Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 1207–1214
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

On the new double integral transform for solving singular system of hyperbolic equations

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Communicated by R. Saadati

Abstract

In this manuscript, we will introduce a new double transform called double Elzaki transform (modification of Smudu transform), where we will study this transform and their theorems on convergence. Also, we will discuss the double new transform and it is convergent. After that, we study the combination of this double transforms and the new method in order to solve the singular system of hyperbolic equations of anomalies in through the examples in this paper. We found that this method is very effective in solving these equations compared to other methods as they need only one step to get the exact solution, while the other methods need more steps.

Keywords: Double new integral, transform, convergence, nonlinear singular system of hyperbolic equations.

2010 MSC: 35A22, 35L81.

1. Introduction

Nonlinear equations are of great importance to our contemporary world. Nonlinear phenomena have important applications in applied mathematics, physics, and issues related to engineering. Despite the importance of obtaining the exact solution of nonlinear partial differential equations in physics and applied mathematics there is still the daunting problem of finding new methods to discover new exact or approximate solutions. In the recent years, many authors have devoted their attention to study solutions of nonlinear partial differential equations using various methods. Among these attempts are the Adomian decomposition method, homotopy perturbation method, variational iteration method [1, 8–10], Laplace variational iteration method [12, 13, 22] differential transform method, Elzaki transform [2, 3, 6], Laplace, double Laplace transforms [2, 4] and projected differential transform method. Many analytical and numerical methods have been proposed to obtain solutions for nonlinear PDEs with fractional derivatives,

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doi: 10.22436/jnsa.011.10.08

Received: 2018-04-17 *Revised:* 2018-06-05 *Accepted:* 2018-06-28

such as local fractional variational iteration method [11], local fractional Fourier method, Yang-Fourier transform and Yang-Laplace transform and other methods [14–18]. Two Laplace variational iteration methods are currently suggested by Wu in [19–21, 23]. In this paper, we will introduce the new method depends on double new integral transform (double Elzaki transform) [7], and it will be employed in a straightforward manner. Also, we study in this paper the combination of this new transform and the new method to solve the singular system of hyperbolic equations. This approach can be taken functions with discontinuities as well as impulse functions effectively.

Elzaki transform, henceforth designated by the operator $E[.]$, is defined by the integral equation,

$$E[\Omega(t)] = T(\beta) = \beta^2 \int_0^\infty \Omega(\beta t)e^{-t}dt. \tag{1.1}$$

By analogy with the double Laplace transform, we shall denote the double Elzaki transform.

2. New double integral transform

In this section and analogy with the double Laplace transform, we will denote the new double transform. Also, in this paper, we will see the importance of this new double transform and its effectiveness in solving some differential equations.

Definition 2.1. Let $\Omega(x,t),\ t,x \in R^+$, be a function which can be expressed as a convergent infinite series, then its new double integral transform given by

$$E_2[\Omega(x,t),\alpha,\beta] = T(\alpha,\beta) = \alpha\beta \int_0^\infty \int_0^\infty \Omega(x,t)e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}dxdt, \quad x,t>0, \tag{2.1}$$

where α,β are complex values.

To find the solution of the singular system of hyperbolic equations by the combination of new double transform and the new method, first we must find the new double transform of partial derivatives as follows:

$$\begin{aligned}
&E_2\left[\frac{\partial\Omega}{\partial x}\right] = \frac{1}{\alpha}T(\alpha,\beta) - \alpha T(0,\beta), \qquad E_2\left[\frac{\partial^2\Omega}{\partial x^2}\right] = \frac{1}{\alpha^2}T(\alpha,\beta) - T(0,\beta) - \alpha\frac{\partial T(0,\beta)}{\partial x},\\
&E_2\left[\frac{\partial\Omega}{\partial t}\right] = \frac{1}{\beta}T(\alpha,\beta) - \beta T(\alpha,0), \qquad E_2\left[\frac{\partial^2\Omega}{\partial t^2}\right] = \frac{1}{\beta^2}T(\alpha,\beta) - T(\alpha,0) - \beta\frac{\partial T(\alpha,0)}{\partial t},\\
&E_2\left[\frac{\partial^2\Omega}{\partial x\partial t}\right] = \frac{1}{\alpha\beta}T(\alpha,\beta) - \frac{\beta}{\alpha}T(\alpha,0) - \frac{\alpha}{\beta}T(0,\beta) + \alpha\beta T(0,0).
\end{aligned} \tag{2.2}$$

Proof.

$$E_2\left[\frac{\partial\Omega}{\partial x}\right] = \alpha\beta\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{\alpha}+\frac{t}{\beta}\right)}\frac{\partial}{\partial x}\Omega(x,t)dxdt = \beta\int_0^\infty e^{-\frac{t}{\beta}}\left\{\alpha\int_0^\infty e^{-\frac{x}{\alpha}}\frac{\partial}{\partial x}\Omega(x,t)dx\right\}dt.$$

The inner integral gives $\frac{1}{\alpha}T(\alpha,t) - \alpha\Omega(0,t)$, and then:

$$E_2\left[\frac{\partial\Omega}{\partial x}\right] = \frac{\alpha}{\beta}\int_0^\infty e^{-\frac{t}{\beta}}T(\alpha,t)dt - \alpha\beta\int_0^\infty e^{-\frac{t}{\beta}}\Omega(0,t)dt = \frac{1}{\alpha}T(\alpha,\beta) - \alpha T(0,\beta).$$

Also,

$$E_2\left[\frac{\partial\Omega}{\partial t}\right] = \frac{1}{\beta}T(\alpha,\beta) - \beta T(\alpha,0).$$

We can prove the formulas mentioned in (2.2) easily by using the same method. □

3. Theorems of convergence of double new integral transform

Here we need to discuss some theorems of convergence of double new integral transform.

Theorem 3.1. *Suppose that* $\beta \int_0^\infty e^{-\frac{t}{\beta}} \Omega(x,t)dt$, *converges at* $\beta = \beta_0$, *then the integral converges at* $\beta < \beta_0$.

Proof. Let

$$p(x,t) = \beta_0 \int_0^t e^{-\frac{u}{\beta_0}} \Omega(x,u)du, \quad 0 < t < \infty,$$

then we have

(i) $p(x,0) = 0$,

(ii)$\lim_{t\to\infty} p(x,t)$exist,

(iii)$p_t(x,t) = \beta_0 e^{-\frac{t}{\beta_0}} \Omega(x,t)$.

Choosing ε_1, R_1, such that $0 < \varepsilon_1 < R_1$, then we have

$$\beta \int_{\varepsilon_1}^{R_1} e^{-\frac{t}{\beta}} \Omega(x,t)dt = \beta \int_{\varepsilon_1}^{R_1} \frac{1}{\beta_0} e^{-\frac{t}{\beta}} P_t(x,t) e^{\frac{t}{\beta_0}} dt = \frac{\beta}{\beta_0} \int_{\varepsilon_1}^{R_1} e^{-\left(\frac{\beta_0-\beta}{\beta\beta_0}\right)} P_t(x,t)dt.$$

Integrating the last integral by parts to gives

$$\frac{\beta}{\beta_0} \int_{\varepsilon_1}^{R_1} e^{-\left(\frac{\beta_0-\beta}{\beta\beta_0}\right)t} P_t(x,t)dt = \frac{\beta}{\beta_0} \left\{ e^{-\left(\frac{\beta_0-\beta}{\beta\beta_0}\right)R_1} p(x,R_1) - e^{-\left(\frac{\beta_0-\beta}{\beta\beta_0}\right)\varepsilon_1} p(x,\varepsilon_1) \right.$$
$$\left. + \left(\frac{\beta_0-\beta}{\beta\beta_0}\right) \int_{\varepsilon_1}^{R_1} e^{-\left(\frac{\beta_0-\beta}{\beta\beta_0}\right)} p(x,t)dt \right\}.$$

Now take, $\varepsilon_1 \to 0, R_1 \to \infty$, and if $\beta < \beta_0$, then we have

$$\beta \int_0^\infty \Omega(x,t)dt = \frac{\beta_0-\beta}{\beta\beta_0} \int_0^\infty e^{-\left(\frac{\beta_0-\beta}{\beta\beta_0}\right)} p(x,t)dt, \quad \beta < \beta_0.$$

Theorem 3.1 is proved if the last integral is converges.

By using the limits test for convergence we get

$$\lim_{t\to\infty} t^2 e^{-\left(\frac{\beta_0-\beta}{\beta\beta_0}\right)t} p(x,t) = 0,$$

finite. Therefore

$$\beta \int_0^\infty e^{-\frac{t}{\beta}} \Omega(x,t)dt,$$

is converges for $\beta < \beta_0$. □

Theorem 3.2. *Let the integral* $Q(x,\beta) = \beta \int_0^\infty e^{-\frac{t}{\beta}} \Omega(x,t)dt$ *converges for* $\beta < \beta_0$ *and the integral*

$$\alpha \int_0^\infty e^{-\frac{x}{\alpha}} \Omega(x,\beta)dx,$$

converges at $\alpha = \alpha_0$. *Then the integral* $\alpha \int_0^\infty e^{-\frac{x}{\alpha}} Q(x,\beta)dx$ *converges for* $\alpha < \alpha_0$.

Proof. The prove of this theorem is same as the method in Theorem 3.1 □

Theorem 3.3. *Let the function* $\Omega(x,t)$ *is continuous in the* xy*-plane, if the integral converges for* $\beta = \beta_0,\ \alpha = \alpha_0$. *Then the integral*

$$\alpha\beta \int_0^\infty \int_0^\infty e^{-\frac{x}{\alpha}-\frac{t}{\beta}} \Omega(x,t)dxdt,$$

is converges for $\alpha < \alpha_0,\ \beta < \beta_0$.

Proof.

$$\alpha\beta\int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-\frac{t}{\beta}}\Omega(x,t)dxdt=\alpha\int_0^\infty e^{-\frac{x}{\alpha}}\left\{\beta\int_0^\infty e^{-\frac{t}{\beta}}\Omega(x,t)dt\right\}dx=\alpha\int_0^\infty e^{-\frac{x}{\alpha}}Q(x,\beta)dx,$$

where $Q(x,\beta)=\beta\int_0^\infty e^{-\frac{t}{\beta}}\Omega(x,t)dt$. By using Theorems 3.1 and 3.2 we see that

$$\alpha\beta\int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-\frac{t}{\beta}}\Omega(x,t)dxdt,$$

is converges for $\alpha<\alpha_0,\ \beta<\beta_0$. □

4. The new method with new double integral transform

To explain this method we will display the singular system of hyperbolic equations,

$$\begin{aligned}\frac{\partial^2 P}{\partial t^2}-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P}{\partial x}\right)-Q=k(x,t),\\ \frac{\partial^2 Q}{\partial t^2}-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q}{\partial x}\right)-P=h(x,t).\end{aligned}\tag{4.1}$$

With the initial conditions

$$\begin{aligned}P(x,0)=k_1(x),\quad P_t(x,0)=k_2(x),\\ Q(x,0)=h_1(x),\quad Q_t(x,0)=h_2(x).\end{aligned}\tag{4.2}$$

To find the solution of the system (4.1), (4.2), firstly we take double new integral transform of (4.1), and single new integral transform of (4.2), we obtain

$$\begin{aligned}\frac{1}{\beta^2}E_2(P(x,t))-K_1(\alpha)-\beta K_2(\alpha)=E_2\left[k(x,t)+\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P}{\partial x}\right)+Q\right],\\ \frac{1}{\beta^2}E_2(Q(x,t))-H_1(\alpha)-\beta H_2(\alpha)=E_2\left[h(x,t)+\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q}{\partial x}\right)+P\right],\end{aligned}\tag{4.3}$$

where $K_1(\alpha),K_2(\alpha),H_1(\alpha),H_2(\alpha)$, are single new integral transform of $k_1(x),\ k_2(x),\ h_1(x),\ h_2(x)$, respectively.

We assume that the solution of a system (4.1) can be written in the series form

$$P(x,t)=\sum_{n=0}^\infty P_n(x,t),\quad Q(x,t)=\sum_{n=0}^\infty Q_n(x,t).\tag{4.4}$$

Now, we take the inverse of double new integral transform of (4.3), and making use of (4.4) to get

$$\begin{aligned}\sum_{n=0}^\infty P_n(x,t)=k_1(x)+tk_2(x)+E_2^{-1}\left\{\beta^2E_2\left[k(x,t)+\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P_n}{\partial x}\right)+Q_n\right]\right\},\\ \sum_{n=0}^\infty Q_n(x,t)=h_1(x)+th_2(x)+E_2^{-1}\left\{\beta^2E_2\left[h(x,t)+\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q_n}{\partial x}\right)+P_n\right]\right\}.\end{aligned}\tag{4.5}$$

This method depends on how to choose the initial iterations, $P_0(x,t),\ Q_0(x,t)$, that leads to the exact solutions in a few steps. For example if we choose

$$P_0(x,t)=k_1(x)+tk_2(x),\quad Q_0(x,t)=h_1(x)+th_2(x).$$

Then the solutions $P(x,t)$, $Q(x,t)$ can be recursively determined by using the relations

$$\begin{aligned} P_{n+1}(x,t) &= k_1(x) + tk_2(x) + E_2^{-1}\left\{\beta^2 E_2\left[k(x,t) + \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P_n}{\partial x}\right) + Q_n\right]\right\},\\ Po(x,t) &= k_1(x) + tk_2(x),\\ Q_{n+1}(x,t) &= h_1(x) + th_2(x) + E_2^{-1}\left\{\beta^2 E_2\left[h(x,t) + \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q_n}{\partial x}\right) + P_n\right]\right\},\\ Q_0(x,t) &= h_1(x) + th_2(x). \end{aligned}$$

From these equations we can find

$$P_0(x,t), P_1(x,t), P_2(x,t), \cdots, \qquad Q_0(x,t), Q_1(x,t), Q_2(x,t), \cdots,$$

and then we can obtain the solutions in a series form (4.4).

5. Application

To illustrate the efficiency and effectiveness of this method in solving the singular system of linear and nonlinear hyperbolic equations by taking only one step, we look at the following examples.

Example 5.1. Let us consider the singular system of linear hyperbolic equations

$$\begin{aligned} -\frac{\partial^2 P}{\partial t^2} + \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P}{\partial x}\right) + Q &= x^2\sin t + 4\sin t + x^2\cos t,\\ -\frac{\partial^2 Q}{\partial t^2} + \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q}{\partial x}\right) + P &= x^2\sin t + 4\cos t + x^2\cos t, \end{aligned} \tag{5.1}$$

with the initial conditions

$$P(x,0) = 0, \quad P_t(x,o) = x^2, \quad Q(x,0) = x^2, \quad Q_t(x,0) = 0. \tag{5.2}$$

Using the same steps in Section 3 to get

$$\begin{aligned} \frac{1}{\beta^2}E_2(P(x,t)) - K_1(\alpha) - \beta K_2(\alpha) &= -E_2[x^2\sin t] + E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P}{\partial x}\right) + Q - 4\sin t - x^2\cos t\right],\\ \frac{1}{\beta^2}E_2(Q(x,t)) - H_1(\alpha) - \beta H_2(\alpha) &= -E_2[x^2\cos t] + E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q}{\partial x}\right) + P - x^2\sin t - 4\cos t\right], \end{aligned} \tag{5.3}$$

where, $K_1(\alpha) = 0$, $K_2(\alpha) = 2\alpha^4$, $H_1(\alpha) = 2\alpha^4$, $H_2(\alpha) = 0$. Then, (5.3) becomes

$$\begin{aligned} E_2(P(x,t)) &= 2\beta^3\alpha^4 - \frac{2\beta^5\alpha^4}{1+\beta^2} + \beta^2 E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P}{\partial x}\right) + Q - 4\sin t - x^2\cos t\right],\\ E_2(Q(x,t)) &= 2\beta^2\alpha^4 - \frac{2\beta^4\alpha^4}{1+\beta^2} + \beta^2 E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q}{\partial x}\right) + P - x^2\sin t - 4\cos t\right]. \end{aligned} \tag{5.4}$$

Applying the inverse double new transform to (5.4), to obtain

$$\begin{aligned} P(x,t) &= x^2\sin t + E_2^{-1}\left\{\beta^2 E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P}{\partial x}\right) + Q - 4\sin t - x^2\cos t\right]\right\},\\ Q(x,t) &= x^2\cos t + E_2^{-1}\left\{\beta^2 E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q}{\partial x}\right) + P - 4\cos t - x^2\sin t\right]\right\}. \end{aligned} \tag{5.5}$$

Then the recursive relations are

$$\begin{aligned} P_{n+1}(x,t) &= E_2^{-1}\left\{\beta^2 E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P_n}{\partial x}\right)+Q_n-4\sin t-x^2\cos t\right]\right\},\\ P_0(x,t) &= x^2\sin t,\\ Q_{n+1}(x,t) &= E_2^{-1}\left\{\beta^2 E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q_n}{\partial x}\right)+P_n-4\cos t-x^2\sin t\right]\right\},\\ Q_0(x,t) &= x^2\cos t. \end{aligned} \tag{5.6}$$

The first few components are given by

$$\begin{aligned} P_0(x,t) &= x^2\sin t,\\ P_1(x,t) &= E^{-1}\left\{\beta^2 E_2[0]\right\}=0,\\ Q_0(x,t) &= x^2\cos t,\\ Q_1(x,t) &= E^{-1}\left\{\beta^2 E_2[0]\right\}=0,\\ &\vdots \end{aligned}$$

Then the exact solutions of a system (5.1) are

$$P(x,t)=\sum_{n=0}^{\infty}P_n(x,t)=x^2\sin t,\quad Q(x,t)=\sum_{n=0}^{\infty}Q_n(x,t)=x^2\cos t.$$

Example 5.2. Here we look at the singular system of nonlinear hyperbolic equations,

$$\begin{aligned} -\frac{\partial^2 P}{\partial t^2}+\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P}{\partial x}\right)+Q\frac{\partial P}{\partial x} &= 2xe^tP+4t,\\ -\frac{\partial^2 Q}{\partial t^2}+\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q}{\partial x}\right)+P\frac{\partial Q}{\partial x} &= 2xtQ-x^2e^t+4e^t, \end{aligned} \tag{5.7}$$

with the initial conditions

$$P(x,0)=0,\quad P_t(x,0)=x^2,\quad Q(x,0)=x^2,\quad Q_t(x,0)=x^2. \tag{5.8}$$

Here we use the same steps which we used as before in Example 5.1, to obtain

$$\begin{aligned} E_2(P(x,t)) &= 2\beta^3\alpha^4+\beta^2E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P}{\partial x}\right)+Q\frac{\partial P}{\partial x}-2xe^tP-4t\right],\\ E_2(Q(x,t)) &= 2\beta^2\alpha^4+2\beta^3\alpha^4+\beta^2E_2[x^2e^t]+\beta^2E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q}{\partial x}\right)+P\frac{\partial Q}{\partial x}-2xtQ-4e^t\right]. \end{aligned} \tag{5.9}$$

Taking the inverse double new transform of (5.9), to find

$$\begin{aligned} P(x,t) &= x^2t+E_2^{-1}\left\{\beta^2E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P}{\partial x}\right)+Q\frac{\partial P}{\partial x}-2xe^tP-4t\right]\right\},\\ Q(x,t) &= x^2e^t+E_2^{-1}\left\{\beta^2E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q}{\partial x}\right)+P\frac{\partial Q}{\partial x}-2xtQ-4e^t\right]\right\}. \end{aligned}$$

Therefore, we can write the recursive relations as

$$\begin{aligned} P_{n+1}(x,t) &= E_2^{-1}\left\{\beta^2E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial P_n}{\partial x}\right)+Q_n\frac{\partial P_n}{\partial x}-2xe^tP_n-4t\right]\right\},\\ P_0(x,t) &= x^2t,\\ Q_{n+1}(x,t) &= E_n^{-1}\left\{\beta^2E_2\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial Q_n}{\partial x}\right)+P_n\frac{\partial Q_n}{\partial x}-2xtQ_n-4e^t\right]\right\},\\ Q_0(x,t) &= x^2e^t. \end{aligned}$$

Then from as before we can find the first few components in the form

$$\begin{aligned}
P_0(x,t) &= x^2 t,\\
P_1(x,t) &= E^{-1}\left\{\beta^2 E_2[0]\right\} = 0,\\
Q_0(x,t) &= x^2 e^t,\\
Q_1(x,t) &= E^{-1}\left\{\beta^2 E_2[0]\right\} = 0,\\
&\vdots
\end{aligned}$$

Then the exact solutions of a system (5.7) are

$$P(x,t) = \sum_{n=0}^{\infty} P_n(x,t) = x^2 t, \quad Q(x,t) = \sum_{n=o}^{\infty} Q_n(x,t) = x^2 e^t.$$

6. Conclusion

This paper examines the convergence of the new double transform, and explain the effectiveness and ease of the method used to solve the singular system of linear and nonlinear hyperbolic equations, as we obtained the exact solutions using only one step. Comparing this method with other methods, such as the Adomian method and the homotopy method, we find that this method is faster and easier for them to reach for exact solutions.

Acknowledgment

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work though General Project under grant number (G.R.P-77-39).

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