



## Numerical methods for solving initial value problems of some kinds of nonlinear impulsive fractional differential equations



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### Abstract

This article is concerned with the numerical solutions for initial value problems of nonlinear impulsive fractional differential equations which are actively studied recently. In this paper we construct numerical schemes for solving initial value problems of I-type impulsive fractional differential equation and II-type impulsive fractional differential equation and estimate their convergence and stability.

**Keywords:** Caputo fractional derivative, impulsive fractional differential equation, difference method, operational matrix method, decomposition method.

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### 1. Introduction

Fractional differential equations have recently proved to be commonly used in different research areas such as engineering, physics, chemistry, economics, etc. Especially, they draw a great application in nonlinear oscillations of earthquakes, see page flow in porous media and in fluid dynamic traffic model, and many physical phenomena [6, 10, 11, 13, 14, 17, 21].

Recently, Mahto et al. [12] investigated the existence and uniqueness of solutions of Caputo impulsive fractional differential equations with  $0 < \alpha < 1$  as below.

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t)), & t \in I = [0, 1], t \neq t_k, \\ \Delta x(t)|_{t=t_k} = I_k(x(t_k^-)), & k = 1, 2, \dots, m, \\ x(0) = 0, \end{cases}$$

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Chang [5] studied the existence of solution for the impulsive fractional differential equations with integral boundary condition as follows.

$$\begin{cases} {}^cD_t^\alpha y(t) = f(t, y(t)), & t \in J = [0, 1], t \neq t_k, \\ \Delta y(t)|_{t=t_k} = I_k(y(t_k^-)), & k = \overline{1, m}, \\ y(0) = \int_0^1 g(s)y(s)ds, \end{cases}$$

In a recent work [3], Belmekki et al. studied the existence of solution to the following periodic boundary value problem for a nonlinear impulsive fractional differential equation by Schaeffer's fixed point theorem.

$$\begin{cases} D_{t_k^+}^\alpha \mu(t) - \lambda \mu(t) = f(t, \mu(t)), & t \in (t_k, t_{k+1}), k = 0, 1, \dots, p, \\ \lim_{t \rightarrow t_k^+} (t - t_k)^{1-\delta} (\mu(t) - \mu(t_k)) = I_k(\mu(t_k)), & k = 1, 2, \dots, p, \\ \lim_{t \rightarrow 0^+} t^{1-\delta} \mu(t) = \mu(1). \end{cases}$$

And they [1, 2, 4, 7, 22, 23] also investigated the existence and uniqueness of solutions for impulsive fractional differential equations.

While almost papers about the impulsive fractional differential equations were written in respect to the existence and uniqueness of solutions, few papers treated the numerical method for solving impulsive fractional differential equations.

Randelic et al. [18] studied a difference approximation algorithm for the integer order impulsive differential equation with initial condition and gave an example, but they did not investigate the numerical analysis for their method.

Yang et al. [24] proposed an effective numerical method for time-space fractional Fokker-Planck equation and proved the stability and convergence of their method, but it is easy to see that the constants of stability and convergence grow as the iteration number increases.

$$\begin{aligned} \frac{\partial^\alpha \mu(x, t)}{\partial t^\alpha} &= \left[ \frac{\partial}{\partial x} \frac{V'(x)}{m \eta_\alpha} + K_\alpha^\mu \frac{\partial^\mu}{\partial |x|^\mu} \right] \mu(x, t) + S(\mu(x, t), x, t), \\ \mu(a, t) &= \mu(b, t) = 0, 0 \leq t \leq T, \\ \mu(x, 0) &= \mu_0(x), a < x < b. \end{aligned}$$

Hariharan et al. [9] showed broadly that Haar wavelet method is a very effective and powerful tool in solving linear and nonlinear differential equations. Habibollah saeedi et al. [20] reduced the fractional Volterra and Abel integral equations to a system of algebraic equations, estimated a global error bound and gave some numerical examples by using Haar wavelet operational matrix. Neamaty et al. [15] proposed a solving method for fractional partial differential equation with Caputo fractional derivative by using Haar wavelet operational matrix and gave a numerical example.

Motivated by the above researches we firstly study the effective difference scheme for the following impulsive fractional differential equation (II-type impulsive fractional differential equation) and estimated the stability and convergence of our scheme

$$\begin{cases} {}^cD_{t_k}^\alpha x(t) = f(t, x(t)), & t \in (t_k, t_{k+1}), k = 0, 1, \dots, m, \\ \Delta x(t)|_{t=t_k} = I_k(\mu(t_k^-)), & k = 1, 2, \dots, m, \\ x(0) = 0. \end{cases}$$

And we propose the wavelet operational matrix method for the other kind of impulsive fractional differential equation (I-type impulsive fractional differential equation), analyze the convergence of our method, and give an numerical example.

$$\begin{cases} {}^cD_0^\alpha x(t) = f(t, x(t)), & t \in (0, 1], t \neq t_k \in (0, 1) \\ \Delta x(t)|_{t=t_k} = I_k(\mu(t_k^-)), & k = 1, 2, \dots, l, \\ x(0) = x_0. \end{cases}$$

In above two equations, we assume that

$$f \in C(I \times X, X), \quad I_k : X \rightarrow X, \quad 0 < \alpha < 1,$$

and put as follows.

$$\begin{aligned} 0 &= t_0 < t_1 < \cdots < t_m < t_{m+1} = 1, \\ \Delta x(t)|_{t=t_k} &:= x(T_k^+) - x(t_k^-), \\ x(t_k^+) &:= \lim_{h \rightarrow +0} x(t_k + h), x(t_k^-) := \lim_{h \rightarrow +0} x(t_k - h). \end{aligned}$$

## 2. II-type impulsive fractional differential equation and the effective difference scheme

### 2.1. II-type impulsive fractional differential equation

**Definition 2.1.** The equation (2.1) is said to be an II-type impulsive fractional differential equation.

$$\begin{cases} {}^cD_{t_k}^\alpha x(t) = f(t, x(t)), t \in (t_k, t_{k+1}), k = 0, 1, \dots, m, \\ \Delta x(t)|_{t=t_k} = I_k(x(t_k^-)), k = 1, 2, \dots, m, \\ x(0) = 0, \end{cases} \quad (2.1)$$

where

$$\begin{aligned} f &\in C(I \times X, X), \\ I_k &: X \rightarrow X, \\ 0 &= t_0 < t_1 < \cdots < t_m < t_{m+1} = 1, \\ \Delta x(t)|_{t=t_k} &= x(t_k^+) - x(t_k^-), \\ x(t_k^+) &= \lim_{h \rightarrow +0} x(t_k + h), x(t_k^-) = \lim_{h \rightarrow +0} x(t_k - h). \end{aligned}$$

In this section, we assume that  $1/2 < \alpha < 1$ .

### 2.2. The effective difference scheme

Denote as follows:

$$T_k := t_{k+1} - t_k, \quad \tau_k := T_k/N_k, \quad t_n^k := t_k + n\tau_k, \quad N_k \in \mathbb{N}, n = \overline{0, N_k}, k = \overline{0, m}.$$

Firstly, adopting the L1-algorithm [16], we discretize the Caputo time fractional derivative as

$${}^cD^\alpha x(t_{n+1}) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^n b_j (x(t_{n+1-j}) - x(t_{n-j})) + O(\tau^{1+\alpha}), \quad (2.2)$$

where  $\tau = T/N$ ,  $t_n = n\tau$ ,  $n = \overline{0, N}$  and  $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$ . The nonlinear function  $f$  can be discretized as

$$f(t_{n+1}^k, x(t_{n+1}^k)) = f(t_n^k, x(t_n^k)) + O(\tau_k). \quad (2.3)$$

Therefore, using (2.2)-(2.3), we have

$$\frac{\tau_k^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^n b_j (x(t_{n+1-j}^k) - x(t_{n-j}^k)) = f(t_n^k, x(t_n^k)) + O(\tau_k). \quad (2.4)$$

Put  $\mu_k := \tau_k^\alpha \Gamma(2 - \alpha)$ . From (2.4) we can get

$$x(t_{n+1}^k) = b_n x(t_0^k) + \sum_{j=0}^{n-1} (b_j - b_{j+1}) x(t_{n-j}^k) + \mu_k f(t_n^k, x(t_n^k)) + R_{n+1}^k, \quad (2.5)$$

where there exists a positive constant  $c_1$  such that  $|R_{n+1}^k| \leq c_1 \tau_k^{1+\alpha}$ . Let  $x_n^k$  be the numerical approximation of  $x(t_n^k)$ , and  $f_n^k$  the numerical approximation of  $f(t_n^k, x(t_n^k))$ . Then we obtain the following effective difference scheme for the first equation of problem (2.1):

$$x_{n+1}^k = b_n x_0^k + \sum_{j=0}^{n-1} (b_j - b_{j+1}) x_{n-j}^k + \mu_k f_n^k, \quad x_1^k = b_0 x_0^k + \mu_k f_0^k, \quad n = \overline{1, N_k - 1}, k = \overline{0, m}. \quad (2.6)$$

The second expression of problem (2.1) can be written as follows:

$$x(t_{k+1} + 0) = x(t_{k+1} - 0) + I_{k+1}(x(t_{k+1} - 0)), k = \overline{0, m}.$$

Thus, we obtain the approximation

$$x_0^k = x_{N_{k-1}}^{k-1} + I_k(x_{N_{k-1}}^{k-1}), k = 1, 2, \dots, m, \quad x_0^0 = x_0. \quad (2.7)$$

Finally, we can get the effective difference scheme (2.6)-(2.7).

In the sequel, we assume that the functions  $f$  and  $I_k$  are Lipschitz continuous, that is,

$$\exists L > 0; \forall x, y \in \mathbb{R}, |f(t, x) - f(t, y)| \leq L|x - y|, \quad (2.8)$$

$$\exists L_k > 0; \forall x, y \in \mathbb{R}, |I_k(x) - I_k(y)| \leq L_k|x - y|. \quad (2.9)$$

### 2.3. Stability of the effective difference scheme

Let  $x_n^k$  and  $\tilde{x}_n^k$  be the approximate solutions of (2.6)-(2.7) with different initial conditions  $x_0^0 = x_0$  and  $\tilde{x}_0^0 = x_0 + \delta$ .

Denote as follows:

$$\rho_n^k := x_n^k - \tilde{x}_n^k, \quad f_n^k := f(t_n^k, x_n^k), \quad \tilde{f}_n^k := f(t_n^k, \tilde{x}_n^k).$$

**Definition 2.2.** The effective difference scheme (2.6)-(2.7) is said to be stable if

$$\exists c_0 > 0; \max_{n,k} |\rho_n^k| \leq c_0 |\rho_0^0|.$$

**Lemma 2.3.** The effective difference scheme (2.6)-(2.7) is stable if

$$\begin{aligned} \tau_k^\alpha &\leq \frac{1}{\Gamma(2-\alpha)L} \min\left\{\sum_{j=1}^n b_j e^{-(n\tau_k)^{1-\alpha}} (e^{((n-j+1)\tau_k)^{1-\alpha}} - e^{((n-j)\tau_k)^{1-\alpha}}), (e^{\tau_k^{1-\alpha}} - 1)\right\}, \\ n &= \overline{1, N_k}, k = \overline{0, m}. \end{aligned} \quad (2.10)$$

*Proof.* Put  $\beta := 1 - \alpha$  and  $\|\rho_n^k\|_{n,k} := e^{-(n\tau_k)^\beta} |\rho_n^k|$ . It is obvious that

$$e^{-1} |\rho_n^k| \leq \|\rho_n^k\|_{n,k} \quad (2.11)$$

holds.

i) The case of  $k = 0$ .

At first we can get  $\|\rho_0^0\|_{0,0} = |\delta|$ , when  $n = 0$ . For  $n = 1$ , after some manipulation, we obtain

$$|\rho_1^0| \leq b_0 |\rho_0^0| + \mu_0 L |\rho_0^0|, \quad \|\rho_1^0\|_{1,0} \leq e^{-\tau_0^\beta} (b_0 + \mu_0 L) |\rho_0^0|.$$

From (2.10) we have

$$e^{-\tau_0^\beta} (b_0 + \mu_0 L) \leq e^{-\tau_0^\beta} (1 + \Gamma(2 - \alpha) \cdot \frac{1}{\Gamma(2 - \alpha)L} (e^{-\tau_0^{1-\alpha}} - 1) \cdot L) = 1.$$

Thus we can see easily that

$$\|\rho_1^0\|_{1,0} \leq \|\rho_0^0\|_{0,0}.$$

Now suppose that

$$\|\rho_n^0\|_{n,0} \leq \|\rho_0^0\|_{0,0}, \quad n = 2, 3, \dots, l,$$

when  $n = l + 1$ , from (2.6) we have

$$|\rho_{l+1}^0| \leq b_l |\rho_0^0| + \sum_{j=0}^{l-1} (b_j - b_{j+1}) |\rho_{l-j}^0| + \mu_0 L |\rho_l^0|.$$

Therefore, we obtain

$$\begin{aligned} \|\rho_{l+1}^0\|_{l+1,0} &= e^{-((l+1)\tau_0)^\beta} |\rho_{l+1}^0| \\ &\leq b_l e^{-((l+1)\tau_0)^\beta} |\rho_0^0| + \sum_{j=0}^{l-1} (b_j - b_{j+1}) e^{-((l+1)\tau_0)^\beta + ((l-j)\tau_0)^\beta} \|\rho_{l-j}^0\|_{l-j,0} \\ &\quad + \mu_0 L e^{-((l+1)\tau_0)^\beta + (l\tau_0)^\beta} \|\rho_l^0\|_{l,0} \\ &\leq \left[ b_l e^{-((l+1)\tau_0)^\beta} + \sum_{j=0}^{l-1} (b_j - b_{j+1}) e^{-((l+1)\tau_0)^\beta + ((l-j)\tau_0)^\beta} + \mu_0 L e^{-((l+1)\tau_0)^\beta + (l\tau_0)^\beta} \right] \|\rho_0^0\|_{0,0} \\ &= \left[ b_l e^{-(l\tau_0)^\beta} + \sum_{j=1}^{l-1} (b_j - b_{j+1}) e^{-(l\tau_0)^\beta + ((l-j)\tau_0)^\beta} = b_0 - b_1 + \mu_0 L \right] e^{-((l+1)\tau_0)^\beta + (l\tau_0)^\beta} \|\rho_0^0\|_{0,0} \\ &\leq \left[ b_l e^{-(l\tau_0)^\beta} + \sum_{j=1}^{l-1} (b_j - b_{j+1}) e^{-(l\tau_0)^\beta + ((l-j)\tau_0)^\beta} - b_1 + \mu_0 L + 1 \right] \|\rho_0^0\|_{0,0}. \end{aligned}$$

Define  $C := b_l e^{-(l\tau_0)^\beta} + \sum_{j=1}^{l-1} (b_j - b_{j+1}) e^{-(l\tau_0)^\beta + ((l-j)\tau_0)^\beta} - b_1 + \mu_0 L$ . Then we have

$$\begin{aligned} C &= b_l e^{-(l\tau_0)^\beta} + \sum_{j=1}^{l-1} b_j e^{-(l\tau_0)^\beta + ((l-j)\tau_0)^\beta} - \sum_{j=1}^{l-1} b_{j+1} e^{-(l\tau_0)^\beta + ((l-j)\tau_0)^\beta} - b_1 + \mu_0 L \\ &= \sum_{j=1}^l b_j e^{-(l\tau_0)^\beta + ((l-j)\tau_0)^\beta} - \sum_{j=0}^{l-1} e^{-(l\tau_0)^\beta + ((l-j)\tau_0)^\beta} + \mu_0 L \\ &= \sum_{j=1}^l b_j e^{-(l\tau_0)^\beta + ((l-j)\tau_0)^\beta} - \sum_{j=1}^l b_j e^{-(l\tau_0)^\beta + ((l-j+1)\tau_0)^\beta} + \mu_0 L \\ &= \sum_{j=1}^l b_j e^{-(l\tau_0)^\beta} \left( e^{((l-j)\tau_0)^\beta} - e^{((l-j+1)\tau_0)^\beta} \right) + \mu_0 L. \end{aligned}$$

Using (2.10), we obtain

$$C \leq -\Gamma(2-\alpha)L\tau_0^\alpha + \mu_0 L = 0.$$

Thus we get

$$\|\rho_{l+1}^0\|_{l+1,0} \leq \|\rho_0^0\|_{0,0}.$$

Consequently, we can see that

$$\|\rho_n^0\|_{n,0} \leq \|\rho_0^0\|_{0,0}, n = \overline{1, N_0}. \quad (2.12)$$

Substituting (2.11) into (2.12) shows that

$$|\rho_n^0| \leq e\|\rho_n^0\|_{n,0} \leq e\|\rho_0^0\|_{0,0} = e|\delta|.$$

ii) The case of  $k = 1$ .

From (2.7), we have

$$x_0^k = x_{N_{k-1}}^{k-1} + I_k(x_{N_{k-1}}^{k-1}), \quad \tilde{x}_0^k = \tilde{x}_{N_{k-1}}^{k-1} + I_k(\tilde{x}_{N_{k-1}}^{k-1}), \quad k = \overline{1, m}.$$

Thus we can see that

$$\rho_0^1 = \rho_{N_0}^0 + I_1(x_{N_0}^0) - I_1(\tilde{x}_{N_0}^0), \quad |\rho_0^1| \leq |\rho_{N_0}^0| + L_1|\rho_{N_0}^0| \leq (1 + L_1)|\rho_{N_0}^0| \leq (1 + L_1)e|\rho_0^0|.$$

Similarly to i), we have

$$|\rho_n^1| \leq (1 + L_1)e^2|\rho_0^0|.$$

iii) The case of  $k \geq 2$ .

For any  $k \geq 2$ , we can see easily that

$$|\rho_n^k| \leq e^{k+1} \prod_{i=1}^k (1 + L_i) \cdot |\rho_0^0|.$$

The proof of Lemma 2.3 is completed.  $\square$

**Lemma 2.4** ([25]). *The coefficients  $b_j$  satisfy*

(i)

$$\forall j \in \mathbb{N}, b_j > 0;$$

(ii)

$$1 = b_0 > b_1 > \dots > b_n, b_n \rightarrow 0 \quad \text{as } n \rightarrow \infty;$$

(iii) when  $0 < \alpha < 1$ ,

$$\lim_{j \rightarrow \infty} \frac{b_j^{-1}}{j^\alpha} = \lim_{j \rightarrow \infty} \frac{j^{-1}}{(1 + j^{-1})^{1-\alpha}} = \frac{1}{1-\alpha}.$$

Thus there is a positive constant  $C$  such that

$$b_j^{-1} \leq CJ^\alpha, j = 0, 1, 2, \dots$$

**Lemma 2.5.** *The coefficients  $b_j$  satisfy*

- (i)  $\forall j \in \mathbb{N}, b_j b_{j+1} \leq b_{j-1} b_{j+1}$ ;
- (ii)  $\min_{j \in \{1, 2, \dots, n\}} = \begin{cases} b_{n'}^2, n = 2n', n' \in \mathbb{N}, \\ b_{n'} b_{n'+1}, n = 2n' + 1, n' \in \mathbb{N}; \end{cases}$
- (iii)  $b_2 < 2^\alpha b_3 < 3^\alpha b_4 < \dots < n^\alpha b_{n+1} < \dots$ .

*Proof.*

(i). We can get

$$\frac{b_{j+1}}{b_j} = \frac{(j+1)^\beta - (j-1)^\beta}{(j+1)^\beta - j^\beta} = \frac{(1+2/j)^\beta - (1+1/j)^\beta}{(1+1/j)^\beta - 1}.$$

Let  $F(h) := \frac{(1+2h)^\beta - (1+h)^\beta}{(1+h)^\beta - 1}$ . Then we obtain

$$\begin{aligned} F'(h) &= \frac{\beta[(2(1+2h)^{\beta-1} - (1+h)^{\beta-1})((1+h)^\beta - 1) - ((1+2h)^\beta - (1+h)^\beta)(1+h)^{\beta-1}]}{((1+h)^\beta - 1)^2} \\ &= \frac{\beta[2(1+2h)^{\beta-1}(1+h)^\beta - 2(1+2h)^{\beta-1} + (1+h)^{\beta-1} - (1+2h)\beta(1+h)^{\beta-1}]}{((1+h)^\beta - 1)^2} \\ &= \frac{\beta[1 + (1+2h)^{1-\beta} - 2(1+h)^{1-\beta}]}{((1+h)^\beta - 1)^2(1+2h)^{1-\beta}(1+h)^{1-\beta}} \\ &= \frac{\beta[((1+2h)^{1-\beta} - (1+h)^{1-\beta}) - ((1+h)^{1-\beta} - 1)]}{((1+h)^\beta - 1)^2(1+2h)^{1-\beta}(1+h)^{1-\beta}} \\ &= \frac{\beta h(1-\beta)[1/(1+h+\theta_1)^\beta - 1/(1+\theta_2 h)^\beta]}{((1+h)^\beta - 1)^2(1+2h)^{1-\beta}(1+h)^{1-\beta}} < 0, \end{aligned}$$

where  $0 < \theta_1, \theta_2 < 1$ .

Therefore we can see that

$$F(1/(j-1)) < F(1/j).$$

Consequently, we have

$$\frac{b_j}{b_{j-1}} \leq \frac{b_{j+1}}{b_j}. \quad (2.13)$$

(ii). In the case of  $n = 2n', n' \in \mathbb{N}$ , we get

$$\bigcup_{j=1}^n b_j b_{n-j} = \{b_1 b_{n-1}, b_2 b_{n-2}, \dots, b_{n'} b_{n'}, b_{n'+1} b_{n'+1}, \dots, b_n b_0\} = \{b_0 b_n, b_1 b_{n-1}, b_2 b_{n-2}, \dots, b_{n'} b_{n'}\}.$$

On the other hand, using (2.13), for  $k = \overline{1, n'-1}$  we obtain

$$\frac{b_{n'-k}}{b_{n'-k-1}} \leq \frac{b_{n'-k+1}}{b_{n'-k}} = \frac{b_{n'-(k-1)}}{b_{n'-k}} \leq \frac{b_{n'-(k-2)}}{b_{n'-(k-1)}} \leq \dots \leq \frac{b_{n'}}{b_{n'-1}} \leq \frac{b_{n'+1}}{b_{n'}} \leq \dots \leq \frac{b_{n'+k+1}}{b_{n'+k}}.$$

Hence it is obvious that

$$b_{n'-k} \cdot b_{n'+k} \leq b_{n'-(k+1)} \cdot b_{n'+(k+1)}$$

holds.

Therefore, we get

$$\min_j \{b_j b_{n-j}\} = b_{n'}^2.$$

In the case of  $n = 2n' + 1, n' \in \mathbb{N}$ , we have

$$\bigcup_{j=1}^n b_j b_{n-j} = \{b_1 b_{n-1}, b_2 b_{n-2}, \dots, b_{n'} b_{n'+1}, b_{n'+1} b_{n'}, \dots, b_n b_0\} = \{b_0 b_n, b_1 b_{n-1}, b_2 b_{n-2}, \dots, b_{n'} b_{n'+1}\}.$$

Using (2.13), we obtain

$$\frac{b_{n'}}{b_{n'-1}} \leq \frac{b_{n'+1}}{b'_n} \leq \frac{b_{n'+2}}{b_{n'+1}}.$$

Thus, we can get

$$b_{n'} \cdot b_{n'+1} \leq b_{n'-1} \cdot b_{n'+2}.$$

Similarly to the case of  $n = 2n'$ , it should be proved that

$$\min_j \{b_j b_{n-j}\} = b_{n'} \cdot b_{n'+1}.$$

(iii). Obviously, we have

$$n^\alpha b_{n+1} = n^\alpha ((n+2)^{1-\alpha} - (n+1)^{1-\alpha}) = \frac{(1+2/n)^{1-\alpha} - (1+1/n)^{1-\alpha}}{1/n}.$$

Let  $E(h) := \frac{(1+2h)^{1-\alpha} - (1+h)^{1-\alpha}}{h}$ , then we get

$$\begin{aligned} E'(h) &= \frac{[2(1-\alpha)/(1+2h)^\alpha - (1-\alpha)/(1+h)^\alpha]h - [(1+2h)^{1-\alpha} - (1+h)^{1-\alpha}]}{h^2} \\ &= \frac{[2(1-\alpha) - (1-\alpha)((1+2h)/(1+h))^\alpha]h - [(1+2h) - (1+h)((1+2h)/(1+h))^\alpha]}{h^2(1+2h)^\alpha} \\ &= \frac{[2(1-\alpha)h - (1+2h)] + [(1+h) - (1-\alpha)h]((1+2h)/(1+h))^\alpha}{h^2(1+2h)^\alpha} \\ &= \frac{(-1-2\alpha h) + (1+\alpha h)(1+h/(1+h))^\alpha}{h^2(1+2h)^\alpha}. \end{aligned}$$

Since  $(1+h/(1+h))^\alpha < 1 + \alpha h/(1+h)$ , we have

$$E'(h) < \frac{(-1-2\alpha h) + (1+\alpha h)(1+\alpha h/(1+h))}{h^2(1+2h)^\alpha} = \frac{\alpha h(-1 + (1+\alpha h)/(1+h))}{h^2(1+2h)^\alpha} < 0.$$

Thus it is obvious that  $n^\alpha b_{n+1}$  is an increasing sequence.

The proof of Lemma 2.5 is completed.  $\square$

**Theorem 2.6.** *Inequality (2.10) holds if the following condition is satisfied:*

$$\frac{e\Gamma(2-\alpha)L}{2(3^{1-\alpha}-2^{1-\alpha})^2} \leq 1. \quad (2.14)$$

*Proof.* We will prove this theorem in three steps.

(i). Inequality (2.10) holds if the following condition is satisfied.

$$\tau_k^{2\alpha-1} \leq \frac{1}{e\Gamma(2-\alpha)L} \min \left\{ \sum_{j=1}^n b_j ((n-j+1)^{1-\alpha} - (h-j)^{1-\alpha}), 1 \right\}, \quad n = \overline{1, N_k}, k = \overline{0, m}. \quad (2.15)$$

In fact, from (2.15) we have

$$\begin{aligned}\tau_k^\alpha &\leq \frac{\tau_k^{1-\alpha}}{2\Gamma(2-\alpha)L} \sum_{j=1}^n b_j((n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}) \\ &= \frac{1}{e\Gamma(1-\alpha)L} \sum_{j=1}^n b_j(((n-j+1)\tau_k)^{1-\alpha} - ((n-j)\tau_k)^{1-\alpha}) \\ &\leq \frac{1}{\Gamma(2-\alpha)L} \sum_{j=1}^n b_j e^{-(n\tau_k)^{1-\alpha}} (((n-j+1)\tau_k)^{1-\alpha} - ((n-j)\tau_k)^{1-\alpha}).\end{aligned}$$

Since  $x > y > 0 \rightarrow e^x - e^y > x - y$ , we get

$$\tau_k^\alpha \leq \frac{1}{\Gamma(2-\alpha)L} \sum_{j=1}^n b_j e^{-(n\tau_k)^{1-\alpha}} (e^{((n-j+1)\tau_k)^{1-\alpha}} - e^{((n-j)\tau_k)^{1-\alpha}}).$$

On the other hand, from (2.15) we can get

$$\tau_k^\alpha \leq \frac{\tau_k^{1-\alpha}}{e\Gamma(2-\alpha)L} \leq \frac{\tau_k^{1-\alpha} e^{-\tau_k^{1-\alpha}}}{\Gamma(2-\alpha)L} = \frac{(2\tau_k^{1-\alpha} - \tau_k^{1-\alpha}) e^{-\tau_k^{1-\alpha}}}{\Gamma(2-\alpha)L} \leq \frac{(e^{2\tau_k^{1-\alpha}} - e^{\tau_k^{1-\alpha}}) e^{-\tau_k^{1-\alpha}}}{\Gamma(2-\alpha)L} = \frac{e^{\tau_k^{1-\alpha}} - 1}{\Gamma(2-\alpha)L}.$$

(ii). Inequality (2.15) holds if the following condition is satisfied.

$$\tau_k^{2\alpha-1} \leq \frac{2N_k^{1-2\alpha} b_2^2}{e\Gamma(2-\alpha)L}. \quad (2.16)$$

It is obvious that

$$\sum_{j=1}^n b_j((n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}) = \sum_{j=1}^n b_j b_{n-j} \geq n \cdot \min_j \{b_j b_{n-j}\}.$$

Using Lemmas 2.4 and 2.5, we can get

$$\min_j \{b_j b_{n-j}\} \geq b_{n'+1}^2, n' = [n/2].$$

Thus we have

$$\sum_{j=1}^n b_j((n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}) \geq n b_{n'+1}^2 b_{n'+1}^2 \geq 2n' b_{n'+1}^2. \quad (2.17)$$

If  $n = 1$ , then we can see that

$$\sum_{j=1}^n b_j((n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}) = b_1 > b_2 > 2b_2^2 > 2N_k^{1-2\alpha} b_2^2.$$

And if  $n \geq 2$ , by using Lemma 2.5 and (2.17) we get

$$\sum_{j=1}^n b_j((n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}) \geq 2n' b_{n'+1}^2 > 2n' \cdot (n')^{-2\alpha} b_2^2 = 2(n')^{1-2\alpha} b_2^2 \geq 2N_k^{1-2\alpha} b_2^2.$$

On the other hand, since  $\tau_k = T_k/N_k \leq 1$ , we can see easily that

$$\tau_k^{2\alpha-1} \leq 1.$$

Therefore, if the inequality (2.16) is satisfied, then the expression (2.15) holds.

(iii). Inequality (2.16) holds if the condition (2.14) is satisfied.

It can be easily shown that

$$\tau_k^{2\alpha-1} \leq (1/N_k)^{2\alpha-1} = N_k^{1-2\alpha} \leq \frac{2N_k^{1-2\alpha} b_2^2}{e\Gamma(2-\alpha)L}.$$

Thus we can see that if the condition (2.14) is satisfied, then inequality (2.10) holds. This completes the proof.  $\square$

Then we conclude the following.

**Theorem 2.7.** Assuming that the functions  $f$  and  $I_k$  satisfy Lipschitz conditions (2.8)-(2.9) and the condition (2.14) is satisfied, the EDS defined by (2.6)-(2.7) is stable.

#### 2.4. Convergence of the effective difference scheme

Let  $x(t_n^k)$  be the exact solution of the problem (2.1) at mesh point  $t_n^k$ , and  $x_n^k$  the numerical solution of the problem (2.1) computed by using the EDS (2.6)-(2.7).

Denote  $\eta_n^k := x(t_n^k) - x_n^k$ .

Subtracting (2.6) from (2.5) leads to

$$\eta_{n+1}^k = b_n \eta_0^k + \sum_{j=0}^{n-1} (b_j - b_{j+1}) \eta_{n-j}^k + \mu_k [f(t_n^k, x(t_n^k)) - f(t_n^k, x_n^k)] + R_{n+1}^k, \quad n = \overline{1, N_k - 1}, k = \overline{0, m}.$$

From the Lipschitz continuity of the function  $f$ , we get

$$|\eta_{n+1}^k| \leq b_n |\eta_0^k| + \sum_{j=1}^{n-1} (b_j - b_{j+1}) |\eta_{n-j}^k| + \mu_k L |\eta_n^k| + |R_{n+1}^k|.$$

Following a similar argument to that presented above for the stability analysis of the EDS (2.6)-(2.7), we have

$$\begin{aligned} |\eta_1^k| &\leq b_0 |\eta_0^k| + \mu_k L |\eta_0^k| + |R_1^k|, \\ \|\eta_1^k\|_{1,k} &\leq \|\eta_0^k\|_0 + e^{-\tau_k^\beta} |R_1^k|, \\ |\eta_2^k| &\leq b_1 |\eta_0^k| + (b_0 - b_1) |\eta_1^k| + \mu_k L |\eta_1^k| + |R_2^k|, \\ \|\eta_2^k\|_{2,k} &\leq b_1 e^{-(2\tau_k)^\beta} \|\eta_0^k\|_{0,k} + (b_0 - b_1 + \mu_k L) e^{-(2\tau_k)^\beta + \tau_k^\beta} \|\eta_1^k\|_{1,k} + e^{-(2\tau_k)^\beta} |R_2^k| \\ &\leq [b_1 e^{-(2\tau_k)^\beta} + (b_0 - b_1 + \mu_k L) e^{-(2\tau_k)^\beta + \tau_k^\beta}] \|\eta_0^k\|_{0,k} + (b_0 - b_1 + \mu_k L) e^{-(2\tau_k)^\beta} |R_1^k| + e^{-(2\tau_k)^\beta} |R_2^k|. \end{aligned}$$

Since from (2.10) the coefficients of  $\|\eta_0^k\|_{0,k}$  and  $e^{-(2\tau_k)^\beta} |R_1^k|$  are smaller than 1, we get

$$\|\eta_2^k\|_{2,k} \leq \|\eta_0^k\|_{0,k} + e^{-(2\tau_k)^\beta} (|R_1^k| + |R_2^k|).$$

By induction, it can be easily shown that

$$\|\eta_n^k\|_{n,k} \leq \|\eta_0^k\|_{0,k} + e^{-(n\tau_k)^\beta} \sum_{j=1}^n |R_i^k|.$$

Hence we can get

$$\|\eta_n^k\|_{n,k} \leq \|\eta_0^k\|_{0,k} + e^{-(n\tau_k)^\beta} n c_1 \tau_k^{1+\alpha} \leq \|\eta_0^k\|_{0,k} + n c_1 \tau_k^{1+\alpha} \leq \|\eta_0^k\|_{0,k} + c_1 \tau_k^\alpha.$$

Therefore, it is obvious that

$$\max_n |\eta_n^k| \leq e (|\eta_0^k| + c_1 \tau_k^\alpha)$$

holds. In the case of  $k = 0$ , by using  $\eta_0^0 = 0$  we have

$$\max_n |\eta_n^0| \leq c_1 e \tau_0^\alpha.$$

Let  $\tau_{\max} := \max_k \tau_k$ . Then we obtain

$$\max_n |\eta_n^0| \leq c_1 e \tau_{\max}^\alpha.$$

When  $k \geq 1$ , in the similar way to the above we get

$$\begin{aligned} |\eta_0^1| &\leq (1 + L_1) c_1 e \tau_{\max}^\alpha, \\ \max_n |\eta_n^1| &\leq e [(1 + L_1) c_1 e \tau_{\max}^\alpha + c_1 \tau_{\max}^\alpha] < (2 + L_1) c_1 e^2 \tau_{\max}^\alpha, \\ \max_n |\eta_n^k| &\leq 2 (|\eta_0^k| + c_1 \tau_k^\alpha) \leq e^{k+1} c_1 \prod_{i=1}^k (2 + L_i) \cdot \tau_{\max}^\alpha. \end{aligned}$$

Denote  $c_2 := e^{m+1} c_1 \prod_{i=1}^m (2 + L_i)$ . Then we have

$$\max_{n,k} |\eta_n^k| \leq c_2 \tau_{\max}^\alpha.$$

Thus we conclude the following.

**Theorem 2.8.** Assuming that the functions  $f$  and  $I_k$  satisfy the Lipschitz conditions (2.8)-(2.9) and inequality (2.14) holds, the EDS defined by (2.6)-(2.7) is convergent and there exists a positive constant  $c_2$  such that

$$\max_{n,k} |\eta_n^k| \leq c_2 \tau_{\max}^\alpha.$$

### 3. I-type impulsive fractional differential equation and decomposition-operational matrix method

#### 3.1. I-type impulsive fractional differential equation

**Definition 3.1.** The equation (3.1) is said to be an I-type impulsive fractional differential equation.

$$\begin{cases} {}^c D_0^\alpha x(t) = f(t, x(t)), t \in (0, 1], t \neq t_k \in (0, 1), \\ \Delta x(t)|_{t=t_k} = I_k(x(t_k^-)), k = 1, 2, \dots, l, \\ x(0) = x_0, \end{cases} \quad (3.1)$$

where

$$\begin{aligned} f &\in C(I \times X, X), \\ I_k : X &\rightarrow X, \\ 0 &= t_0 < t_1 < \dots < t_l < t_{l+1} = 1, \\ \Delta x(t)|_{t=t_k} &= x(t_k^+) - x(t_k^-), \\ x(t_k^+) &= \lim_{h \rightarrow +0} x(t_k + h), x(t_k^-) = \lim_{h \rightarrow +0} x(t_k - h). \end{aligned}$$

**Lemma 3.2.** Assume that  $g \in C(0, 1) \cap L(0, 1)$  with a  $n^{\text{th}}$  derivative that belongs  $C(0, 1) \cap L(0, 1)$ . Then,

$$I_{0+}^\alpha {}^c D_{\alpha+}^\alpha g(t) = g(t) + c_1 + c_2 t + c_3 t^2 + \dots + c_n t^{n-1},$$

where  $c_i \in \mathbb{R}, i = 1, \dots, n, n = [\alpha] + 1$ .

**Lemma 3.3.** Let  $q \in (0, 1)$  and  $h : J \rightarrow \mathbb{R}$  be continuous. A function  $\mu \in C(J, \mathbb{R})$  is a solution of the fractional integral equation

$$\mu(t) = \mu_0 - \frac{1}{\Gamma(q)} \int_0^a (a-s)^{q-1} h(s) ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds,$$

if and only if  $\mu$  is a solution of the following fractional Cauchy problems

$$\begin{cases} {}^c D_t^q \mu(t) = h(t), t \in J, \\ \mu(a) = \mu_0, a > 0. \end{cases}$$

For the convenience, put  $l = 1$ .

And then the problem we considered is

$${}^c D_0^\alpha x(t) = f(t, x(t)), t \in (0, 1], t \neq t_* \in (0, 1), \quad (3.2)$$

$$\Delta x(t)|_{t_*} = I_*(x(t_*^-)), \quad (3.3)$$

$$x(0) = x_0, \quad (3.4)$$

where  $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ .

Operating the fractional integral operator  $I_0^\alpha$  on both sides of (3.2), we get

$$I_0^\alpha \circ {}^c D_0^\alpha x(t) = I_0^\alpha \circ f(t, x(t)), t \in (0, 1], t \neq t_* \in (0, 1).$$

Then from  $0 < \alpha < 1$  and Lemma 3.2 we have

$$I_{0+}^\alpha {}^c D_0^\alpha x(t) = x(t) + c_1.$$

Therefore we can obtain

$$x(t) = I_0^\alpha \circ f(t, x(t)) - c_1, t \in [0, 1], t \neq t_* \in (0, 1),$$

where  $x(t)$  satisfies (3.3) and (3.4).

When  $t \in (0, t_*)$ , from initial condition (3.4) we have

$$x(0) = I_0^\alpha \circ f(t, x(t))|_{t=0+} - c_1 = x_0.$$

Since  $I_0^\alpha \circ f(t, x(t))|_{t=0+} = 0$ , we get

$$c_1 = -x_0.$$

Thus, we have

$$x(t) = I_0^\alpha f(t, x(t)) + x_0. \quad (3.5)$$

On the other hand, when  $t \in (t_*, 1)$ , from (3.3) we obtain

$$\Delta x(t_*) = x(t_*^+) - x(t_*^-) = I_*(x(t_*^-)), \quad x(t_*^+) = x(t_*^-) + I_*(x(t_*^-)).$$

Using Lemma 3.3 we get

$$x(t) = x_0 + I_*(x(t_*^-)) + I_0^\alpha f(t, x(t)). \quad (3.6)$$

From (3.5) and (3.6) we can obtain the following integral equation

$$x(t) = x_0 + I_*(x(t_*^-))\chi(t - t_*) + I_0^\alpha \circ f(t, x(t)), \quad (3.7)$$

where  $\chi(t) = \begin{cases} 1, t > 0 \\ 0, t \leq 0 \end{cases}$ ,  $x_{0*} := \begin{cases} x_0, t < t_*, \\ x_*^-, t > t_*. \end{cases}$

Therefore we can see that if we can solve the integral equation (3.7), we can also get a solution of the problem (3.2)-(3.4).

The purpose of this paper is to propose a decomposition-operational matrix method that is a fast computing method by which we can compute a numerical solution of (3.7) at some discrete points.

### 3.2. Decomposition-operational matrix method

$f, I_*$  are nonlinear functions, so we will use decomposition method.

$$(H3.1) \quad x(t) := \sum_{n=0}^{\infty} x_n(t). \quad (3.8)$$

Equation (3.8) is satisfied if  $f$  is an analytic function in respect to second parameter.

From (3.7) we get

$$\sum_{n=0}^{\infty} x_n(t) = x_0 + I_*(\sum_{n=0}^{\infty} x_n(t_*^-))\chi(t - t_*) + I_0^\alpha \circ f(t, \sum_{n=0}^{\infty} x_n(t)).$$

And then we can rewrite above expression as

$$\begin{aligned} \sum_{n=0}^{\infty} x_n(t) &= x_0 + I_*(\sum_{n=0}^{\infty} x_n(t_*^-))\chi(t - t_*) + I_0^\alpha \circ f(t, \sum_{n=0}^{\infty} x_n(t))x_0 \\ &\quad + \sum_{n=1}^{\infty} (I_*(\sum_{j=0}^n x_j(t_*^-)) - I_*(\sum_{j=0}^{n-1} x_j(t_*^-)))\chi(t - t_*) \\ &\quad + I_*(x_0(t_*^-))\chi(t - t_*) + I_0^\alpha \circ \sum_{n=1}^{\infty} (f(t, \sum_{i=0}^n x_i(t)) - f(t, \sum_{i=0}^{n-1} x_i(t))) + I_0^\alpha \circ f(t, x_0(t)). \end{aligned}$$

From this we construct the following decomposition algorithm.

$$x_0(t) = x_0, t \in (0, 1],$$

$$x_1(t) = I_0^\alpha \circ f(t, x_0(t)) + I_*(x_0(t_*^-))\chi(t - t_*),$$

$$x_{n+1}(t) = I_0^\alpha \circ (f(t, \sum_{j=0}^n x_j(t)) - f(t, \sum_{j=0}^{n-1} x_j(t))) + (I_*(\sum_{j=0}^n x_j(t_*^-)) - I_*(\sum_{j=0}^{n-1} x_j(t_*^-)))\chi(t - t_*), n = 1, 2, \dots$$

Define  $\Delta t := 1/m = 1/2^r, r \in \mathbb{N}, t_k := (k - 0.5)\Delta t, k = \overline{1, m}$ . For the convenience, put  $t_* = t_{m/2}$ .

**Definition 3.4 ([8]).** The  $m$ -set of Block-Pulse functions(BPF) is given by

$$b_i(t) = \begin{cases} 1, & \frac{i}{m} \leq t < \frac{i+1}{m}, i = \overline{0, m-1}, \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 3.5 ([8]).** A pair of integers  $(j, k)$  satisfying the following conditions is said to be a integer decomposition of index  $i$ .

$$i = k + 2^j - 1, \quad 0 \leq j < i, \quad 1 \leq k < 2^j + 1.$$

**Definition 3.6 ([9]).** The Haar wavelet function is given by

$$h_0(t) = \frac{1}{\sqrt{m}}, \quad h_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 2^{\frac{j}{2}}, & \frac{k-1}{2^j} < t \leq \frac{k-1/2}{2^j}, \\ -2^{\frac{j}{2}}, & \frac{k-1/2}{2^j} < t \leq \frac{k}{2^j}, \\ 0, & \text{otherwise,} \end{cases}$$

where  $(j, k)$  is a integer decomposition of  $i$ .

**Definition 3.7.** The Haar wavelet matrix for a set of collocation points  $(t_k)$  is given by

$$H_{\text{matrix}} := \begin{pmatrix} h_0(t_1) & h_0(t_2) & \cdots & h_0(t_m) \\ h_1(t_1) & h_1(t_2) & \cdots & h_1(t_m) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ h_{m-1}(t_1) & h_{m-1}(t_2) & \cdots & h_{m-1}(t_m) \end{pmatrix}.$$

Similarly we can define Block-Pulse function matrix. From the definition of Block-Pulse function, we can easily see that  $B_{\text{matrix}} = I$ .

**Lemma 3.8 ([8]).**  $F_B^\alpha$ , the operational matrix of Block-Pulse function vector  $B(t) = (B_1(t), B_2(t), \dots, B_m(t))^T$ , is given by

$$F_B^\alpha = \frac{1}{m^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where  $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$ .

**Theorem 3.9.**  $F_H^\alpha$ , the operational matrix of Haar wavelet function vector  $H(t) = (h_0(t), h_1(t), \dots, h_{m-1}(t))^T$ , is given by

$$F_H^\alpha = H_{\text{matrix}} F_B^\alpha H_{\text{matrix}}^T.$$

*Proof.* We have

$$\exists C \in \mathbb{R}^{m \times m}; H(t) = CB(t).$$

Then we can easily get

$$H_{\text{matrix}} = CB_{\text{matrix}}.$$

Since  $B_{\text{matrix}} = I$ , we have

$$C = H_{\text{matrix}}.$$

We obtain

$$I_0^\alpha H(t) = I_0^\alpha B(t) = CI_0^\alpha B(t) = CF_B^\alpha B(t) = H_{\text{matrix}} F_B^\alpha B(t), \quad I_0^\alpha H(t) = F_H^\alpha H(t).$$

And we get

$$F_H^\alpha H(t) = H_{\text{matrix}} F_B^\alpha B(t).$$

Therefore we can see

$$F_H^\alpha H_{\text{matrix}} = H_{\text{matrix}} F_B^\alpha B_{\text{matrix}} = H_{\text{matrix}} F_B^\alpha, \quad F_H^\alpha = H_{\text{matrix}} F_B^\alpha H_{\text{matrix}}^T.$$

The proof is completed.  $\square$

Denote as follows:

$$H_m(t) := (h_0(t), h_1(t), \dots, h_{m-1}(t))^T, \quad C_m^T := (c_0, c_1, \dots, c_{m-1}).$$

If  $y(t)$  is a piecewise constant function,  $y(t)$  can be approximated by a number of Haar functions. That is, we have

$$y(t) = \sum_{k=0}^{m-1} c_k h_k(t) = C_m^T H_m(t).$$

Therefore collocation equation is as follows:

$$y(t_k) = C_m^T H_m(t_k), k = \overline{1, m}.$$

Put  $Y := (y(t_1), y(t_2), \dots, y(t_m))$ . Then, we obtain

$$Y = (y(t_1), y(t_2), \dots, y(t_m)) = C_m^T (H_m(t_1), h_m(t_2), \dots, H_m(t_m)).$$

And we can rewrite this as

$$Y = C_m^T H_{\text{matrix}}.$$

So, we have

$$C_m^T = Y H_{\text{matrix}}^T.$$

On the other hand, using  $I_0^\alpha y(t) = I_0^\alpha C_m^T H_m(t) = C_m^T F_H^\alpha H_m(t)$ , we can get

$$(I_0^\alpha y(t_1), I_0^\alpha y(t_2), \dots, I_0^\alpha y(t_m)) = C_m^T F_H^\alpha H_{\text{matrix}}.$$

Denote  $Y := (x_0, x_1, \dots, x_n)$ ,  $C_0^T := Y \circ H_{\text{matrix}}^T$ . Then  $\hat{x}_0(t)$ , a collocation approximation of  $x_0(t)$  is given by

$$\hat{x}_0(t) = C_0^T \circ H(t).$$

Put as follows:

$$\begin{aligned} Y_1 &:= (f(t_1, \hat{x}_0(t_1)), f(t_2, \hat{x}_0(t_2)), \dots, f(t_m, \hat{x}_0(t_m))), \\ Y_2 &:= (I_*(\hat{x}_0(t_1)) \cdot \chi(t_1 - t_*), I_*(\hat{x}_0(t_2)) \cdot \chi(t_2 - t_*), \dots, I_*(\hat{x}_0(t_m)) \cdot \chi(t_m - t_*)), \\ C_1^T &:= Y_1 H_{\text{matrix}}^T F_H^\alpha + Y_2 H_{\text{matrix}}^T, \\ \hat{x}_1(t) &:= C_1^T \circ H(t). \end{aligned}$$

Then  $\hat{x}_1(t)$  is a collocation approximation of  $x_1(t)$ .

Put  $\hat{x}_{n+1}(t) := C_{n+1}^T \circ H(t)$ . And then we have

$$U_{n+1}(t) := \sum_{j=0}^n \hat{x}_j(t) = (\sum_{j=0}^n C_j^T) \cdot H(t),$$

$$\hat{x}_{n+1}(t) = U_{n+1}(t) - U_n(t),$$

$$U_{n+1}(t) = U_n(t) + I_0^\alpha \circ (f(t, U_n(t)) - f(t, U_{n-1}(t))) + (I_*(U_n(t_*^-)) - I_*(U_{n-1}(t_*^-))) \chi(t - t_*), n = 1, 2, \dots$$

Denote  $S_{n+1} := \sum_{j=0}^n C_j^T$ . From this we have

$$U_{n+1}(t) = S_{n+1} \circ H(t).$$

Define  $\bar{U}_{n+1} := (U_{n+1}(t_1), \dots, U_{n+1}(t_m))$ . We can easily see that

$$\bar{U}_{n+1} = S_{n+1} \circ H_{\text{matrix}}.$$

In order to obtain  $P$  which satisfies that  $I_0^\alpha \circ f(t, U_n(t)) = P \circ H(t)$ , denote  $f(t, U_n(t)) = \tilde{C}^T \circ H(t)$ . Then we can get

$$(f(t_1, U_n(t_1)), f(t_2, U_n(t_2)), \dots, f(t_m, U_n(t_m))) = \tilde{C}^T \circ H_{\text{matrix}}.$$

So we have

$$\begin{aligned} \tilde{C}^T &= (f(t_1, U_n(t_1)), f(t_2, U_n(t_2)), \dots, f(t_m, U_n(t_m))) \circ H_{\text{matrix}}^T, \\ I_0^\alpha \circ f(t, U_n(t)) &= \tilde{C}^T \circ I_0^\alpha H(t) = \tilde{C}^T F_H^\alpha H(t), \\ I_0^\alpha \circ f(t, U_n(t)) &= \tilde{C}^T \circ I_0^\alpha H(t) = \tilde{C}^T F_H^\alpha H(t), \\ P &= \tilde{C}^T F_H^\alpha = (f(t_1, U_n(t_1)), f(t_2, U_n(t_2)), \dots, f(t_m, U_n(t_m))) \circ H_{\text{matrix}}^T \circ F_H^\alpha, \\ I_0^\alpha \circ f(t, U_n(t)) &= (f(t_1, U_n(t_1)), f(t_2, U_n(t_2)), \dots, f(t_m, U_n(t_m))) \circ H_{\text{matrix}}^T \circ F_H^\alpha \circ H(t), \end{aligned}$$

that is,

$$\begin{aligned} (I_0^\alpha f(t, U_n(t))|_{t_1}, I_0^\alpha f(t, U_n(t))|_{t_2}, \dots, I_0^\alpha f(t, U_n(t))|_{t_m}) \\ = (f(t_1, U_n(t_1)), f(t_2, U_n(t_2)), \dots, f(t_m, U_n(t_m))) \circ H_{\text{matrix}}^T \circ F_H^\alpha \circ H_{\text{matrix}}. \end{aligned}$$

For the convenience, denote

$$F(\bar{U}_n) := (f(t_1, U_n(t_1)), f(t_2, U_n(t_2)), \dots, f(t_m, U_n(t_m))) \cdot I(\bar{U}_n) := I_*(U_n(t_*)).$$

On the other hand we should get  $Q^T$  that satisfies

$$\chi(t - t_*) = Q^T \circ H(t).$$

Since  $(\chi(t_1 - t_*), \chi(t_2 - t_*), \dots, \chi(t_m - t_*)) = Q^T \circ H_{\text{matrix}}$ , we have

$$Q^T = (\chi(t_1 - t_*), \chi(t_2 - t_*), \dots, \chi(t_m - t_*)) \circ H_{\text{matrix}}^T.$$

Thus, we get

$$\chi(t - t_*) = (\chi(t_1 - t_*), \chi(t_2 - t_*), \dots, \chi(t_m - t_*)) \circ H_{\text{matrix}}^T \circ H(t).$$

Now, put  $\bar{\chi} := (\chi(t_1 - t_*), \chi(t_2 - t_*), \dots, \chi(t_m - t_*))$ . Then we get

$$\bar{U}_{n+1} = \bar{U}_n + (F(\bar{U}_n) - F(\bar{U}_{n-1})) \circ H_{\text{matrix}}^T \circ H_{\text{matrix}}^T \circ F_H^\alpha \circ H_{\text{matrix}} + (I(\bar{U}_n) - I(\bar{U}_{n-1})) \circ \bar{\chi}.$$

From this, we have

$$\begin{aligned} S_{n+1} &= S_n + (F(\bar{U}_n) - F(\bar{U}_{n-1})) \circ H_{\text{matrix}}^T \circ F_H^\alpha \circ H_{\text{matrix}} + (I(\bar{U}_n) - I(\bar{U}_{n-1})) \circ \bar{\chi} \circ H_{\text{matrix}}^T, \\ F(\bar{U}_n) &= (f(t_1, U_n(t_1)), f(t_2, U_n(t_2)), \dots, f(t_m, U_n(t_m))). \end{aligned}$$

Considering that  $U_n(t_i) = S_n \circ H(t_i) = S_n \circ H_{\text{matrix}}^i$ , we can obtain

$$F(\bar{U}_n) = (f(t_1, S_n \circ H_{\text{matrix}}^1), f(t_2, S_n \circ H_{\text{matrix}}^2), \dots, f(t_m, S_n \circ H_{\text{matrix}}^m)) =: \bar{F}(S_n).$$

Similarly, we have

$$I(\bar{U}_n) = I_*(U_n(t_*)) = I_*(S_n \circ H(t_*)).$$

Using the assumption that  $t_* = t_{m/2}$ , we get

$$I(\bar{U}_n) = I_*(S_n \circ H_{\text{matrix}}^{m/2}) =: \bar{I}(S_n).$$

Therefore, we have

$$S_{n+1} = S_n + (\bar{F}(S_n) - \bar{F}(S_{n-1})) \circ H_{\text{matrix}}^T \circ F_H^\alpha + (\bar{I}(S_n) - \bar{I}(S_{n-1})) \circ \bar{\chi} \circ H_{\text{matrix}}^T.$$

Algorithm (decomposition-operational matrix method):

Step 1. Compute  $S_0 := C_0^T = (x_0, x_0, \dots, x_0) \circ H_{\text{matrix}}^T$ .

Step 2. Compute  $S_1 := C_0^T + C_1^T = C_0^T + (Y_1 H_{\text{matrix}}^T F_H^\alpha + Y_2 H_{\text{matrix}}^T)$ , where

$$\hat{x}_0(t) = C_0^T \circ H_{\text{matrix}},$$

$$Y_1 := (f(t_1, \hat{x}_0(t_1)), f(t_2, \hat{x}_0(t_2)), \dots, f(t_m, \hat{x}_0(t_m))),$$

$$Y_2 := (I_*(\hat{x}_0(t_1)) \cdot \chi(t_1 - t_*), I_*(\hat{x}_0(t_2)) \cdot \chi(t_2 - t_*), \dots, I_*(\hat{x}_0(t_m)) \cdot \chi(t_m - t_*)).$$

Step 3. Compute N times

$$S_{n+1} = S_n + (\bar{F}(S_n) - \bar{F}(S_{n-1})) \circ H_{\text{matrix}}^T F_H^\alpha + (\bar{I}(S_n) - \bar{I}(S_{n-1})) \circ \bar{\chi} \circ H_{\text{matrix}}^T.$$

Step 4. Compute  $\bar{U}_n = S_n \circ H_{\text{matrix}}$ .

### 3.3. Convergence of the approximate solution

**Lemma 3.10** ([6]). Assume that  $M := \sup |f(t, x)|$ ,  $t_2 > t_1$ . Then the following inequality holds:

$$|I_0^\alpha f(t, x(t))|_{t_1} - |I_0^\alpha f(t, x(t))|_{t_2} \leq \begin{cases} \frac{2M}{\Gamma(1+\alpha)} \Delta t^\alpha, & \alpha \leq 1, \\ \frac{M}{\Gamma(1+\alpha)} ((t_2 - t_1)^\alpha + t_2^\alpha - t_1^\alpha), & \alpha > 1. \end{cases}$$

**H 3.2**  $\forall x_1, x_2 \in R, |f(t, x_1) - f(t, x_2)| \leq L_f |x_1 - x_2|$ .

**H 3.3**  $\forall x_1, x_2 \in R, |I_*(x_1) - I_*(x_2)| \leq L_I |x_1 - x_2|$ .

**H 3.4**  $\omega := \frac{L_f}{\Gamma(1+\alpha)} + L_I, 0 < \omega < 1$ .

**Theorem 3.11.** The following inequality holds.

$$\|x - U_N\|_{PC[0,1]} \leq \frac{2M}{\Gamma(1+\alpha)(1-\omega)} \left( \frac{\Delta t}{2} \right)^\alpha + \omega^N \|x - U_0\|_{PC[0,1]}.$$

*Proof.* Put  $\forall t \in [0, 1], \exists t_k : t_k \in U_{\Delta t}(t), |x(t) - U_N(t)| = |x(t) - x(t_k) + x(t_k) - U_N(t_k) + U_N(t_k) - U_N(t)|$ . From the construction of  $U_N(t)$ , we get

$$|U_N(t_k) - U_N(t)| = |U_N(t_k) - U_N(t_k)| = 0.$$

We can easily see that

$$x(t) = x_0 + I_*(x(t_*^-))\chi(t - t_*) + I_0^\alpha \circ f(t, x(t)), \quad x(t_k) = x_0 + I_*(x(t_*^-))\chi(t_k - t_*) + I_0^\alpha \circ f(t, x(t))|_{t_k}.$$

Using  $t_k \in \bigcup_{\frac{1}{2}}(t)$  and Lemma 3.10, we have

$$|x(t) - x(t_k)| = |I_0^\alpha \circ f(t, x(t)) - I_0^\alpha \circ f(t, x(t))|_{t_k} \leq \frac{2M}{\Gamma(1+\alpha)} \left( \frac{\Delta t}{2} \right)^\alpha.$$

On the other hand, we have

$$\begin{aligned} U_N(t_k) &= \sum_{j=0}^N \hat{x}_j(t_k) = x_0 + I_0^\alpha f(t, \sum_{j=0}^{N-1} \hat{x}_j(t_k)) + I_*(\sum_{j=0}^{N-1} \hat{x}_j(t_*^-))\chi(t_k - t_*) \\ &= x_0 + I_0^\alpha f(t, U_{N-1}(t_k)) + I_*(U_{N-1}(t_*^-))\chi(t_k - t_*). \end{aligned}$$

Thus, we obtain the following estimate inequality:

$$\begin{aligned} |x(t_k) - U_N(t_k)| &= |I_0^\alpha(f(t, x(t)) - f(t, U_{N-1}(t)))|_{t_k}| + |I_*(x(t_*^-)) - I_*(U_{N-1}(t_*^-))| \\ &\leq \frac{L_f}{\Gamma(1+\alpha)} \|x - U_{N-1}\|_{PC[0,1]} + L_{I_*} \|x - U_{N-1}\|_{PC[0,1]}, \\ |x(t) - U_N(t)| &\leq \frac{2M}{\Gamma(1+\alpha)} \left(\frac{\Delta t}{2}\right)^\alpha + \frac{L_f}{\Gamma(1+\alpha)} \|x - U_{N-1}\|_{PC[0,1]} + L_{I_*} \|x - U_{N-1}\|_{PC[0,1]} \\ &= \frac{2M}{\Gamma(1+\alpha)} \Delta t^\alpha + \omega \|x - U_{N-1}\|_{PC[0,1]} \leq \dots \leq \frac{2M}{\Gamma(1+\alpha)(1-\omega)} \left(\frac{\Delta t}{2}\right)^\alpha + \omega^N \|x - U_0\|_{PC[0,1]}. \end{aligned}$$

Therefore we get

$$\|x - U_N\|_{PC[0,1]} \leq \frac{2M}{\Gamma(1+\alpha)(1-\omega)} \left(\frac{\Delta t}{2}\right)^\alpha + \omega^N \|x - U_0\|_{PC[0,1]}.$$

The proof is completed.  $\square$

### 3.4. Numerical example

#### 3.4.1. Problem

Consider the following nonlinear impulsive fractional differential equation:

$${}^cD_0^\alpha x(t) = f(t, x(t)), t \in (0, 1], t \neq t_* = 1/2, \quad \Delta x(t_*) = -0.2x(t_*^-), \quad x(0) = 1,$$

where  $\alpha = 0.8$ .

Put  $f(t, x(t))$  as follows:

$$\begin{aligned} g_1(t) &= (2(2t-1)^{-\alpha}(4(2-1/t)^\alpha t^2 + 2^\alpha(-1+\alpha-2t)(-1+2t)))/(4\Gamma(1-\alpha)(2+(-3+\alpha)\alpha)), \\ g_2(t) &= (-2^{-2+\alpha}3(-1+2t)^{3-\alpha})/(\Gamma(1-\alpha)^*(6-11\alpha+6\alpha^2-\alpha^3)), \\ f_1(t) &= \begin{cases} 2t^{(2-\alpha)}/\Gamma(3-\alpha), & t \leq \frac{1}{2}, \\ g_1(t) + g_2(t), & t > \frac{1}{2}, \end{cases} \\ f_2(t) &= \begin{cases} (t^2+1)^2, & t \leq \frac{1}{2}, \\ (1-(t-1/2)^3)^2, & t > \frac{1}{2}, \end{cases} \\ f(t, x(t)) &= x^2(t) + f_1(t) - f_2(t). \end{aligned}$$

#### 3.4.2. Results of computation

We put  $r = 7$  and divided  $[0, 1]$  into  $2^r = 2^7 = 128$  equal parts.

The mesh points were taken as the midpoints of all subintervals and we used the 11<sup>th</sup> order of decomposition.

The exact solution of this equation is as follows:

$$x(t) = \begin{cases} t^2 + 1, & t \leq \frac{1}{2}, \\ 1 - (t-1/2)^3, & t > \frac{1}{2}. \end{cases}$$

Table 1: The maximum norm error of the exact solution and approximate solution.

	Mesh Point	Approximate	Exact	Error	Mesh Point	Approximate	Exact	Error	Mesh Point	Approximate	Exact	Error	Mesh Point	Approximate	Exact	Error
1	0.00391	1.00003	1.00002	1.38471E-05	0.253906	1.03449	1.06447	2.61114E-05	0.503906	1.00003	1	0.0003026	0.753906	0.984927	0.98363	0.3012963
2	0.01172	1.00015	1.00014	0.000013649	0.261719	1.03852	1.0685	2.66832E-05	0.511719	1.00049	0.99998	0.0004957	0.761719	0.983396	0.98207	0.001323
3	0.01953	1.0004	1.00038	1.38303E-05	0.269531	1.07267	1.07265	2.72693E-05	0.519531	1.00055	0.99993	0.0005556	0.769531	0.981769	0.98042	0.3013499
4	0.02734	1.00076	1.00075	1.40846E-05	0.27734	1.07695	1.07692	2.78703E-05	0.527344	1.00058	0.99998	0.000596	0.777344	0.980044	0.97867	0.3013771
5	0.03513	1.00125	1.00124	1.43693E-05	0.285156	1.03134	1.08131	2.84867E-05	0.535156	1.00059	0.999957	0.0006285	0.785156	0.978218	0.97681	0.3014047
6	0.04297	1.00186	1.00185	1.46713E-05	0.292969	1.03586	1.08583	0.000029119	0.542969	1.00058	0.999921	0.000657	0.792969	0.976287	0.97483	0.3014327
7	0.05073	1.00259	1.00258	1.49853E-05	0.300781	1.1905	1.09047	0.000029768	0.550781	1.00055	0.999869	0.0006832	0.800781	0.974249	0.97273	0.3014609
8	0.05859	1.00345	1.00343	1.53089E-05	0.308594	1.03526	1.09523	0.000030434	0.558594	1.00051	0.999799	0.0007078	0.808594	0.972102	0.97061	0.3014895
9	0.06641	1.00443	1.00441	1.56407E-05	0.316406	1.13014	1.10011	3.11179E-05	0.566406	1.00044	0.999707	0.0007316	0.816406	0.969842	0.96832	0.3015184
10	0.07422	1.00552	1.00551	1.59801E-05	0.324219	1.13515	1.10512	3.18202E-05	0.574219	1.00035	0.999591	0.0007546	0.824219	0.967466	0.96592	0.3015476
11	0.08203	1.00675	1.00673	1.63268E-05	0.332031	1.11028	1.1.024	3.25415E-05	0.582031	1.00023	0.999448	0.0007773	0.832031	0.96972	0.9634	0.3015772
12	0.08984	1.00809	1.00807	1.66806E-05	0.339844	1.11553	1.1.549	3.28272E-05	0.589844	1.00007	0.99975	0.0007996	0.839844	0.962357	0.96075	0.301607
13	0.09763	1.00955	1.00954	1.70415E-05	0.347656	1.1209	1.12086	3.40443E-05	0.597656	0.999891	0.999069	0.0008218	0.847656	0.959618	0.95793	0.3016372
14	0.10547	1.01114	1.01112	1.74097E-05	0.355469	1.12639	1.12636	3.48272E-05	0.605469	0.999671	0.998827	0.000844	0.855469	0.956751	0.95503	0.3016676
15	0.11323	1.01285	1.01283	1.77853E-05	0.363281	1.13201	1.13197	3.56321E-05	0.613281	0.999412	0.998564	0.0008661	0.863281	0.953755	0.95203	0.3016984
16	0.12103	1.01468	1.01466	1.81682E-05	0.371094	1.13775	1.13771	3.64598E-05	0.621094	0.999113	0.998224	0.0008882	0.871094	0.950626	0.9483	0.3017294
17	0.12891	1.01664	1.01662	1.85593E-05	0.378906	1.14361	1.14357	0.000037311	0.628906	0.998768	0.997858	0.0009105	0.878906	0.947361	0.9453	0.3017607
18	0.13672	1.01871	1.01869	1.89582E-05	0.386719	1.14959	1.14955	3.81867E-05	0.636719	0.998377	0.997444	0.0009328	0.886719	0.943958	0.94217	0.3017923
19	0.14453	1.02091	1.02089	1.93653E-05	0.394531	1.15569	1.15565	3.90876E-05	0.644531	0.997936	0.996981	0.0009553	0.894531	0.940413	0.93853	0.3018241
20	0.15234	1.02323	1.02321	0.000019781	0.40234	1.13192	1.16188	4.00147E-05	0.652344	0.997442	0.996464	0.000978	0.902344	0.936725	0.93487	0.3018562
21	0.16013	1.02867	1.02865	2.02055E-05	0.410156	1.13827	1.16823	4.09687E-05	0.660156	0.996893	0.995892	0.0010009	0.910156	0.932889	0.931	0.3018885
22	0.16797	1.02823	1.02821	2.06392E-05	0.417969	1.17474	1.1747	4.19506E-05	0.667969	0.996285	0.995261	0.001024	0.917969	0.928903	0.92693	0.001921
23	0.17573	1.03092	1.03092	2.10822E-05	0.425781	1.13133	1.18129	4.29613E-05	0.675781	0.995616	0.994569	0.0010474	0.925781	0.924764	0.92281	0.3019537
24	0.18359	1.03373	1.03371	2.15351E-05	0.433594	1.13805	1.188	4.40016E-05	0.683594	0.994883	0.993812	0.001071	0.933594	0.920469	0.91843	0.3019866
25	0.19141	1.03666	1.03664	0.000021998	0.441406	1.13488	1.19484	4.50723E-05	0.691406	0.994082	0.992988	0.0010949	0.941406	0.916016	0.914	0.3020197
26	0.19922	1.03971	1.03969	2.24713E-05	0.449219	1.21184	1.2018	4.61745E-05	0.699219	0.993212	0.992093	0.0011119	0.949219	0.911402	0.90935	0.3020529
27	0.20703	1.04288	1.04286	2.29557E-05	0.457031	1.23892	1.20888	4.73088E-05	0.707031	0.99227	0.991126	0.0011434	0.957031	0.906623	0.90454	0.3020862
28	0.21484	1.04618	1.04616	2.34512E-05	0.464844	1.21613	1.2.608	0.000048416	0.714844	0.991251	0.990083	0.0011681	0.964844	0.901676	0.89953	0.3021196
29	0.22265	1.0496	1.04954	2.39583E-05	0.472656	1.22345	1.2234	4.96769E-05	0.722656	0.990155	0.988962	0.0011931	0.972656	0.89856	0.89441	0.3021532
30	0.23047	1.05314	1.05312	2.44774E-05	0.480469	1.2309	1.23085	0.000050912	0.730469	0.988977	0.987758	0.0012185	0.980469	0.89127	0.88903	0.3021868
31	0.23828	1.05668	1.05678	0.000025009	0.488281	1.23847	1.23842	5.21819E-05	0.738281	0.987715	0.986471	0.0012441	0.988281	0.885805	0.88353	0.3022204
32	0.24609	1.06059	1.06056	2.55535E-05	0.496094	1.24616	1.24611	5.34869E-05	0.746094	0.986366	0.985096	0.0012701	0.996094	0.880161	0.87791	0.3022541

Figure 1 shows the exact solution and approximate solution.

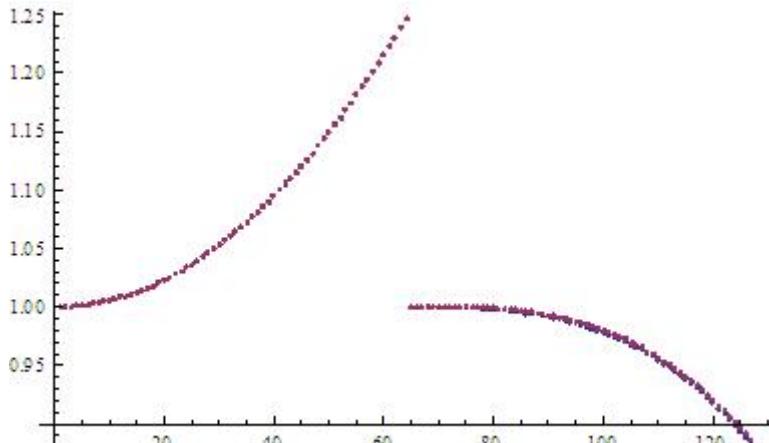


Figure 1: Exact solution and approximate solution

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