Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 1096–1098
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Coincidence for morphisms based on compactness principles

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Communicated by R. Saadati

Abstract

We present some general coincidence results based on coincidence principles for compact morphisms.

Keywords: Coincidence, noncompact morphisms.

2010 MSC: 54H25, 55M20.

1. Introduction

Morphisms (Vietoris fractions) in the sense of Gorniewicz and Granas was introduced in [4] and discussed in detail in the books [3, 6] and in the papers [1, 5, 7, 8]. In this paper we present two general coincidence results for morphisms defined on Hausdorff topological vector spaces. Our theory is based on coincidence principles for compact morphisms.

Now we present some ideas needed in Section 2. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$, where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geqslant 1$, and $H_0(X) \approx K$.

Let X, Y, and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written as $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;
(ii) p is a perfect map, i.e., p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $D(X, Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q). Given two diagrams (p, q) and (p', q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we

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doi: 10.22436/jnsa.011.09.08

Received: 2018-04-10 Revised: 2018-05-28 Accepted: 2018-05-31

write $(p, q) \sim (p', q')$ if there are continuous maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$, and $p \circ g = p'$. The equivalence class of a diagram $(p, q) \in D(X, Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p, q)]$ and is called a morphism from X to Y. We let $M(X, Y)$ be the set of all such morphisms. Note if $(p, q), (p_1, q_1) \in D(X, Y)$ (where $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ and $X \overset{p_1}{\Leftarrow} \Gamma' \overset{q_1}{\to} Y$) and $(p, q) \sim (p_1, q_1)$ then it is easy to see (use $q \circ g = q_1$ and $p \circ g = p_1$ where $g : \Gamma' \to \Gamma$) that for $x \in X$ we have $q_1(p_1^{-1}(x)) = q(p^{-1}(x))$. For any $\phi \in M(X, Y)$ a set $\phi(x) = q\, p^{-1}(x)$ where $\phi = [(p, q)]$ is called an image of x under a morphism ϕ. Let $\phi \in M(X, Y)$ and (p, q) a representative of ϕ. We define $\phi(X) \subseteq Y$ by $\phi(X) = q(p^{-1}(X))$. Note $\phi(X)$ does not depend on the representative of ϕ. Now $\phi \in M(X, Y)$ is called compact provided the set $\phi(X)$ is relatively compact in Y. Note we will identify a map $f : X \to Y$ with the morphism $f = \{X \overset{Id_X}{\Leftarrow} X \overset{f}{\to} Y\} : X \to Y$. Let $X \subseteq Y$. A point $x \in X$ is called a fixed point of a morphism $\phi \in M(X, Y)$ if $x \in \phi(x)$.

Let $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$ be a morphism. We define the coincidence set

$$\text{Coin}(p, q) = \{y \in \Gamma : p(y) = q(y)\}.$$

We say ϕ has a coincidence provided the set $C(\phi) = p(\text{Coin}(p, q))$ is nonempty (i.e., there exists $x \in p(\text{Coin}(p, q))$, i.e., there exists $y \in \Gamma$ with $x = p(y) = q(y)$). Let (p', q') be another representation of ϕ, say $\phi = \{X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y\}$. Note $p(\text{Coin}(p, q)) = p'(\text{Coin}(p', q'))$; to see this note if $x \in p(\text{Coin}(p, q))$ then $x = p(y) = q(y)$ for some $y \in \Gamma$ and now since $(p, q) \sim (p', q')$ and with $f : \Gamma \to \Gamma'$ we have $x = q(y) = q'(f(y))$ and $x = p(y) = p'(f(y))$ so $f(y) \in \Gamma'$ and $x = q'(f(y)) = p'(f(y))$, i.e., $x \in p'(\text{Coin}(p', q'))$. Thus the above definition does not depend on the choice of a representation (p, q). Also $C(\phi) \neq \emptyset$ iff $\text{Coin}(p, q) \neq \emptyset$ for any representation (p, q) of ϕ.

Suppose $\phi \in M(X, X)$ (here $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X\}$) has a coincidence point for (p, q), i.e., suppose there exists $y \in \Gamma$ with $p(y) = q(y)$. Now since p is surjective, there exists $w \in X$ with $y \in p^{-1}(w)$ (note $p(w) = y$) and so $w \in q(p^{-1}(w)) = \phi(w)$ (note $pp^{-1}(w) = w$ and the set $q(p^{-1}(w))$ is the image of w under ϕ) i.e., ϕ has a fixed point. As a result

$$p(y) = q(y),\ y \in \Gamma \ \text{ (and let } w = p(y)) \Leftrightarrow w \in q(p^{-1}(w));$$

note if $w \in q(p^{-1}(w))$, then there exists $y \in p^{-1}(w)$ with $w = q(y)$ so $p(y) \in pp^{-1}(w) = w$ (i.e., $p(y) = w$) and so $p(y) = q(y)$. In particular if the morphism $\phi \in M(X, X)$ (here (p, q) is a representation of ϕ) has a fixed point (say w, i.e., $w \in q(p^{-1}(w))$) then there exists $y \in p^{-1}(w)$ with $q(y) = p(y)$, so ϕ has a coincidence point for (p, q), and now since we can do this argument for any representation (p, q) of ϕ (recall if (p_1, q_1) is another representation of ϕ then $(p, q) \sim (p_1, q_1)$ and as above $q(p^{-1}(w)) = q_1(p_1^{-1}(w))$ so $w \in q_1(p_1^{-1}(w))$ so there exists $y_1 \in p_1^{-1}(w)$ with $q_1(y_1) = p_1(y_1)$), then $\text{Coin}(p, q) \neq \emptyset$ for any representation (p, q) of ϕ, i.e., ϕ has a coincidence.

2. Coincidence theory

We present immediately our main result.

Theorem 2.1. *Let X be a Hausdorff topological vector space, $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X\} \in M(X, X)$ (here Γ is a Hausdorff topological space), and $x_0 \in p(\Gamma)$. Assume the following conditions hold:*

(1) $A \subseteq \Gamma,\ A = p^{-1}(\overline{co}(\{x_0\} \cup q(A)))$ *implies* $\overline{co}(q(A))$ *is compact;*

(2) *for any nonempty convex compact subset K of X and any $\psi \in M(K, K)$ we have that ψ has a coincidence.*

Then ϕ has a coincidence.

Remark 2.2. Conditions (i.e., spaces X and sets K) to guarantee (2) can be found in [4, 7].

Proof. Let $\mathcal{F}$ be the family of all subsets D of Γ with $p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D)\right)\right) \subseteq D$. Note $\mathcal{F} \neq \emptyset$ since $\Gamma \in \mathcal{F}$ (recall p is surjective). Let

$$D_0 = \cap_{D \in \mathcal{F}} D \quad \text{and} \quad D_1 = p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_0)\right)\right). \tag{2.1}$$

We now show $D_1 = D_0$. Now for any $D \in \mathcal{F}$ we have since $D_0 \subseteq D$ that $q(D_0) \subseteq q(D)$, so

$$D_1 = p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_0)\right)\right) \subseteq p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D)\right)\right) \subseteq D,$$

and as a result $D_1 \subseteq D_0$. Next since $D_1 \subseteq D_0$ we have $q(D_1) \subseteq q(D_0)$, so

$$p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_1)\right)\right) \subseteq p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_0)\right)\right) = D_1,$$

and as a result $D_1 \in \mathcal{F}$, so $D_0 \subseteq D_1$. Consequently (see (2.1))

$$D_0 = p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_0)\right)\right). \tag{2.2}$$

Now (1) guarantees that $\overline{co}\left(q(D_0)\right)$ is compact. For convenience let $K = \overline{co}\left(q(D_0)\right)$. Note from (2.2) that $p^{-1}(K) \subseteq D_0$ so $q(p^{-1}(K)) \subseteq q(D_0) \subseteq K$. Also note $\phi \in M(K, K)$ since $\phi|_K = \{K \overset{p_0}{\Leftarrow} p^{-1}(K) \overset{q_0}{\to} K\}$, where p_0 and q_0 denote contractions of the appropriate maps p and q (see [2, pp 214])). Now (2) guarantees that ϕ has a coincidence. □

Theorem 2.3. *Let X be a Hausdorff topological vector space, $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X\} \in M(X, X)$ (here Γ is a Hausdorff topological space), and $x_0 \in p(\Gamma)$. Let*

$$D_0 = p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(\Gamma)\right)\right), \; D_{n+1} = p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_n)\right)\right) \quad \textit{for} \quad n \in \{0, 1, 2, \ldots\}$$

and $D = \cap_{n=0}^{\infty} D_n$. Suppose

$$\overline{co}\left(q(D)\right) \quad \textit{is compact}$$

and assume (2) in Theorem 2.1 holds. Then ϕ has a coincidence.

Proof. The result follows once we show

$$p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D)\right)\right) \subseteq D.$$

To see this, note $D \subseteq \Gamma$, so $q(D) \subseteq q(\Gamma)$ and as a result

$$p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D)\right)\right) \subseteq p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(\Gamma)\right)\right) = D_0.$$

Also $D \subseteq D_0$ implies $q(D) \subseteq q(D_0)$ and as a result

$$p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D)\right)\right) \subseteq p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_0)\right)\right) = D_1.$$

Continuing, we obtain $p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D)\right)\right) \subseteq D_k$ for each $k \in \{0, 1, 2, \ldots\}$ and as a result

$$p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D)\right)\right) \subseteq \cap_{n=0}^{\infty} D_n = D.$$ □

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