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A viscosity iterative algorithm for split common fixed-point problems of demicontractive mappings

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Abstract

In this paper, we firstly introduce a new viscosity cyclic iterative algorithm for the split common fixed-point problem (SCFP) of demicontractive mappings. Next we prove the strong convergence of the sequence generated recursively by such a viscosity cyclic algorithm to a solution of the SCFP, which improves and extends some recent corresponding results.

Keywords: Multiple-set split equality common fixed-point problem, demicontractive mapping, viscosity cyclic iterative algorithm, strong convergence.

2010 MSC: 47H05, 47H09, 47J10, 47H20.

1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) which originally introduced in Censor and Elfving [1] is to find a point $x^* \in C$ with the property:

$$x^* \in C \text{ and } Ax^* \in Q. \tag{1.1}$$

It serves as a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in this operator's ranges. There are a number of significant applications of the SFP in intensity-modulated radiation therapy, signal processing, image reconstruction and so on.

In the case where C and Q in the SFP (1.1) are the intersections of finitely many fixed-point sets of nonlinear operators, the problem (1.1) is called by Censor and Segal [2] the split common fixed-point problem (SCFP). More precisely, the SCFP requires to seek an element $x^* \in H_1$ satisfying

$$x^* \in \cap_{i=1}^{p} \mathrm{Fix}(U_i) \text{ and } Ax^* \in \cap_{j=1}^{q} \mathrm{Fix}(T_j), \tag{1.2}$$

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where $p, q \geqslant 1$ are integers, and $\mathrm{Fix}(U_i)$ and $\mathrm{Fix}(T_j)$ denote the fixed point sets of two classes of nonlinear operators $U_i : H_1 \to H_1$ $(i = 1, 2, \cdots, p)$, $T_j : H_2 \to H_2$ $(j = 1, 2, \cdots, q)$. In particular, if $p = q = 1$, the problem (1.2) is reduced to find a point x^* with the property:

$$x^* \in \mathrm{Fix}(U) \ \text{ and } \ Ax^* \in \mathrm{Fix}(T), \tag{1.3}$$

which is usually called the two-set SCFP. To solve the two-set SCFP (1.3), Censor and Segal [2] proposed the following iterative method: for any initial guess $x_1 \in H_1$, define $\{x_n\}$ recursively by

$$x_{n+1} = U(x_n - \lambda A^*(I - T)Ax_n),$$

where U and T are directed operators. The further generalization of this algorithm has been studied by Moudafi [10] for demicontractive operators. Under suitable conditions he proved that the sequence $\{x_n\}$ converges weakly to a point of the two-set SCFP (1.3).

Recall that, for a fixed positive integer p and each $n \geqslant 0$, the p-mod function $[n]$ is defined by

$$[n] = \begin{cases} p, & \text{if } r = 0, \\ r, & \text{if } 0 < r < p, \end{cases}$$

whenever $n = kp + r$ for some $k \geqslant 0$. Afterwards, the p-mod function will be sometimes written as $[n] = n \pmod p$ in case distinction of p is needed. Recently, Wang and Xu [14] proposed the following cyclic algorithm:

$$x_{n+1} = U_{[n]}(x_n - \lambda A^*(I - T_{[n]})Ax_n), \tag{1.4}$$

where U_i and T_i are directed operators for $i = 1, 2, \ldots, p$. They proved that the sequence $\{x_n\}$ generated by the algorithm (1.4) converges weakly to a solution of the problem (1.2) in a case when $p = q$.

Noticing that the existing algorithm for the SCFP (1.2) have only weak convergence in infinite dimensional spaces (see [10, 14]), in 2013, Cui et al. [3] constricted the following cyclic iterative procedure, motivated by Eicke's damped projection algorithm [5], so that strong convergence is guaranteed: given $x_1 \in H_1$ and a positive integer p, define a sequence $\{x_n\}$ by the iterative procedure

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n U_{[n]}[(1 - \alpha_n)(x_n - \lambda_n A^*(I - T_{[n]})Ax_n)], \quad n \geqslant 1, \tag{1.5}$$

where U_i and T_i are directed operators for $i = 1, 2, \ldots, p$, $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset \mathbb{R}^+$ are properly chosen real sequences. Under some suitable conditions of parameters, they proved that the sequence $\{x_n\}$ generated recursively by (1.5) converges strongly to a solution of the problem (1.2) provided $p = q$.

Very recently, He et al. [6] developed the following viscosity algorithm to approximate the solution of the two-set SCFP (1.3) for demicontractive mappings

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)U_\lambda(x_n - \rho_n A^*(I - T)Ax_n), \quad n \geqslant 0 \tag{1.6}$$

equipped with the step size

$$\rho_n = \begin{cases} \frac{(1-\eta)\|(I-T)Ax_n\|^2}{2\|A^*(I-T)Ax_n\|^2}, & Ax_n \neq T(Ax_n), \\ 0, & \text{otherwise}, \end{cases}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are μ and η-demicontractive mappings, respectively, $U_\lambda = (1 - \lambda)I + \lambda U$ for $\lambda \in (0, 1 - \mu)$, f denotes a fixed contraction in $\mathrm{Fix}(U)$ and $\{\alpha_n\} \subset (0, 1)$ is a real sequence satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then they established that the sequence $\{x_n\}$ generated recursively by (1.6) converges strongly to a solution $\hat{x}$ of the two-set SCFP (1.3), and the $\hat{x}$ solves the following variational inequality:

$$\langle \hat{x} - f(\hat{x}), \hat{x} - z \rangle \leqslant 0, \quad \forall z \in \Lambda,$$

where Λ denotes the set of all solutions of the two-set SCFP (1.3).

In this paper, inspired and motivated by [6, 14], we first consider the following cyclic algorithm of the SCFP (1.2) for demicontractive mappings: given an initial guess $x_0 \in H_1$ and two positive integers p and q, let a sequence $\{x_n\}$ generated recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U_{\lambda_n}(x_n - \rho_n A^*(I - T_{[n]})Ax_n), \quad \forall n \geqslant 0, \tag{1.7}$$

where U_i is μ_i-demicontractive, T_j is η_j-demicontractive for $i \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$, $\mu = \max_{1\leqslant i\leqslant p} \mu_i$, $U_{\lambda_n} = (1-\lambda_n)I + \lambda_n U_{[n]}$ for $\lambda_n \in (0, 1-\mu)$, $U_{[n]} = U_{n \,(\mathrm{mod}\, p)}$, and $T_{[n]} = T_{n\,(\mathrm{mod}\, q)}$. Under the conditions of $\{\alpha_n\}$ in (1.6), we next prove that the sequence $\{x_n\}$ defined recursively by (1.7) converges strongly to a solution $\hat{x}$ of the SCFP (1.2), and the $\hat{x}$ solves the following variational inequality:

$$\langle \hat{x} - f(\hat{x}), \hat{x} - z\rangle \leqslant 0, \quad \forall z \in \Omega,$$

where

$$\Omega := (\cap_{i=1}^{p} \mathrm{Fix}(U_i)) \cap A^{-1}(\cap_{j=1}^{q} \mathrm{Fix}(T_j)) \tag{1.8}$$

denotes the solution set of the SCFP (1.2).

2. Preliminaries

Let H be a real Hilbert space with the norm $\|\cdot\|$ induced by the inner product $\langle\cdot,\cdot\rangle$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote by $\mathrm{Fix}(T)$ the set of fixed points of T. We use $\omega_w(x_n) = \{x : \exists\, x_{n_k} \rightharpoonup x\}$ to stand for weak ω-limit set of $\{x_n\}$. Also we need the following inequality which is very crucial for our argument:

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{2.1}$$

Definition 2.1. An operator $T : H \to H$ is said to be:

(i) nonexpansive if
$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in H;$$

(ii) quasi-nonexpansive if
$$\|Tx - z\| \leqslant \|x - z\|, \quad \forall (x, z) \in H \times \mathrm{Fix}(T);$$

(iii) directed if
$$\langle z - Tx, x - Tx\rangle \leqslant 0, \quad \forall (x, z) \in H \times \mathrm{Fix}(T),$$
equivalently,
$$\|Tx - z\|^2 \leqslant \|x - z\|^2 - \|x - Tx\|^2, \quad \forall (x, z) \in H \times \mathrm{Fix}(T);$$

(iv) μ-demicontractive if $\mathrm{Fix}(T) \neq \emptyset$ and there exists a constant $\mu \in (-\infty, 1)$ such that
$$\|Tx - z\|^2 \leqslant \|x - z\|^2 + \mu\|x - Tx\|^2, \quad \forall (x, z) \in H \times \mathrm{Fix}(T),$$
which is equivalent to
$$\langle z - Tx, x - Tx\rangle \leqslant \frac{1+\mu}{2}\|x - Tx\|^2, \quad \forall (x, z) \in H \times \mathrm{Fix}(T).$$

It is worth noting that the class of demicontractive mappings contain important operators such as quasi-nonexpansive mappings and directed mappings.

Remark 2.2. Notice that 0-demicontractive is exactly quasi-nonexpansive. In particular, we say that $T : H \to H$ is quasi-strictly pseudo-contractive [9] if (iv) in Definition 2.1 is satisfied with $0 \leqslant \mu < 1$. Moreover, if $\mu \leqslant 0$, every μ-demicontractive mapping becomes quasi-nonexpansive. So, it seems to be sufficient to only take $\mu \in (0, 1)$ in (iv) of Definition 2.1 in Hilbert spaces. However, as seen in (iii) of Definition 2.1, every directed operator is (-1)-demicontractive.

Definition 2.3. Let $T : H \to H$ be an operator, then $I - T$ is said to be demiclosed at zero whenever, for any sequence $\{x_n\} \subset H$ satisfying that $x_n \rightharpoonup x \in H$ and $(I-T)x_n \to 0$, it results $x = Tx$.

Lemma 2.4 ([15, Lemma 2.1]). *$\{\beta_n\}$ is a sequence of nonnegative real numbers such that*

$$\beta_{n+1} \leqslant (1-\gamma_n)\alpha_n + \delta_n, \quad n \geqslant 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| \leqslant \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.5 ([4, Lemmas 2.5 and 2.6]). *$A : H_1 \to H_2$ be a bounded linear operator and $T : H_2 \to H_2$ be a η-demicontractive, $\eta < 1$, if $A^{-1}\mathrm{Fix}(T) \neq \emptyset$, then,*

(a) $(I-T)Ax = 0 \Leftrightarrow A^*(I-T)Ax = 0, \quad \forall x \in H_1$;
(b) *in addition, for $z \in A^{-1}\mathrm{Fix}(T)$,*

$$\|x - \rho A^*(I-T)Ax - z\|^2 \leqslant \|x-z\|^2 - \frac{(1-\eta)^2}{4}\frac{\|(I-T)Ax\|^4}{\|A^*(I-T)Ax\|^2}, \tag{2.2}$$

where $x \in H_1$, $Ax \neq T(Ax)$ and

$$\rho := \frac{1-\eta}{2}\frac{\|(I-T)Ax\|^2}{\|A^*(I-T)Ax\|^2}.$$

Lemma 2.6 ([10, (1.7)] or [4, Lemma 2.4]). *Let $U : H_1 \to H_1$ be a μ-demicontractive operator with $\mu < 1$. Denote $U_\lambda := (1-\lambda)I + \lambda U$ for $\lambda \in (0, 1-\mu)$. Then for any $x \in H_1$ and $z \in \mathrm{Fix}(U)$,*

$$\|U_\lambda x - z\|^2 \leqslant \|x-z\|^2 - \lambda(1-\mu-\lambda)\|(I-U)x\|^2. \tag{2.3}$$

Lemma 2.7 ([9, Proposition 2.1]). *Assume C is a nonempty closed convex subset of a Hilbert space H. If $T : C \to C$ is a μ-demicontractive mapping (which is also called μ-quasi-strict pseudo-contraction in [9]), then the fixed point set $F(T)$ is closed and convex.*

Lemma 2.8 ([8, Lemma 3.1]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \geqslant 0$. Define the sequence $\{\tau(n)\}_{n \geqslant n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leqslant n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leqslant n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then the following properties hold

(i) $\tau(n_0) \leqslant \tau(n_0+1) \leqslant \cdots$ *and* $\tau(n) \to \infty$;
(ii) $\Gamma_{\tau(n)} \leqslant \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leqslant \Gamma_{\tau(n)+1}$, $\forall\, n \geqslant n_0$.

Recall that if C is a nonempty closed convex subset of a Hilbert space H, the metric (or nearest point) projection from H onto C is the mapping $P : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\|.$$

Lemma 2.9 ([11, Lemma 3.1.3 and Theorem 3.1.4]). *Let C be a nonempty closed convex subset of a Hilbert space H. Then P_C is a nonexpansive mapping from H onto C and $P_C x$ is characterized by the following inequality*

$$\langle y - P_C x, x - P_C x \rangle \leqslant 0, \quad \forall\, y \in C. \tag{2.4}$$

Lemma 2.10 ([7, Theorem 1.1.1]). *Let X and Y be Banach spaces, A be a continuous linear operator from X to Y. Then A is weakly continuous.*

Finally, we need the following result for proving our main theorem in section 3.

Lemma 2.11 ([13, Lemma 3.1]). *Let $\{u_n\}$ be a bounded sequence of a Hilbert space H. Let s be a positive integer and $I = \{1, 2, \ldots, s\}$. If $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$ and $x^* \in \omega_w(u_n)$, then for any $i \in I$, there exists a subsequence $\{u_{m_k}\}$ of $\{u_n\}$, depending on i, such that $u_{m_k} \rightharpoonup x^*$ and $[m_k] = i$ for all k, where $[n]$ denotes the s-mod function for each $n \geqslant 1$.*

3. Main results

In this section, we establish the strong convergence of the viscosity iterative algorithm (1.7) to a solution of SCFP (1.2) for demicontractive mappings.

Assumption 3.1. Let H_1, H_2 be two real Hilbert spaces. We assume the following conditions:

(i) the solution set Ω of (1.8) is nonempty;
(ii) $U_i : H_1 \to H_1$ $(1 \leqslant i \leqslant p)$ and $T_j : H_2 \to H_2$ $(1 \leqslant j \leqslant q)$ are μ_i-demicontractive and η_j-demicontractive, respectively;
(iii) $I - U_i$ $(1 \leqslant i \leqslant p)$ and $I - T_j$ $(1 \leqslant j \leqslant q)$ are demiclosed at origin.

Let $\mu = \max_{1\leqslant i\leqslant p} \mu_i$ and $\eta = \max_{1\leqslant j\leqslant q} \eta_i$. Clearly U_i is μ-demicontractive for all $1 \leqslant i \leqslant p$ and T_j is η-demicontractive for all $1 \leqslant j \leqslant q$.

Algorithm 3.2. Let f be a fixed contraction on $U := \cap_{i=1}^{p} \text{Fix}(U_i) \neq \emptyset$ with coefficient α and $\lambda_n \in (0, 1 - \mu)$. Given arbitrary initial guess x_0 and two positive integers p, q, on assuming that the nth iterate x_n has been constructed, we can define the $(n + 1)$th iterate by the following formula

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) U_{\lambda_n}(x_n - \rho_n A^*(I - T_{[n]})Ax_n), \quad n \geqslant 0, \tag{3.1}$$

where $U_{\lambda_n} = (1 - \lambda_n)I + \lambda_n U_{[n]}$, $U_{[n]} = U_{n \,(\text{mod } p)}$, $T_{[n]} = T_{n\,(\text{mod } q)}$, A^* is the adjoint of a bounded linear operator A, and the step size ρ_n is chosen in the following way

$$\rho_n = \begin{cases} \frac{(1-\eta)\|(I-T_{[n]})Ax_n\|^2}{2\|A^*(I-T_{[n]})Ax_n\|^2}, & Ax_n \neq T_{[n]}(Ax_n), \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Lemma 3.3. *Let Assumption 3.1 be satisfied. Given a bounded linear operator $A : H_1 \to H_2$, let $\Omega \neq \emptyset$ and let $\{x_n\} \subset H_1$ be the sequence defined as in Algorithm 3.2. Assume that the sequence $\{x_n\}$ is bounded and all the sequences $\{\|x_n - y_n\|\}$, $\{\|y_{n+1} - y_n\|\}$, $\{\|(I - U_{[n]})y_n\|\}$, and $\{\|(I - T_{[n]})Ax_n\|\}$ converge to zero, where $y_n := x_n - \rho_n A^*(I - T_{[n]})Ax_n$. Then $\emptyset \neq \omega_w(x_n) \subset \Omega$.*

Proof. Since $\{x_n\}$ is bounded, $\omega_w(x_n) \neq \emptyset$ and it also follows from the assumption $\|x_n - y_n\| \to 0$ that $\omega_w(x_n) = \omega_w(y_n)$. Now let $x^* \in \omega_w(x_n) = \omega_w(y_n)$. In view of $\|y_{n+1} - y_n\| \to 0$, for any fixed $i \in \{1, 2, \ldots, s\}$ with $s = \max\{p, q\}$, use Lemma 2.11 with $u_n = y_n$ to get a subsequence $\{y_{m_k}\}$ of $\{y_n\}$ (depending on i) such that $y_{m_k} \rightharpoonup x^*$ and $[m_k] = i$ for all k. Based on $\|(I - U_i)y_{m_k}\| = \|(I - U_{[m_k]})y_{m_k}\| \to 0$ and the demiclosedness of $I - U_i$ at the origin it results $x^* \in \text{Fix}(U_i)$ for any fixed $i \in \{1, 2, \ldots, p\}$; hence $x^* \in \cap_{i=1}^{p}\text{Fix}(U_i)$. Observing that $x_{m_k} \rightharpoonup x^*$, apply Lemma 2.10 to see that $Ax_{m_k} \rightharpoonup Ax^*$. Based on $\|(I - T_i)Ax_{m_k}\| = \|(I - T_{[m_k]})Ax_{m_k}\| \to 0$ and the demiclosedness property of $I - T_i$ at the origin, it follows that $Ax^* \in \text{Fix}(T_i)$ for any $i \in \{1, 2, \ldots, q\}$ and so $Ax^* \in \cap_{i=1}^{q}\text{Fix}(T_i)$. Therefore $x^* \in \cap_{i=1}^{p}\text{Fix}(U_i) \cap A^{-1}(\cap_{j=1}^{q}\text{Fix}(T_j)) = \Omega$, completing the proof. □

Theorem 3.4. *Let Assumption 3.1 be satisfied. Given a bounded linear operators $A : H_1 \to H_2$, assume the SCFP (1.2) is consistent ($\Omega \neq \emptyset$). If the sequences $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\}$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \lambda_n \leqslant \limsup_{n\to\infty} \lambda_n < 1 - \mu$.

The sequence $\{x_n\}$ generated by explicit algorithm (3.1) converges strongly to a point $\hat{x} = P_\Omega f(\hat{x})$, i.e., $\hat{x}$ satisfies the following variational inequality:

$$\langle \hat{x} - f(\hat{x}), \hat{x} - z\rangle \leqslant 0, \quad \forall z \in \Omega. \tag{3.3}$$

Proof. By Lemma 2.7, $U = \cap_{i=1}^{p} \mathrm{Fix}(U_i)$ is closed convex in H_1. Further, by Lemma 2.9, $P_\Omega f : U \to \Omega$ is a contraction and therefore admits a unique fixed point $\hat{x}$ of $P_\Omega f$, namely, $\hat{x} = P_\Omega f(\hat{x})$ is equivalent to the variational inequality (3.3) by the immediate aid of (2.4). Now from now on the proof is divided into three steps.

Step 1. We show that sequence $\{x_n\}$ is bounded. Let $y_n = x_n - \rho_n A^*(I - T_{[n]})Ax_n$, take $z \in \Omega$, it follows from (3.1) that

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(U_{\lambda_n} y_n - z)\| \\ &\leqslant \alpha_n\|f(x_n) - f(z)\| + (1-\alpha_n)\|U_{\lambda_n} y_n - z\| + \alpha_n\|f(z) - z\| \\ &\leqslant \alpha\alpha_n\|x_n - z\| + (1-\alpha_n)\|U_{\lambda_n} y_n - z\| + \alpha_n\|f(z) - z\|. \end{aligned} \tag{3.4}$$

(a) If $\rho_n \neq 0$, from (2.2) and (2.3), we have

$$\begin{aligned} \|U_{\lambda_n} y_n - z\|^2 &\leqslant \|y_n - z\|^2 - \lambda_n(1-\mu-\lambda_n)\|(I - U_{[n]})y_n\|^2 \\ &= \|x_n - \rho_n A^*(I - T_{[n]})Ax_n - z\|^2 - \lambda_n(1-\mu-\lambda_n)\|(I - U_{[n]})y_n\|^2 \\ &\leqslant \|x_n - z\|^2 - \frac{(1-\eta)^2}{4}\frac{\|(I - T_{[n]})Ax_n\|^4}{\|A^*(I - T_{[n]})Ax_n\|^2} - \lambda_n(1-\mu-\lambda_n)\|(I - U_{[n]})y_n\|^2. \end{aligned} \tag{3.5}$$

Thus, we get

$$\|U_{\lambda_n} y_n - z\| \leqslant \|x_n - z\|. \tag{3.6}$$

By substituting (3.6) into (3.4), we have

$$\begin{aligned} \|x_{n+1} - z\| &\leqslant \alpha\alpha_n\|x_n - z\| + (1-\alpha_n)\|x_n - z\| + \alpha_n\|f(z) - z\| \\ &\leqslant [1 - (1-\alpha)\alpha_n]\|x_n - z\| + \alpha_n\|f(z) - z\| \leqslant \max\{\|(x_n - z)\|, \frac{1}{1-\alpha}\|f(z) - z\|\} \end{aligned}$$

for all sufficiently large n. By induction, we arrive at

$$\|x_n - z\| \leqslant \max\{\|x_0 - z\|, \frac{1}{1-\alpha}\|f(z) - z\|\}.$$

Thus the sequence $\{x_n\}$ is bounded, so is $\{f(x_n)\}$.
(b) If $\rho_n = 0$, then $y_n = x_n$. In view of (2.3), we observe

$$\|U_{\lambda_n} x_n - z\| \leqslant \|x_n - z\|. \tag{3.7}$$

By applying the inequality (3.7) to (3.4), we conclude that the sequence $\{x_n\}$ and $\{f(x_n)\}$ are also bounded in a similar way as before.

Step 2. We show that the following inequality holds for $\hat{x} = P_\Omega f(\hat{x})$:

$$\|x_{n+1} - \hat{x}\|^2 \leqslant (1-\alpha_n)\|x_n - \hat{x}\|^2 + 2\alpha_n\langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle. \tag{3.8}$$

(a) If $\rho_n = 0$, it follows from (2.1) and (2.3) that

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &\leqslant (1-\alpha_n)\|U_{\lambda_n} x_n - z\|^2 + 2\alpha_n\langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle \\ &\leqslant (1-\alpha_n)[\|x_n - z\|^2 - \lambda_n(1-\lambda_n-\mu)\|(I - U_{[n]})x_n\|^2] + 2\alpha_n\langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle, \end{aligned} \tag{3.9}$$

which immediately yields

$$\|x_{n+1} - \hat{x}\|^2 \leqslant (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle.$$

Thus the inequality (3.8) is obtained.

(b) If $\rho_n \neq 0$, by (2.1) and (3.5) replaced with $z = \hat{x}$, we obtain

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &\leqslant (1-\alpha_n)\|U_{\lambda_n} y_n - \hat{x}\|^2 + 2\alpha_n \langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle \\ &\leqslant (1-\alpha_n)\Big[\|x_n - \hat{x}\|^2 - \frac{(1-\eta)^2}{4} \frac{\|(I - T_{[n]})Ax_n\|^4}{\|A^*(I - T_{[n]})Ax_n\|^2} \\ &\quad - \lambda_n(1-\lambda_n - \mu)\|(I - U_{[n]})y_n\|^2\Big] + 2\alpha_n \langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle, \end{aligned} \tag{3.10}$$

which quickly gives the inequality (3.8).

Step 3. We show that $x_n \to \hat{x}$. Setting $s_n := \|x_n - \hat{x}\|^2$, the proof of this step is divided into two cases.

Case I. Assume that there is a positive integer n_0 such that the sequence $\{s_n\}$ is decreasing for all $n \geqslant n_0$, then the sequence $\{s_n\}$ is obviously convergent. First, we claim that

$$\limsup_{n\to\infty} \langle f(\hat{x}) - \hat{x}, x_n - \hat{x}\rangle \leqslant 0. \tag{3.11}$$

(a) If $\rho_n = 0$, i.e., $x_n = y_n$, by a simple inequality eliminating $(1-\alpha_n)$ in (3.9) and based on the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, we obtain

$$\lambda_n(1-\lambda_n-\mu)\|(I - U_{[n]})x_n\|^2 \leqslant s_n - s_{n+1} + \alpha_n K,$$

where $K := \sup_{n\in N}\{2\langle f(x_n) - \hat{x}, x_{n+1} - \hat{x}\rangle\}$. By the aids of convergence of the sequence $\{s_n\}$ and the conditions (i) and (ii), it follows that

$$\|(I - U_{[n]})x_n\| \to 0. \tag{3.12}$$

Since $Ax_n = T_{[n]}Ax_n$ in (3.2), we also have

$$\|(I - T_{[n]})Ax_n\| \to 0.$$

Next we claim that $\|x_{n+1} - x_n\| \to 0$. Indeed, since $U_{\lambda_n} x_n - x_n = \lambda_n(U_{[n]}x_n - x_n)$, an easy calculation yields

$$\begin{aligned} \|x_{n+1} - x_n\| &\leqslant \alpha_n\|f(x_n) - x_n\| + (1-\alpha_n)\|x_n - U_{\lambda_n}x_n\| \\ &\leqslant \alpha_n\|f(x_n) - x_n\| + +\lambda_n\|(I - U_{[n]})x_n\| \to 0 \end{aligned}$$

by the help of (3.12) and $\alpha_n \to 0$. Now choose a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup u \in H_1$ and

$$\limsup_{n\to\infty}\langle f(\hat{x}) - \hat{x}, x_n - \hat{x}\rangle = \lim_{k\to\infty}\langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle \tag{3.13}$$

by boundedness of $\{x_n\}$. Obviously, $u \in \omega_w(x_n) \subset \Omega$ because all hypotheses of Lemma 3.3 are fulfilled with $x_n = y_n$. Therefore, it follows from (3.13), (3.3), and $x_{n_k} \rightharpoonup u \in \Omega$ that

$$\limsup_{n\to\infty}\langle f(\hat{x}) - \hat{x}, x_n - \hat{x}\rangle = \langle f(\hat{x}) - \hat{x}, u - \hat{x}\rangle \leqslant 0,$$

which proves the inequality (3.11).

(b) If $\rho_n \neq 0$, by using a simple inequality with no $(1-\alpha_n)$ in (3.10) and the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, we have

$$\lambda_n(1-\lambda_n-\mu)\|(I - U_{[n]})y_n\|^2 + \frac{(1-\eta)^2}{4}\left(\frac{\|(I - T_{[n]})Ax_n\|^2}{\|A^*(I - T_{[n]})Ax_n\|}\right)^2 \leqslant s_n - s_{n+1} + \alpha_n K.$$

Using the convergence of $\{s_n\}$ and the conditions (i) and (ii) we obtain that

$$\|(I-U_{[n]})y_n\| \to 0 \tag{3.14}$$

and

$$\frac{\|(I-T_{[n]})Ax_n\|^2}{\|A^*(I-T_{[n]})Ax_n\|} \to 0. \tag{3.15}$$

Moreover,

$$\frac{1}{\|A\|}\|(I-T_{[n]})Ax_n\| = \frac{\|(I-T_{[n]})Ax_n\|}{\|A\|} = \|(I-T_{[n]})Ax_n\| \frac{\|(I-T_{[n]})Ax_n\|}{\|A\|\,\|(I-T_{[n]})Ax_n\|} \leqslant \frac{\|(I-T_{[n]})Ax_n\|^2}{\|A^*(I-T_{[n]})Ax_n\|},$$

and so

$$\|(I-T_{[n]})Ax_n\| \to 0.$$

On one hand, since

$$\|x_n-y_n\| = \rho_n\|A^*(I-T_{[n]})Ax_n\| = \frac{(1-\eta)}{2}\frac{\|(I-T_{[n]})Ax_n\|^2}{\|A^*(I-T_{[n]})Ax_n\|} \to 0 \tag{3.16}$$

by (3.15), it follows that

$$\begin{aligned}\|x_{n+1}-x_n\| &\leqslant \alpha_n\|f(x_n)-x_n\| + (1-\alpha_n)\|x_n - U_{\lambda_n}y_n\|\\ &\leqslant \alpha_n\|f(x_n)-x_n\| + \|x_n-y_n\| + \|y_n - U_{\lambda_n}y_n\|\\ &= \alpha_n\|f(x_n)-x_n\| + \|x_n-y_n\| + \lambda_n\|(I-U_{[n]})y_n\| \to 0\end{aligned}$$

by the aids of $\alpha_n \to 0$, (3.14), and (3.16). Then we also have

$$\|y_{n+1}-y_n\| \leqslant \|y_{n+1}-x_{n+1}\| + \|x_{n+1}-x_n\| + \|x_n-y_n\| \to 0.$$

For employing the proof in Case I, choose the subsequence $\{x_{n_k}\} \subset \{x_n\}$ which satisfies (3.13) and $x_{n_k} \rightharpoonup u \in H_1$. Since all the assumptions of Lemma 3.3 are fulfilled, we conclude that $u \in \omega_w(x_n) = \omega_w(y_n) \subset \Omega$, which immediately gives the required inequality (3.11). Now we prove that $x_n \to \hat{x}$. In fact, use $\|x_{n+1}-x_n\| \to 0$ and (3.11) to induce that

$$\limsup_{n\to\infty}\langle f(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle \leqslant 0.$$

Then since all the assumptions of Lemma 2.4 are fulfilled, we conclude that $x_n \to \hat{x}$.

Case II. Suppose that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $s_{n_i} < s_{n_i+1}$ for all $i \geqslant 0$. By applying Lemma 2.8, we can take a nondecreasing sequence $\{\tau(n)\}_{n\geqslant n_0}$ of integers such that $\tau(n) \to \infty$ and

$$s_{\tau(n)} \leqslant s_{\tau(n)+1},\ s_n \leqslant s_{\tau(n)+1}, \quad \forall n \geqslant n_0. \tag{3.17}$$

First, we show that

$$\limsup_{n\to\infty}\langle f(\hat{x})-\hat{x}, x_{\tau(n)}-\hat{x}\rangle \leqslant 0. \tag{3.18}$$

(a) If $\rho_{\tau(n)} = 0$, by using a simple inequality with no $(1-\alpha_n)$ in (3.9), (3.17), and the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, we have

$$\lambda_{\tau(n)}(1-\mu-\lambda_{\tau(n)})\|(I-U_{[\tau(n)]})x_{\tau(n)}\|^2 \leqslant s_{\tau(n)} - s_{\tau(n)+1} + \alpha_{\tau(n)}K_0 \leqslant \alpha_{\tau(n)}K_0,$$

where $K_0 := \sup_{n\geqslant n_0}\{2\langle f(x_{\tau(n)})-\hat{x}, x_{\tau(n)+1}-\hat{x}\rangle\}$. So,

$$\|(I-U_{[\tau(n)]})x_{\tau(n)}\| \to 0.$$

Since $Ax_{\tau(n)} = T_{[\tau(n)]}Ax_{\tau(n)}$ in (3.2), it is obvious that

$$\|(I - T_{[\tau(n)]})Ax_{\tau(n)}\| \to 0.$$

By slightly modifying the proof of (a) in Case I we could prove

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0.$$

Now use Lemma 3.3, after equipped with $\{x_{\tau(n)}\}$ in place of $\{x_n\}$, to establish (3.18).

(b) If $\rho_{\tau(n)} \neq 0$, it follows from (3.10) and (3.17) that

$$\begin{aligned}\lambda_{\tau(n)}(1-\lambda_{\tau(n)}-\mu)\|(I-U_{[\tau(n)]})y_{\tau(n)}\|^2 + \frac{(1-\eta)^2}{4}\frac{\|(I-T_{[\tau(n)]})Ax_{\tau(n)}\|^4}{\|A^*(I-T_{[\tau(n)]})Ax_{\tau(n)}\|^2}\\ \leqslant s_{\tau(n)} - s_{\tau(n)+1} + \alpha_{\tau(n)}K_0 \leqslant \alpha_{\tau(n)}K_0 \to 0\end{aligned}$$

by the boundedness of $\{x_{\tau(n)}\}$ and $\{f(x_{\tau(n)})\}$ and $\alpha_{\tau(n)} \to 0$. In view of two conditions (i) and (ii), the above inequality yields

$$\|(I-U_{[\tau(n)]})y_{\tau(n)}\| \to 0 \ \text{ and } \ \frac{\|(I-T_{[\tau(n)]})Ax_{\tau(n)}\|^2}{\|A^*(I-T_{[\tau(n)]})Ax_{\tau(n)}\|} \to 0.$$

Now mimicking the proof of (b) in Case 1 we easily prove that all the sequences $\{\|(I-T_{[\tau(n)]})Ax_{\tau(n)})\|\}$, $\{\|x_{\tau(n)} - y_{\tau(n)}\|\}$, $\{\|x_{\tau(n)+1} - x_{\tau(n)}\|\}$, and $\{\|y_{\tau(n)} - y_{\tau(n)+1}\|\}$ converge to zero. Since all the hypotheses of Lemma 3.3 are fulfilled, if we choose a subsequence $\{\tau(k_n)\}$ of $\{\tau(n)\}_{n\geqslant n_0}$ such that $x_{\tau(k_n)} \rightharpoonup \nu \in H_1$ and

$$\limsup_{n\to\infty}\langle f(\hat{x}) - \hat{x}, x_{\tau(n)} - \hat{x}\rangle = \lim_{n\to\infty}\langle f(\hat{x}) - \hat{x}, x_{\tau(k_n)} - \hat{x}\rangle,$$

then $\nu \in \omega_w(x_{\tau(n)}) = \omega_w(y_{\tau(n)}) \subset \Omega$; so this equality becomes

$$\limsup_{n\to\infty}\langle f(\hat{x}) - \hat{x}, x_{\tau(n)} - \hat{x}\rangle = \langle f(\hat{x}) - \hat{x}, \nu - \hat{x}\rangle \leqslant 0$$

for $\hat{x} = P_\Omega(f(\hat{x}))$; (3.18) is thus obtained. Since $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0$, it follows from (3.18) that

$$\limsup_{n\to\infty}\langle f(\hat{x}) - \hat{x}, x_{\tau(n)+1} - \hat{x}\rangle \leqslant 0. \tag{3.19}$$

Secondly we show that $x_n \to 0$. Indeed, since $s_{\tau(n)} \leqslant s_{\tau(n)+1}$ for all $n \geqslant n_0$, a slight transformation of (3.8) yields

$$\alpha_{\tau(n)}s_{\tau(n)+1} + (1-\alpha_{\tau(n)})(s_{\tau(n)+1} - s_{\tau(n)}) \leqslant 2\alpha_{\tau(n)}\langle f(x_{\tau(n)}) - \hat{x}, x_{\tau(n)+1} - \hat{x}\rangle$$

and so

$$\alpha_{\tau(n)}s_{\tau(n)+1} \leqslant 2\alpha_{\tau(n)}\langle f(x_{\tau(n)}) - \hat{x}, x_{\tau(n)+1} - \hat{x}\rangle \ \Rightarrow \ 0 \leqslant s_{\tau(n)+1} \leqslant 2\langle f(x_{\tau(n)}) - \hat{x}, x_{\tau(n)+1} - \hat{x}\rangle$$

because $\alpha_n \in (0,1)$. Now taking the limit superior on both sides as $n \to \infty$ and using (3.19), we obtain $s_{\tau(n)+1} \to 0$; hence $s_n \to 0$ because of $s_n \leqslant s_{\tau(n)+1}$ for all $n \geqslant n_0$ in (3.17), completing the proof. □

Remark 3.5. The main result of Theorem 3.4 is a cyclic explicit version of Theorem 3.2 in [6]. If we take $p = q = 1$, the algorithm (3.1) equipped with $\lambda_n = \lambda$ for all n reduces to (1.6).

Finally we shall give an example which satisfies all the conditions of the solution set Ω of the MCFP (1.2), the mappings $\{U_i\}_{i=1}^p$, and $\{T_j\}_{j=1}^q$ in Assumption 3.1.

Example 3.6. Let $H_1 = H_2 = H_3 = \ell_2$ and let $i \in \{1, 2, \dots, p\}$ and $j \in \{1, 2, \dots, q\}$ be arbitrarily fixed. Let U_i, $T_j : \ell_2 \to \ell_2$ be defined by $U_i x = -2ix$ and $T_j x = -(2j+1)x$ for all $x \in \ell_2$. Then it is easy to see that $\cap_{i=1}^{p} \mathrm{Fix}(U_i) = \{0\} = \cap_{j=1}^{q} \mathrm{Fix}(T_j)$ and $A0 = 0$. Thus $\Omega = \{0\} \neq \emptyset$. Also U_i is μ_i-demicontractive and T_j is η_j-demicontractive by Example 2.5 in [12], where $\mu_i = \frac{2i-1}{2i+1}$, $\mu = \max_{1 \leqslant i \leqslant p}, \frac{2i-1}{2i+1}, \mu_i = \frac{2p-1}{2p+1}$, $\eta_j = \frac{j}{j+1}$, and $\eta = \max_{1 \leqslant j \leqslant q} \eta_j = \frac{q}{q+1}$; then $I - U_i$ and $I - T_j$ are demiclosed at 0 by Remark 2.12 in [12].

Next we give an example which satisfies the conditions (i) and (ii) in Theorem 3.4.

Example 3.7. We can take $\alpha_n = \frac{1}{n}$ and $\lambda_n = \frac{k}{k+1}(1-\mu) + (-1)^n \frac{1}{n}$ for all n, where $k \in \mathbb{N}$ is arbitrarily fixed. Then $\lim_{n \to \infty} \lambda_n = \frac{k}{k+1}(1-\mu) < 1-\mu$, which satisfies the condition (ii) of Theorem 3.4.

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