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Relative strongly harmonic convex functions and their characterizations

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Abstract

In this paper, we introduce a new class of harmonic convex functions with respect to an arbitrary non-negative function, which is called the strongly general harmonic convex function. We discuss some characterizations of strongly general harmonic convex functions. Relationship with other classes of convex functions are also discussed. Some special cases are discussed as applications of the main results. The ideas and techniques of this paper may be starting point for further research.

Keywords: Harmonic convex function, strongly harmonic convex function, strongly general convex functions.

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1. Introduction

Recently, different ideas and techniques have been used to consider some novel and significant generalization of convex functions and convex sets. These new concepts have played important role in the developments of several branches of pure and applied sciences. A significant class of convex functions is that of strongly convex functions introduced by Polyak [26]. Strongly convex functions are being used to consider to construct some iterative methods for solving variational inequalities and related optimization problems. Merentes and Nikodem [13] obtained the Hermite-Hadamard inequality for strongly convex functions. Azócar et. al. [5] have refined the Hermite-Hadamard type inequalities for strongly convex functions. For various applications of strongly convex functions an variational inequalities, see [1, 3–5, 7, 12–18, 21–23] and the references therein.

Anderson et al. [2] and Iscan [10] have considered and investigated harmonic convex functions, which has appeared as a significant class of convex functions. Noor and Noor [19] have shown that the minimum of the differential harmonic convex functions by a class of variational inequalities, which are

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called harmonic variational inequalities. This shows that the harmonic convex have nice properties as the convex functions enjoy. It is worth mentioning that the field of harmonic variational inequalities is a new subject and needs further development. See also Noor and Noor [20]. Noor et al. [21–23] considered the strongly harmonic convex functions. This class includes the class of harmonic functions as special case [10]. In [21], the authors present relations between strongly harmonic convex (strongly harmonic Jensen convex) and harmonic convex (harmonic Jensen convex) functions. They have proved that each strongly harmonic convex (strongly harmonic Jensen convex function) is in the form $g(x)+ \parallel \frac{1}{x} \parallel$, where $g(x)$ is a harmonic convex function (harmonic Jensen convex function).

Adamek [1] introduced and investigated a new class of strongly convex functions with respective to an arbitrary non-negative function, which is called the strongly general convex function. We would like to point out that strongly general convex functions include strongly convex functions considered by Polyak [26] as a special case. Motivated and inspired by the work of Adamek [1] and Merentes and Nikodem [13], we introduce a new concept of strongly harmonic convex function with respect to an arbitrary non-negative function $F : X \setminus \{0\} \to [0, \infty)$, which is called strongly general harmonic convex function. We discuss some characterizations of strongly general harmonic convex functions (strongly general harmonic J-convex functions). This is the main motivation of this paper. The technique and ideas of this paper may stimulate further research in this dynamic field.

Let $(X, \|\cdot\|)$ be a real normed space, I be a harmonic convex subset of X and c be a positive constant.

Definition 1.1 ([2]). A set $I \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic set, if

$$\frac{xy}{tx + (1-t)y} \in I, \quad \forall x, y \in I, t \in [0, 1].$$

We now consider a new concept of strongly harmonic convex function with respect to an arbitrary function, which is the main motivation of this paper.

Definition 1.2. A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be strongly general harmonic convex function, if there exists a non-negative function $F : X \setminus \{0\} \to [0, \infty)$, such that

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leqslant (1-t)f(x) + tf(y) - t(1-t)F\left(\frac{xy}{x-y}\right), \quad \forall x, y \in I, t \in (0, 1). \tag{1.1}$$

If (1.1) is assumed only for $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{x+y}\right) \leqslant \frac{f(x) + f(y)}{2} - \frac{1}{4}F\left(\frac{xy}{x-y}\right),$$

which is called strongly general harmonic J-convex function.

Remark 1.3. Let $F : X \setminus \{0\} \to [0, \infty)$ be a given function. A function $f : I = [a, b] \subseteq X \setminus \{0\} \to \mathbb{R}$ is strongly general harmonic J-convex (strongly general harmonic convex), if and only if, a function f is strongly general harmonic J-convex (strongly general harmonic convex), where $F^* := \max\{F(-x), F(x)\}$.

If we take $F\left(\frac{xy}{x-y}\right) = c \parallel \frac{x-y}{xy} \parallel^2$ in (1.1), then it reduces to the Definition of strongly harmonic convex functions introduced by Noor et al. [21].

Definition 1.4 ([21]). A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be strongly harmonic convex function with modulus $c > 0$, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leqslant (1-t)f(x) + tf(y) - ct(1-t) \parallel \frac{x-y}{xy} \parallel^2, \quad \forall x, y \in I, t \in (0, 1). \tag{1.2}$$

If (1.2) is assumed only for $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{x+y}\right) \leqslant \frac{f(x) + f(y)}{2} - \frac{c}{4} \parallel \frac{x-y}{xy} \parallel^2,$$

which is called strongly harmonic J-convex function.

2. Main results

We need the following Lemma's in order to prove our main results.

Lemma 2.1. *Let $F : I = [a, b] \subseteq X \setminus \{0\} \to [0, \infty)$ be an even function. If the function F is strongly general harmonic convex function, then*

$$F\left(\frac{x}{1-t}\right) = (1-t)^2 F(x) \quad \forall t \in (0,1)\ x \in I.$$

Proof. Since F is strongly general harmonic convex function, so we have

$$F\left(\frac{1}{\frac{1-t}{x} + \frac{t}{y}}\right) \leqslant (1-t)F(x) + tF(y) - t(1-t)F\left(\frac{1}{\frac{1}{y} - \frac{1}{x}}\right).$$

Putting $y = \infty$ and using the fact that $F(\infty) = 0$ and evenness of F, we have

$$F\left(\frac{x}{1-t}\right) \leqslant (1-t)^2 F(x). \tag{2.1}$$

In order to show the reverse inequality, we put $\frac{1}{x} = -\frac{t}{u}$, $\frac{1}{y} = \frac{1-t}{u}$ and using the fact that $F(\infty) = 0$, we have

$$0 = F(\infty) \leqslant (1-t)F\left(\frac{-u}{t}\right) + tF\left(\frac{u}{1-t}\right) - t(1-t)F(u).$$

Now, using the above inequality, (2.1), and evenness of F, we have

$$t(1-t)F(u) \leqslant (1-t)F\left(\frac{-u}{t}\right) + tF\left(\frac{u}{1-t}\right) \leqslant t^2(1-t)F(u) + tF\left(\frac{u}{1-t}\right).$$

Dividing this inequality by t, we have

$$(1-t)F(u) \leqslant t(1-t)F(u) + F\left(\frac{u}{1-t}\right).$$

Thus

$$(1-t)^2 F(u) \leqslant F\left(\frac{u}{1-t}\right). \tag{2.2}$$

Hence from (2.1) and (2.2), we conclude that

$$F\left(\frac{x}{1-t}\right) = (1-t)^2 F(x), \quad \forall t \in (0,1)\ x \in I,$$

which is the required result. □

If $t = \frac{1}{2}$ in Lemma 2.1, then we have the following result.

Lemma 2.2. *Let $F : I = [a, b] \subseteq X \setminus \{0\} \to [0, \infty)$ be an even function. If the function F is strongly general harmonic J-convex function, then $F(x) = 4F(2x)$ for all $x \in I$.*

We now consider the concept of quadratic harmonic equations.

Definition 2.3. A function $F : X \setminus \{0\} \to [0, \infty)$ is said to be a quadratic harmonic function, if and only if, $F(\frac{xy}{x+y}) + F(\frac{xy}{x-y}) = 2F(x) + 2F(y)$.

Lemma 2.4. *Let* $F : I = [a, b] \subseteq X \setminus \{0\} \to [0, \infty)$ *be an even function. The function* F *is strongly general harmonic* J*-convex function, if and only if,* F *is a quadratic harmonic function.*

Proof. Since F is strongly general harmonic J-convex function, then we have

$$F\left(\frac{2xy}{x+y}\right) \leqslant \frac{F(x)+F(y)}{2} - \frac{1}{4}F\left(\frac{xy}{x-y}\right).$$

Now using Lemma 2.2, we have

$$F\left(\frac{xy}{x+y}\right) + F\left(\frac{xy}{x-y}\right) \leqslant 2F(x) + 2F(y).$$

Now putting $u = \frac{xy}{x+y}$, $v = \frac{xy}{x-y}$ and using once again Lemma 2.2, we have

$$F\left(\frac{uv}{u+v}\right) + F\left(\frac{uv}{u-v}\right) \geqslant 2F(u) + 2F(v).$$

Thus

$$F\left(\frac{xy}{x+y}\right) + F\left(\frac{xy}{x-y}\right) = 2F(x) + 2F(y).$$

The converse is obvious. □

Theorem 2.5. *Let* $F : X \setminus \{0\} \to [0, \infty)$ *be a given quadratic harmonic function. The function* $f : I = [a, b] \subseteq X \setminus \{0\} \to \mathbb{R}$ *is strongly general harmonic* J*-convex function, if and only if the function* $g = f - F$ *is harmonic* J*-convex function.*

Proof. Since F is a quadratic harmonic function, thus using Lemma 2.2, it can be rewritten as

$$\frac{1}{4}F\left(\frac{xy}{x-y}\right) = \frac{F(x)}{2} + \frac{F(y)}{2} - F\left(\frac{2xy}{x+y}\right).$$

Now consider f is strongly general harmonic convex function. Then

$$\begin{aligned}
g\left(\frac{2xy}{x+y}\right) &= f\left(\frac{2xy}{x+y}\right) - F\left(\frac{2xy}{x+y}\right) \\
&\leqslant \frac{f(x)+f(y)}{2} - \frac{1}{4}F\left(\frac{xy}{x-y}\right) - F\left(\frac{2xy}{x+y}\right) \\
&= \frac{f(x)}{2} + \frac{f(y)}{2} - \frac{F(x)}{2} - \frac{F(y)}{2} + F\left(\frac{2xy}{x+y}\right) - F\left(\frac{2xy}{x+y}\right) \\
&= \frac{f(x)-F(x)}{2} + \frac{f(y)-F(y)}{2} \\
&= \frac{g(x)+g(y)}{2}.
\end{aligned}$$

This shows that g is strongly general harmonic J-convex function.

Conversely, if g is harmonic J-convex function, then we have

$$\begin{aligned}
f\left(\frac{2xy}{x+y}\right) &= g\left(\frac{2xy}{x+y}\right) + F\left(\frac{2xy}{x+y}\right) \\
&\leqslant \frac{g(x)+g(y)}{2} + F\left(\frac{2xy}{x+y}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{f(x)-F(x)}{2} + \frac{f(y)-F(y)}{2} + F\left(\frac{2xy}{x+y}\right)\\
&= \frac{f(x)+f(y)}{2} - \frac{F(x)+F(y)}{2} + F\left(\frac{2xy}{x+y}\right)\\
&= \frac{f(x)+f(y)}{2} - \frac{1}{4}F\left(\frac{xy}{x-y}\right).
\end{aligned}$$

This shows that F is strongly general harmonic J-convex function. □

Theorem 2.6. *Let $F : I = [a,b] \subseteq X \setminus \{0\} \to [0,\infty)$ be an even function. The function F is strongly general harmonic convex function, if and only if F is strongly general harmonic affine function.*

Proof. Since F is strongly general harmonic convex function. Thus

$$F\left(\frac{xy}{tx+(1-t)y}\right) \leqslant (1-t)F(x) + tF(y) - t(1-t)F\left(\frac{xy}{x-y}\right). \tag{2.3}$$

Now, putting $\frac{xy}{tx+(1-t)y} = \frac{u}{\sqrt{1-t}}$ and $\frac{xy}{y-x} = v\sqrt{1-t}$ in (2.3) and using Lemma 2.1, we have

$$(1-t)F(u) \leqslant F\left(\frac{uv}{tu+(1-t)v}\right) + t(1-t)F\left(\frac{uv}{u-v}\right) - tF(v),$$

or equivalently

$$F\left(\frac{uv}{tu+(1-t)v}\right) \geqslant (1-t)F(u) + tF(v) - t(1-t)F\left(\frac{uv}{u-v}\right).$$

This together with (2.3) implies that

$$F\left(\frac{uv}{tu+(1-t)v}\right) = (1-t)F(u) + tF(v) - t(1-t)F\left(\frac{uv}{u-v}\right).$$

This implies that the function F is strongly general harmonic affine function.

The converse is obvious. □

Theorem 2.7. *Let $F : I = [a,b] \subseteq X \setminus \{0\} \to [0,\infty)$ be a strongly general harmonic affine function. The function $f : I = [a,b] \subseteq X \setminus \{0\} \to \mathbb{R}$ is strongly general harmonic convex function, if and only if the function $g = f - F$ is harmonic convex function.*

Proof. Since F is strongly general harmonic affine function. Thus

$$t(1-t)F\left(\frac{xy}{x-y}\right) = -F\left(\frac{xy}{tx+(1-t)y}\right) + (1-t)F(x) + tF(y).$$

Now the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant (1-t)f(x) + tf(y) - t(1-t)F\left(\frac{xy}{x-y}\right)$$

can be written in an equivalent form

$$f\left(\frac{xy}{tx+(1-t)y}\right) - F\left(\frac{xy}{tx+(1-t)y}\right) \leqslant (1-t)[f(x)-F(x)] + t[f(y)-F(y)].$$

Taking $g := f - F$, we have

$$g\left(\frac{xy}{tx+(1-t)y}\right) \leqslant (1-t)g(x) + tg(y).$$

The converse is obvious. □

Theorem 2.8. *Let $F : X \setminus \{0\} \to [0, \infty)$ be a given even function and $F^*(x) = \max\{F(-x), F(x)\}$. Then the following conditions are equivalent.*

1. *The function f is strongly general harmonic J-convex, if and only if, the function $g = f - F^*$ is harmonic J-convex function.*
2. *The function F^* is strongly general harmonic J-convex function.*
3. *The function F^* is a quadratic harmonic function.*

Proof. Assuming (1) and taking $g = 0$, we have $F^* = f$. Thus F^* is strongly general harmonic J-convex and from Remark 1.3, F^* is strongly general harmonic J-convex. So, we have (2). Using Theorem 2.5, it follows the implication $(2) \Rightarrow (3)$ and from Lemma 2.2 and Remark 1.3 we deduce the implication $(3) \Rightarrow (2)$. □

Theorem 2.9. *Let $F : X \setminus \{0\} \to [0, \infty)$ be a given even function. The following conditions are equivalent.*

1. *The function f is strongly general harmonic convex, if and only if, the function $g = f - F^*$ is harmonic convex function.*
2. *The function F^* is strongly general harmonic convex function.*
3. *The function F^* is strongly general harmonic affine function.*

Proof. Assuming (1) and taking $g = 0$, we have $F^* = f$. Thus F^* is strongly general harmonic convex and from Remark 1.3, F^* is strongly general harmonic convex. So, we have (2). Using Theorem 2.6, it follows the implication $(2) \Rightarrow (3)$ and from Theorem 2.7 and Remark 1.3 we deduce the implication $(3) \Rightarrow (2)$. □

Conclusion

In this paper, we have defined a new class of harmonic convex functions with respect to a non-negative arbitrary function, which is called the strongly general harmonic convex functions. With appropriate and suitable choice of the arbitrary function, one can obtain several new classes of harmonic convex functions and its variant classes as special cases. Some characterizations of this new class of harmonic convex functions are discussed. Further efforts are needed to establish some integral inequalities for relative strongly harmonic convex functions.

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