



## Differentiability of pseudo-dual-quaternionic functions with a differential operator



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### Abstract

This paper introduces the new concept of pseudo-dual-quaternions and some of their basic properties based on matrices. We extend the concept of differentiability to pseudo-dual-quaternionic functions. Also, we propose a corresponding Cauchy-Riemann formulas induced the properties of a holomorphic function of pseudo-dual-quaternionic variables.

**Keywords:** Dual-quaternion, differential operators, differentiability, Cauchy-Riemann formulas.

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### 1. Introduction

Dual numbers are based on the non-zero and non-real term  $\varepsilon$  satisfying  $\varepsilon^2 = 0$ , instead of  $i$  which constitutes complex numbers and satisfies  $i^2 = -1$ . Dual numbers are in the two-dimensional form of  $x + \varepsilon y$  with base 1 and  $\varepsilon$  associative-commutative algebras over the field of real numbers. In 1873, dual numbers introduced by William Clifford [2] are a type of generalized complex numbers. Unlike the field of complex numbers, The algebra of dual numbers comprises zero divisors and is considered over an arbitrary field or commutative ring. By using the structure of dual numbers and the properties of  $\varepsilon$ , it has been studied in a variety of fields including non-Euclidean geometry. Study [13] defined dual angles having the form of dual numbers to represent the relation between two skew lines in the Euclidean space. Messelmi [10] generalized holomorphicity to dual-complex functions simplifying manipulation of dual-complex numbers, based on matrices. Since then, dual numbers have been applied to study the kinematics, dynamics and open-chain robot manipulators. Dual numbers are useful for geometrical treatments with analytical methods in kinematics and dynamics of spatial mechanisms. Thus, the application of dual numbers in the study of problems in science and engineering is being studied (see [1, 3, 12, 15]).

The theory of functions over the complex field has induced the theory for the non-trivial real associative and invertible algebra. In 1843, Hamilton [6] discovered a 4-dimensional division algebra called

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the quaternions with 3 imaginary values which are multiplication provided proper inverses. The quaternionic derivatives of the functions of a quaternion variable can be represented by quaternionic power series similar to power series in four real variables. In 1935, Fueter [4] proposed a definition of regular for quaternionic functions by using the analogue Cauchy-Riemann equations. Also, Fueter [5] led to analogues of Cauchy's theorem and Cauchy's integral formula. Sudbery [14] gave simple proofs of the main theorems of quaternionic analysis and made clear the relationship between quaternionic analysis and complex analysis by means of the exterior differential calculus. Since then, quaternionic analysis has been investigated by various approaches and calculation processes (see [7, 9, 11]).

In this paper, we combine the unit  $\varepsilon$  of dual numbers with the structure of quaternions. From this, we obtain the modified quaternions which is called a pseudo-dual-quaternion (PDQ). Quaternions possess the noncommutative rule of the product and preserve each part of their units after computing operators defined on quaternions. However, PDQs preserve the noncommutative rule of the product and can express the projection of an arbitrary PDQ onto a plane due to the property of the product for PDQs. We have investigated the properties of the unit of dual numbers (see [8]) and we deal with quaternions by means of the unit of dual numbers. This paper introduces PDQs and some of their basic properties. Also, we give an expression of PDQs based on matrices. We extend the concept of holomorphicity and a corresponding Cauchy-Riemann formulas for pseudo-dual-quaternionic functions.

## 2. Preliminaries

The ordinary quaternion is a four-parameter of complex number systems which has two components. Let  $\mathbb{D}_p$  denote the set of PDQs, which has elements of the form

$$p = x_0 + ix_1 + jx_2 + kx_3,$$

where  $x_r$  ( $r = 0, 1, 2, 3$ ) are real,  $i$  is the imaginary unit, and  $j$  is the unit element such that

$$i^2 = -1, \quad j^2 = 0$$

and let  $k := ij$ , then

$$ij = -ji = k, \quad jk = kj = 0, \quad ki = -ik = j, \tag{2.1}$$

which is isomorphic to  $\mathbb{R}^4$ . By the properties (2.1) of  $i$  and  $j$ , we have the following rules for an addition and a product that are

$$p + q = (x_0 + y_0) + i(x_1 + y_1) + j(x_2 + y_2) + k(x_3 + y_3)$$

and

$$pq = (x_0y_0 - x_1y_1) + i(x_1y_0 + x_0y_1) + j(x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3) + k(x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3),$$

respectively. From the definition of the product for PDQs, we give a conjugation of that numbers. The conjugation of each complex number is important to play a role for algebraic and geometric properties of  $\mathbb{C}$ . For PDQs, we can also extend this notion. Let  $p = x_0 + ix_1 + jx_2 + kx_3$  be a PDQ. Then we define the conjugate of  $p$  by the formula

$$p^* = x_0 - ix_1 - jx_2 - kx_3,$$

where  $p$  has the following relation

$$pp^* = p^*p \in \mathbb{R}.$$

Thus, we have the modulus, denoted by  $\mathcal{N}(p)$ , and the inverse element, denoted by  $\frac{1}{p}$  of  $p \in \mathcal{D}_p$ :

$$\mathcal{N}(p) := pp^* = x_0^2 + x_1^2$$

and

$$\frac{1}{p} = \frac{p^*}{N(p)} = \frac{x_0}{x_0^2 + x_1^2} - i \frac{x_1}{x_0^2 + x_1^2} - j \frac{x_2}{x_0^2 + x_1^2} - k \frac{x_3}{x_0^2 + x_1^2}, \quad (x_0, x_1 \neq 0),$$

respectively. Furthermore,  $\sqrt{N(p)}$  is expressed by Figure 1.

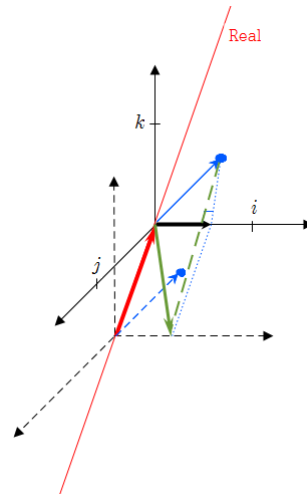


Figure 1:  $\sqrt{N(p)}$  is the green vector which is the orthographic projection of the blue line onto the plane consisting of the red line and the line containing  $i$ .

PDQs form a non-commutative ring give the structure of the four dimensional real Algebra. The algebra  $\mathcal{D}_p$  is not a division algebra since if it has the elements satisfying  $x_0^2 + x_1^2 = 0$ , that is, if  $x_0 = x_1 = 0$ , the element is not invertible. All elements with the above property are zero divisors. Then, we introduce the set  $\mathcal{D}_0$  of zero divisors of  $\mathcal{D}_p$  which can be called the null part of  $\mathcal{D}_p$  as

$$\mathcal{D}_0 = \{p \in \mathcal{D}_p \mid p = x_0 + ix_1 + jx_2 + kx_3, x_0^2 + x_1^2 = 0\} = \{p \in \mathcal{D}_p \mid p = jx_2 + kx_3, x_2, x_3 \in \mathbb{R}\}.$$

There can be also the representation such as

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}.$$

So, PDQs can be also written by using matrices with the above units:

$$p = \begin{pmatrix} z_1 & z_2 \\ 0 & \bar{z}_1 \end{pmatrix},$$

where  $z_1 = x_0 + ix_1$ ,  $z_2 = x_2 + ix_3$  and  $\bar{z}_1 = x_0 - ix_1$ .

We let the set of PDQs expressed by matrices, which is denoted by  $\mathbb{M}_\delta$ ,

$$\mathbb{M}_\delta = \left\{ A \in M_2(\mathbb{C}) \mid A = \begin{pmatrix} z_1 & z_2 \\ 0 & \bar{z}_1 \end{pmatrix} \right\}.$$

Thus, the set  $\mathbb{M}_\delta$  is a subring of  $M_2(\mathbb{C})$  which forms a four dimensional real associative and non-commutative Algebra. Let us now define the map

$$\mathcal{T}_\delta : \mathcal{D}_p \rightarrow \mathbb{M}_\delta, \quad \mathcal{T}_\delta(p) = \begin{pmatrix} z_1 & z_2 \\ 0 & \bar{z}_1 \end{pmatrix}.$$

From the definition of the map  $\mathcal{T}_\delta$ , we can give a correspondence between the two algebras  $\mathcal{D}_p$  and  $\mathbb{M}_\delta$  via the map  $\mathcal{T}_\delta$ . Furthermore, the map  $\mathcal{T}_\delta$  is an isomorphism of algebras.

**Proposition 2.1.** *We also note that  $p \in \mathcal{D}_0$  if and only if  $\det(\mathcal{T}_\delta(p)) = 0$ .*

*Proof.* Suppose that  $p \in \mathcal{D}_0$ . Then  $p = jx_2 + kx_3$ ,  $x_2, x_3 \in \mathbb{R}$ , that is, for  $p = x_0 + ix_1 + jx_2 + kx_3$ ,  $x_0 = x_1 = 0$ . Hence, we have

$$\det(\mathcal{T}_\delta(p)) = \begin{vmatrix} z_1 & z_2 \\ 0 & \bar{z}_1 \end{vmatrix} = x_0^2 + x_1^2 = 0.$$

Conversely, by the definition of  $\mathcal{D}_0$  and  $\mathcal{T}_\delta$ , if  $\det(\mathcal{T}_\delta(p)) = 0$ , then  $x_0 = x_1 = 0$ . Thus,  $p$  has the form  $p = jx_2 + kx_3 \in \mathcal{D}_0$ . □

Now, we give the map, denoted by  $|\cdot|_\delta$ :

$$|\cdot|_\delta : \mathcal{D}_p \rightarrow \mathbb{R}^+, \quad |p|_\delta = |z_1 \bar{z}_1|^{1/2} = \sqrt{x_0^2 + x_1^2},$$

where  $\mathbb{R}^+$  is the set of positive real numbers.

**Proposition 2.2.** *The map  $|\cdot|_\delta$  has the following properties:*

- (i)  $|p|_\delta = |\det(\mathcal{T}_\delta(p))|^{1/2}$  for  $p \in \mathcal{D}_p$ ;
- (ii)  $|p + q|_\delta \leq |p|_\delta + |q|_\delta$  for  $p, q \in \mathcal{D}_p$ ;
- (iii)  $|pq|_\delta = |p|_\delta |q|_\delta$  for  $p, q \in \mathcal{D}_p$ ;
- (iv)  $|\alpha p|_\delta = |\alpha| |p|_\delta$  for  $p \in \mathcal{D}_p$ ;
- (v)  $|p|_\delta = 0$  if and only if  $p \in \mathcal{D}_0$ ,

where  $|\cdot|$  is the usual norm in complex analysis.

The proofs of the above properties are an immediate consequence of the definition of  $|\cdot|_\delta$  and their calculations. Thus, we can induce a structure of a corresponding topology over the algebra  $\mathcal{D}_p$ . So, we construct a pseudo-dual disk and pseudo-dual sphere of center  $p_0 = x_0^o + ix_1^o + jx_2^o + kx_3^o$  and radius  $r > 0$ , in  $\mathcal{D}_p$ , by

$$D_\delta(p_0, r) := \{p = x_0 + ix_1 + jx_2 + kx_3 \in \mathcal{D}_p \mid |p - p_0|_\delta < r\}$$

and

$$S_\delta(p_0, r) := \{p = x_0 + ix_1 + jx_2 + kx_3 \in \mathcal{D}_p \mid |p - p_0|_\delta = r\},$$

respectively.

### 3. Hyperholomorphicity of pseudo-dual-quaternionic functions

We consider some properties of functions of pseudo-dual-quaternionic variable. We investigate the continuity of pseudo-dual-quaternionic functions and the holomorphicity in those functions.

**Definition 3.1.** Let  $\Omega$  be an open subset of  $\mathcal{D}_p$ . Then for  $p = x_0 + ix_1 + jx_2 + kx_3$ , a function  $f : \Omega \rightarrow \mathcal{D}_p$ ,

$$f(p) = u_0 + iu_1 + ju_2 + ku_3,$$

where  $u_t = u_t(x_0, x_1, x_2, x_3)$  ( $t=0, 1, 2, 3$ ) are real-valued functions, is said to be a pseudo-dual-quaternionic function (PDQ-function) if it is a mapping from a subset  $\Omega$  of  $\mathcal{D}_p$  to  $\mathcal{D}_p$ .

**Definition 3.2.** Let  $\Omega$  be an open subset of  $\mathcal{D}_p$ . Then for  $p \in \Omega$ , a PDQ-function  $f : \Omega \rightarrow \mathcal{D}_p$  is said to be continuous at  $p_0 = x_0^o + ix_1^o + jx_2^o + kx_3^o$  if there exists the limit satisfying

$$\lim_{p \rightarrow p_0} f(p) = f(p_0),$$

where the limit is calculated for each component,

$$\lim_{p \rightarrow p_0} f(p) = \lim_{\substack{x_t \rightarrow x_t^0 \\ (t=0,1,2,3)}} f(p) = f(p_0).$$

That is, for given  $\varepsilon > 0$  there exists  $\delta > 0$  such that each  $|x_t - x_t^0| < \delta$  ( $t = 0, 1, 2, 3$ ) implies  $|u_r - u_r^0| < \varepsilon$  ( $r = 0, 1, 2, 3$ ), where  $u_r^0 = u_r(x_0^0, x_1^0, x_2^0, x_3^0)$ . Moreover, the function  $f$  is continuous in  $\Omega$  if  $f$  is continuous at every point of  $\Omega$ .

**Definition 3.3.** Let  $\Omega$  be an open subset of  $\mathcal{D}_p$ . Then for  $p \in \Omega$ , a PDQ-function  $f : \Omega \rightarrow \mathcal{D}_p$  is said to be left-differentiable at  $p_0 = x_0^0 + ix_1^0 + jx_2^0 + kx_3^0$  if the limit exists

$$\frac{df}{dp}(p_0)_L := \lim_{p \rightarrow p_0} \frac{1}{p - p_0} \{f(p) - f(p_0)\},$$

where the limit is also calculated for each component, and  $f$  is said to be left-differentiable at  $p_0$  if the limit exists

$$\frac{df}{dp}(p_0)_R := \lim_{p \rightarrow p_0} \{f(p) - f(p_0)\} \frac{1}{p - p_0},$$

where the limit is also calculated for each component. Here  $\frac{df}{dp}(p_0)_L$  is called the left-derivative and  $\frac{df}{dp}(p_0)_R$  is called the right-derivative of  $f$  at the point  $p_0$ , respectively. Moreover, if  $f$  is left(right)-differentiable for every point in a neighborhood of the point  $p_0$ , then  $f$  is called left(right)-holomorphic at  $p_0$ . The function  $f$  is left(right)-holomorphic in  $\Omega$  if  $f$  is left(right)-holomorphic at every point of  $\Omega$ .

Since PDQs have the non-commutative rule of product, we need to consider the order of multiplication. So we give two definitions of differentiable functions such as the left-differentiability and the right-differentiability. The right differentiability is similar to the left differentiability with respect to the property and the process of calculations, so we deal with the left-differentiability. In addition, the derivative that can be expressed later is the left-differentiable, and for convenience it is expressed as

$$\frac{df}{dp}(p_0)_L := \frac{df}{dp}(p_0) \quad \text{and} \quad \lim_{p \rightarrow p_0} \frac{1}{p - p_0} \{f(p) - f(p_0)\} := \lim_{p \rightarrow p_0} \frac{f(p) - f(p_0)}{p - p_0}.$$

Now, we generalize a corresponding Cauchy-Riemann formulas to PDQ-functions.

**Theorem 3.4.** Let  $\Omega$  be an open subset of  $\mathcal{D}_p$  and  $f$  be a PDQ-function in  $\Omega$ . If the function  $f$  is holomorphic at  $p_0$ , then  $f$  satisfies

$$\frac{df}{dp}i = \frac{\partial f}{\partial x_1}, \quad \frac{df}{dp}j = \frac{\partial f}{\partial x_2}, \quad \frac{df}{dp}k = \frac{\partial f}{\partial x_3}, \quad \text{and} \quad \frac{df}{dp} = \frac{\partial f}{\partial x_0}.$$

*Proof.* Suppose  $f$  is holomorphic at  $p_0$ . Then there exists the limit  $\frac{df}{dp}(p_0)$ . So, we can express as follows:

$$\begin{aligned} \frac{\partial f}{\partial x_0} &= \lim_{\substack{x_t \rightarrow x_0^0 \\ (t=0;1;2;3)}} \frac{f(x_0 + ix_1^0 + jx_2^0 + kx_3^0) - f(x_0^0 + ix_1^0 + jx_2^0 + kx_3^0)}{x_0 - x_0^0} \\ &= \lim_{\substack{x_t \rightarrow x_0^0 \\ (t=0;1;2;3)}} \frac{f(x_0 + ix_1 + jx_2 + kx_3) - f(x_0^0 + ix_1^0 + jx_2^0 + kx_3^0)}{x_0 - x_0^0} \\ &\quad + \lim_{\substack{x_t \rightarrow x_0^0 \\ (t=0;1;2;3)}} \frac{f(x_0 + ix_1^0 + jx_2^0 + kx_3^0) - f(x_0 + ix_1 + jx_2^0 + kx_3^0)}{x_0 - x_0^0} \\ &\quad + \lim_{\substack{x_t \rightarrow x_0^0 \\ (t=0;1;2;3)}} \frac{f(x_0 + ix_1 + jx_2^0 + kx_3^0) - f(x_0 + ix_1 + jx_2 + kx_3^0)}{x_0 - x_0^0} \\ &\quad + \lim_{\substack{x_t \rightarrow x_0^0 \\ (t=0;1;2;3)}} \frac{f(x_0 + ix_1 + jx_2 + kx_3^0) - f(x_0 + ix_1 + jx_2 + kx_3)}{x_0 - x_0^0} \end{aligned}$$

$$\begin{aligned}
 &= \lim_{p \rightarrow p_0} \frac{f(p) - f(p_0)}{p_0 - p_0} \frac{p_0 - p_0}{x_0 - x_0^0} - \lim_{\substack{x_t \rightarrow x_0^0 \\ (t=0;1;2;3)}} \frac{f(x_0 + ix_1 + jx_2^0 + kx_3^0) - f(x_0 + ix_1^0 + jx_2^0 + kx_3^0)}{x_1 - x_1^0} \frac{x_1 - x_1^0}{x_0 - x_0^0} \\
 &\quad - \lim_{\substack{x_t \rightarrow x_0^0 \\ (t=0;1;2;3)}} \frac{f(x_0 + ix_1 + jx_2 + kx_3^0) - f(x_0 + ix_1 + jx_2^0 + kx_3^0)}{x_2 - x_2^0} \frac{x_2 - x_2^0}{x_0 - x_0^0} \\
 &\quad - \lim_{\substack{x_t \rightarrow x_0^0 \\ (t=0;1;2;3)}} \frac{f(x_0 + ix_1 + jx_2 + kx_3) - f(x_0 + ix_1 + jx_2 + kx_3^0)}{x_3 - x_3^0} \frac{x_3 - x_3^0}{x_0 - x_0^0}.
 \end{aligned}$$

Since we have

$$\lim_{p \rightarrow p_0} \frac{f(p) - f(p_0)}{p_0 - p_0} \frac{p_0 - p_0}{x_0 - x_0^0} = \lim_{p \rightarrow p_0} \frac{f(p) - f(p_0)}{p_0 - p_0} \frac{(x_0 - x_0^0) + (x_1 - x_1^0)i + (x_2 - x_2^0)j + (x_3 - x_3^0)k}{x_0 - x_0^0},$$

we obtain

$$\frac{\partial f}{\partial x_0} = \frac{df}{dp} + \left( \frac{df}{dp} i - \frac{\partial f}{\partial x_1} \right) \frac{x_1 - x_1^0}{x_0 - x_0^0} + \left( \frac{df}{dp} j - \frac{\partial f}{\partial x_2} \right) \frac{x_2 - x_2^0}{x_0 - x_0^0} + \left( \frac{df}{dp} k - \frac{\partial f}{\partial x_3} \right) \frac{x_3 - x_3^0}{x_0 - x_0^0}.$$

Since  $f$  is holomorphic at  $p_0$ , in order to determine the existence of the limit  $\frac{df}{dp}$ , the limits

$$\left( \frac{df}{dp} i - \frac{\partial f}{\partial x_1} \right), \quad \left( \frac{df}{dp} j - \frac{\partial f}{\partial x_2} \right), \quad \text{and} \quad \left( \frac{df}{dp} k - \frac{\partial f}{\partial x_3} \right)$$

should be independent to ratios

$$\frac{x_1 - x_1^0}{x_0 - x_0^0}, \quad \frac{x_2 - x_2^0}{x_0 - x_0^0}, \quad \text{and} \quad \frac{x_3 - x_3^0}{x_0 - x_0^0},$$

respectively, and then

$$\frac{\partial f}{\partial x_0} = \frac{df}{dp}.$$

Thus, the equations

$$\frac{df}{dp} i = \frac{\partial f}{\partial x_1}, \quad \frac{df}{dp} j = \frac{\partial f}{\partial x_2}, \quad \frac{df}{dp} k = \frac{\partial f}{\partial x_3}, \quad \text{and} \quad \frac{\partial f}{\partial x_0} = \frac{df}{dp}$$

are obtained. □

**Theorem 3.5.** Let  $\Omega$  be an open subset of  $\mathcal{D}_p$  and  $f$  be a PDQ-function in  $\Omega$ . A function  $f$  is holomorphic on  $\Omega$  if and only if the function  $f$  satisfies

$$D^*f = 0,$$

where the differential operator  $D^*$  is

$$D^* = \vec{v} \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}$$

with  $\vec{v} = i + j + k$  satisfying  $\vec{v}^2 = -1$ .

*Proof.* Since the function  $f$  is holomorphic on  $\Omega$ , the following equations is derived from Theorem 3.4.

$$\frac{\partial f}{\partial x_0} i = \frac{\partial f}{\partial x_1}, \quad \frac{\partial f}{\partial x_0} j = \frac{\partial f}{\partial x_2}, \quad \frac{\partial f}{\partial x_0} k = \frac{\partial f}{\partial x_3}.$$

If the above equations are detailed, then we have

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial u_r}{\partial x_r} \quad (r = 1, 2, 3),$$

$$\begin{aligned} \frac{\partial u_r}{\partial x_t} &= 0 \quad (r = 0, 1; t = 2, 3), \\ \frac{\partial u_0}{\partial x_1} + \frac{\partial u_1}{\partial x_0} &= 0, \quad \frac{\partial u_2}{\partial x_0} + \frac{\partial u_3}{\partial x_1} = 0, \quad \frac{\partial u_1}{\partial x_0} + \frac{\partial u_2}{\partial x_3} = 0, \\ \frac{\partial u_3}{\partial x_2} - \frac{\partial u_1}{\partial x_0} &= 0, \quad \frac{\partial u_3}{\partial x_0} - \frac{\partial u_2}{\partial x_1} = 0. \end{aligned}$$

That is, we have

$$\left\{ \begin{aligned} \frac{\partial u_0}{\partial x_0} &= \frac{\partial u_r}{\partial x_r}, \quad (r = 1, 2, 3), \\ \frac{\partial u_r}{\partial x_t} &= 0, \quad (r = 0, 1; t = 2, 3), \\ \frac{\partial u_0}{\partial x_1} &= \frac{\partial u_2}{\partial x_3} = -\frac{\partial u_3}{\partial x_2} = -\frac{\partial u_1}{\partial x_0}, \\ \frac{\partial u_2}{\partial x_0} &= -\frac{\partial u_3}{\partial x_1}, \\ \frac{\partial u_3}{\partial x_0} &= \frac{\partial u_2}{\partial x_1}. \end{aligned} \right. \tag{3.1}$$

Comparing the result of the calculation of

$$D^*f = \vec{v} \frac{\partial f}{\partial x_0} - \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} - \frac{\partial f}{\partial x_3},$$

with the equations (3.1), we can obtain that the holomorphic function  $f$  satisfies  $D^*f = 0$ . □

**Corollary 3.6.** *Let  $\Omega$  be an open subset of  $\mathcal{D}_p$ . Suppose that  $f$  is holomorphic and continuously differentiable in a domain  $\Omega$  with differentiable boundary. Then the function  $f$  satisfies*

$$\int_{\partial\Omega} D^*f = 0.$$

*Proof.* Cauchy's theorem holds for any contour of integration. The integral theorems, for holomorphic PDQ-functions with the values of the interior of a region in terms of its values on the boundary, holds for a general rectifiable boundary. From this, applying the Stokes theorem, it follows that if  $f$  is holomorphic and continuously differentiable in a domain  $\Omega$  with differentiable boundary, then the function  $f$  satisfies  $D^*f = 0$  and thus, we obtain

$$\int_{\partial\Omega} D^*f = 0. \tag{□}$$

#### 4. Conclusion

In this paper, the algebraic and analytic properties of the modified quaternions, called a pseudo-dual-quaternion, is obtained. Like the rule of the product for quaternions, and PDQs preserve the noncommutative rule of the product. However, since PDQs can express the projection of an arbitrary PDQ onto a plane due to the property of the product for PDQs, the function of PDQ variables can be extended to the new concept of holomorphicity of PDQ functions. Thus, we have defined derivatives to be applied to the PDQ system. A calculation method using a formula called a corresponding Cauchy-Riemann formulas can be also substituted for the definition of derivative. And, the theorem related to this definition of derivative can be proved using the corresponding Cauchy-Riemann formulas. Therefore, the mathematical approach based on the function over the PDQ system and its application to the rate of change such as the velocity of objects and the slope of the tangent line can be induced.

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