$ $(m_1 + \dots + m_s = n)$, $1 \leqslant p \leqslant \infty$, and $0 < h < 1$. Suppose $x^{(0)} \in \mathbb{K}^s$ is an initial approximation which satisfies the condition*

$$\|x^{(0)} - \xi\|_p \leqslant c\, h \tag{3.19}$$

for some $h \in (0, 1)$ and $c \in [0, \mathcal{R}_h \operatorname{sep}(f)]$, where $\operatorname{sep}(f)$ is defined by (1.6) and $\mathcal{R}_h$ is defined by

$$\mathcal{R}_h = \frac{2}{b + 1 + \sqrt{(b-1)^2 + 4a(1 + 1/h^2)}}\,. \tag{3.20}$$

Then the Gargantini-Farmer-Loizou method (1.1) is well defined and converges cubically to ξ with error estimates

$$\|x^{(k+1)} - \xi\| \preceq (h^2)^{3^k} \|x^{(k)} - \xi\| \quad \textit{and} \quad \|x^{(k)} - \xi\| \preceq h^{3^k-1} \|x^{(0)} - \xi\| \quad \textit{for all } k \geqslant 0, \tag{3.21}$$

and

$$\|x^{(k)} - \xi\|_p \leqslant c\, h^{3^k} \quad \textit{for all } k \geqslant 0. \tag{3.22}$$

Proof. It follows from (3.19) and the inequalities $c \leqslant \mathcal{R}_h \operatorname{sep}(f)$ and $h < 1$ that

$$E(x^{(0)}) = \left\| \frac{x^{(0)} - \xi}{d(\xi)} \right\|_p \leqslant \frac{\|x^{(0)} - \xi\|_p}{\operatorname{sep}(f)} \leqslant \frac{c\,h}{\operatorname{sep}(f)} \leqslant h\,\mathcal{R}_h \leqslant \mathcal{R}_h\,.$$

Therefore, applying Theorem (3.6) with h^2 instead of h, we conclude that the iteration (1.1) is well defined and converges cubically to ξ with error estimates (3.21). Combining the second estimate of (3.21) with the initial condition (3.20), we deduce the estimate (3.22). □

Setting $p = \infty$ in Corollary 3.7, we obtain the following result.

Corollary 3.8. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geqslant 2$ which splits over $\mathbb{K}$, $\xi_1, \ldots, \xi_s$ be all distinct zeros of f with multiplicity $m_1, \ldots, m_s$ $(m_1 + \ldots + m_s = n)$, and $0 < h < 1$. Suppose $x^{(0)} \in \mathbb{K}^s$ is an initial approximation which satisfies the condition (1.4) for some $h \in (0, 1)$ and $c \in [0, R_h \operatorname{sep}(f)]$, where $\mathcal{R}_h$ is defined by*

$$\mathcal{R}_h = \frac{2}{3 + \sqrt{1 + 4(n/m - 1)(1 + 1/h^2)}}\,,$$

$\operatorname{sep}(f)$ *is defined by (1.6), and m is defined in (3.2). Then the Gargantini-Farmer-Loizou method (1.1) is well defined and converges cubically to ξ with error estimates (3.18) and (1.5).*

Remark 3.9. Corollary 3.8 improves in several direction the corresponding result of Kyurkchiev, Andreev and Popov [9] and Iliev [5, 6] for the method (1.1). In these works the authors obtained results of the type of Corollary 3.8 with only the estimate (1.5) and with a smaller R_h which were not obtained in an explicit form.

4. Local convergence theorem of the second type

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geqslant 2$ which splits over $\mathbb{K}$ and let $\xi_1, \ldots, \xi_s$ be all distinct zeros of f with multiplicity $m_1, \ldots, m_s$, respectively.

In this section, we study the local convergence of the Gargantini-Farmer-Loizou iteration (1.1) with respect to the function of initial conditions $E\colon \mathcal{D} \subset \mathbb{K}^s \to \mathbb{R}_+$ defined by (2.5). We begin with a technical lemma.

Lemma 4.1 ([17, Lemma 7.1])**.** *Let $x, \xi \in \mathbb{K}^s$ and $1 \leqslant p \leqslant \infty$ $(s \geqslant 2)$. If x has pairwise distinct components, then for $i \neq j$ we have*

$$|x_i - \xi_j| \geqslant (1 - E(x))\, d_i(x) \quad \textit{and} \quad |x_i - x_j| \geqslant d_j(x),$$

where $E\colon \mathcal{D} \to \mathbb{R}_+$ is defined by (2.5).

We shall derive a convergence theorem for the method (1.1) by applying Theorem 2.4 to the operator $\Phi\colon \mathscr{D} \subset \mathbb{K}^s \to \mathbb{K}^s$. In the next lemma we find a quasi-homogeneous function $\beta\colon [0, \mu) \to \mathbb{R}_+$ of the second degree, which satisfies the conditions (a) and (b) of Theorem 2.4.

Lemma 4.2. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geqslant 2$ which splits over $\mathbb{K}$, and $\xi_1, \ldots, \xi_s$ be all distinct zeros of f with multiplicity $m_1, \ldots, m_s$ and $1 \leqslant p \leqslant \infty$. Suppose $x \in \mathbb{K}^s$ is a vector satisfying the following condition*

$$E(x) = \left\| \frac{x - \xi}{d(x)} \right\|_p < \mu = \frac{2}{1 + \sqrt{1 + 4a}}\,, \tag{4.1}$$

where the function $E\colon \mathcal{D} \subset \mathbb{K}^s \to \mathbb{R}_+$ is defined by (2.5) and a is defined by (3.1). Then $x \in \mathscr{D}$ and

$$\|\Phi(x) - \xi\| \preceq \beta(E(x))\, \|x - \xi\|, \tag{4.2}$$

where the function $\beta\colon [0, \mu) \to \mathbb{R}_+$ is defined by

$$\beta(t) = \frac{a t^2}{1 - t - a t^2}\,. \tag{4.3}$$

Proof. First, we prove that $x \in \mathscr{D}$. Let $f(x_i) \neq 0$ for some i. Let $\sigma_i \in \mathbb{K}$ be again defined by (3.9). From Lemma 4.1 and (3.11), we obtain

$$|\sigma_i| \leqslant \frac{|x_i - \xi_i|}{m_i} \sum_{j \neq i} \frac{m_j\, |x_j - \xi_j|}{|x_i - \xi_j|\, |x_i - x_j|} \leqslant \frac{E(x)}{1 - E(x)}\, \frac{1}{m_i} \sum_{j \neq i} \frac{m_j\, |x_j - \xi_j|}{d_j(\xi)} \leqslant \frac{aE(x)^2}{1 - E(x)}. \tag{4.4}$$

It follows from (4.4) and (4.1) that $|\sigma_i| < 1$ which means that $\sigma_i \neq 1$. Then it follows from (3.8) that (3.7) is satisfied. Therefore, $x \in \mathscr{D}$.

Now, we shall prove (4.2). In other words, we have to prove that for every $i = 1, \ldots, s$, we have

$$|\Phi_i(x) - \xi_i| \leqslant \beta(E(x))\, |x_i - \xi_i|. \tag{4.5}$$

If $x_i = \xi_i$ for some i, then $\Phi_i(x) = \xi_i$ and so (4.5) becomes an equality. In the case $x_i \neq \xi_i$, it follows from Lemma 4.1 that $f(x_i) \neq 0$. Combining (3.14) and (4.4), we get

$$|\Phi_i(x) - \xi_i| = \frac{|\sigma_i|}{|1 - \sigma_i|} |x_i - \xi_i| \leqslant \frac{|\sigma_i|}{1 - |\sigma_i|} |x_i - \xi_i| \leqslant \beta(E(x))\, |x_i - \xi_i|,$$

which proves (4.5). This completes the proof. □

We can now state the main result of this section. This result generalizes and improves the results for Ehrlich's method due to Wang and Zhao [25], Tilli [24], and Proinov [17]. According to Theorem 2.4, using the function β defined by (4.3), we have to define the functions ψ and ϕ by (2.6) and (2.8), respectively. It is easy to calculate that

$$\psi(t) = \frac{(1 - t)(1 - bt) - at^2}{1 - t - at^2} \tag{4.6}$$

and that ϕ is again defined by (3.6).

Theorem 4.3. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geqslant 2$ which splits over $\mathbb{K}$, and $\xi_1, \ldots, \xi_s$ be all distinct zeros of f with multiplicity $m_1, \ldots, m_s (m_1 + \ldots + m_s = n)$ and $1 \leqslant p \leqslant \infty$. Suppose $x^{(0)} \in \mathcal{D}$ is an initial approximation satisfying*

$$E(x^{(0)}) = \left\| \frac{x^{(0)} - \xi}{d(x^{(0)})} \right\|_p \leqslant R = \frac{2}{b + 1 + \sqrt{(b-1)^2 + 8a}}, \tag{4.7}$$

where a and b are defined by (3.1). Then the Gargantini-Farmer-Loizou iteration (1.1) is well defined and converges to ξ with error estimates

$$\|x^{(k+1)} - \xi\| \preceq \theta \lambda^{3^k} \|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \theta^k \lambda^{(3^k - 1)/2} \|x^{(0)} - \xi\| \quad \text{for all } k \geqslant 0, \tag{4.8}$$

where $\lambda = \phi(E(x^{(0)}))$, $\theta = \psi(E(x^{(0)}))$ and the real functions ϕ and ψ are defined by (3.6) and (4.6), respectively. Moreover, the method converges cubically to ξ provided that $E(x^{(0)}) < R$.

Proof. We shall apply Theorem 2.4 to the iteration function $\Phi \colon \mathscr{D} \subset \mathbb{K}^s \to \mathbb{K}^s$ defined by (1.2). Let us define the functions β and Ψ on the interval $J = [0, \mu)$ by (4.3) and (2.7), respectively. It is easy to show that β is quasi-homogeneous of degree $m = 2$. By Lemma 4.2, the conditions (a) and (b) of Theorem 2.4 are fulfilled. It is easy to show that R is a zero of the polynomial $g(t) = (b - 2a)t^2 - (b + 1)t + 1$. This implies that $\Psi(R) = 0$ since $\Psi(t) = g(t)/(1 - at - at^2)$. Then, it follows from (4.7) that $x^{(0)}$ satisfies condition (2.10). Now it follows from Theorem 2.4 that the iteration (1.1) is well defined and converges to ξ with order $r = 3$ and with error estimates (4.8). Besides, the iteration converges cubically to ξ if $E(x^{(0)}) < R$. This completes the proof. □

Remark 4.4. If all zeros of the polynomial f are simple, then Theorem 4.3 coincides with Theorem 3.1 of Proinov [17] for Ehrlich's method. Therefore, it generalizes and improves as well the previous convergence result of the second type for Ehrlich's method, which are due to Wang and Zhao [25] and Tilli [24].

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