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Efficient approximations of finite and infinite real alternating p-series

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Abstract

For $n \in \mathbb{N}$ and $p \in \mathbb{R}$ the nth partial sum of the alternating p-series, known also as alternating generalized harmonic number of order p,

$$H^*(n,p) := \sum_{i=1}^{n} (-1)^{i+1} \frac{1}{i^p}$$

is given in the form

$$H^*(n,p) = S_q(k,n,p) + r_q^*(k,n,p),$$

where $k, q \in \mathbb{N}$ with $k < \lfloor n/2 \rfloor$ are parameters, controlling the magnitude of the error term $r_q^*(k,n,p)$. The function $S_q(k,n,p)$ consists of $2(k+1)+q$ simple summands and $r_q^*(k,n,p)$ is estimated for $q > -p+1$, as

$$\left| r_q^*(k,n,p) \right| < \frac{|p|(|p|+1)\cdots(|p|+q-1)\pi^{p+1}}{3(p+q-1)(2k\pi)^{p+q-1}}.$$

Additionally, for $p \in \mathbb{R}^+$ and $k, q \in \mathbb{N}$, we have

$$\left| r_q^*(k,\infty,p) \right| \leqslant \frac{p(p+1)\cdots(p+q-2)\pi^{p+1}}{3(2k\pi)^{p+q-1}}.$$

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1. Introduction

The finite p-series

$$H(k,p) := \sum_{i=1}^{k} \frac{1}{i^p}, \quad p \in \mathbb{R},$$

known also as the kth (generalized) harmonic number of order p, has been studied recently by Abalo [1],

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Hansheng [7], Chlebus [3], and Lampret [9]. Also, a great deal of authors have been investigating the properties of these numbers involved in a wide range of mathematics, see for example [4–6, 8, 15, 16, 21, 22, 25, 26].

However, the finite alternating p-series

$$H^*(k,p) := \sum_{i=1}^{k} (-1)^{i+1} \frac{1}{i^p}, \quad p \in \mathbb{R},$$

known also as the alternating (generalized) kth harmonic number of order p, has been studied less intensively. Actually, some papers investigate several properties of alternating (generalized) harmonic numbers, see e.g. [18–20, 23].

Twelve years ago, it was shown in [23] that for the nth alternating harmonic number (in short the nth alternating harmonic) $H^*(n,1)$ the best constants a and b, satisfying the double inequality

$$\frac{1}{2n+a} \leqslant \left| H^*(\infty,1) - H^*(n,1) \right| \leqslant \frac{1}{2n+b}$$

for every $n \in \mathbb{N}$, are $a = \frac{1}{2n - \ln 2}$ and $b = 1$. Recently, Sintamarian [17] published the paper approximating the sequence of partial sums $H^*(k,1)$ of alternating harmonic series:

$$\frac{1}{4k^2+2} < \left| H^*(\infty,1) - (-1)^{k-1}\frac{1}{2k} - \ln 2 \right| \leqslant \frac{1}{4k^2+1.77\ldots}, \quad k \in \mathbb{N}.$$

An extremely accurate estimate of the sequence of alternating harmonic numbers $H^*(n,1)$, derived using the Euler-Mascheroni constant, was presented recently in [11].

In [12] the sum of conditionally convergent Leibniz series was estimated by collecting its positive and negative summands. This way the sum of Leibniz series becomes equal to the sum of some absolutely convergent series, which can be approximated using standard Euler-Maclaurin summation formula. The similar technique was used also in [14].

Fortunately, we have at our disposal the Euler-Boole summation formula which is suitable precisely for alternating series. Using this formula we shall accurately estimate the sums of finite and infinite alternating p-series; also in case $p > 0$ and $p \approx 0$ when the convergence is very slow. We remark that some treatments of alternating sums of powers can also be found in [8].

2. Preliminaries (summation formulas)

For finite and infinite alternating p-series we shall use the Euler-Boole summation formula presented in [13, Lemma 3].

Lemma 2.1 (The Euler-Boole summation formula)**.** *For integers $1 \leqslant m < n$ and $q \geqslant 1$, and for a function $f \in C^q[2m, 2n]$, the following summation formula*[1] *holds:*

$$\sum_{i=2m}^{2n-1} (-1)^{i+1} f(i) = \frac{1}{2}\left[f(2n) - f(2m)\right] - \sum_{i=1}^{\lfloor q/2 \rfloor} \left(4^i - 1\right) \frac{B_{2i}}{(2i)!} \left[f^{(2i-1)}(2n) - f^{(2i-1)}(2m)\right] + R_q(m,n),$$

where the remainder $R_q(m,n)$ is given as

$$R_q(m,n) = \frac{1}{2}(-2)^q \int_{2m}^{2n} \left[W_q\left(\tfrac{1}{2}t\right) - W_q\left(\tfrac{1}{2}t - \tfrac{1}{2}\right)\right] f^{(q)}(t)\, dt$$

and is roughly estimated in the following way

$$|R_q(m,n)| \leqslant 2^q \cdot \nu_q \int_{2m}^{2n} \left| f^{(q)}(t) \right| \, dt \leqslant \frac{1}{3\pi^{q-2}} \int_{2m}^{2n} \left| f^{(q)}(t) \right| \, dt. \tag{2.1}$$

[1] By definition we have $\sum_{i=m}^{n} x_i = 0$ for $m > n$.

Above, $B_k := B_k(0)$ are known as kth Bernoulli coefficient (or Bernoulli number), where the symbol $B_k(x)$ denotes the kth Bernoulli polynomial ($k \geqslant 0$), defined by using the identity $\frac{te^{xt}}{e^t-1} \equiv \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}$ ($|t| < 2\pi$). The symbol $W_q(x)$ stands for the standardized qth Bernoulli 1-periodic function, i.e. the 1-periodic continuation of $\frac{1}{q!}B_q(x)\big|_{[0,1)}$ (a restriction) from the interval $[0,1)$ to $\mathbb{R}$. More precisely[2], for $q \geqslant 0$, we have $W_q(x) \equiv \frac{1}{q!}B_q(x - \lfloor x \rfloor)$.

Lemma 2.2. *Let f be a polynomial of degree $\nu \in \mathbb{N}$. Then the remainder in Lemma 2.1 is equal to zero, $R_q(m,n) = 0$, for any $m, n, q \in \mathbb{N}$ with $m < n$ and $q \geqslant \nu$.*

Proof. If f is a polynomial of degree $\nu \in \mathbb{N}$, then $f^{(q)}(t) \equiv \text{const}$, (possibly 0), for $q \geqslant \nu$. Therefore, according to Lemma 2.1, we have

$$R_q(m,n) = C\left[\int_m^n W_q(x)\,dx - \int_m^n W_q\left(x - \tfrac{1}{2}\right)\,dx\right] = C(0-0) = 0$$

for some constant C. Indeed, $W_q(x)$ is 1-periodic and $\int_0^1 W_q(x)\,dx = \int_0^1 B_q(x)\,dx = 0$. □

Direct consequence of Lemma 2.1, considering the identity $\sum_{i=1}^{2k} x_i = \sum_{i=1}^{2k-1} x_i + x_{2k}$, is the following lemma.

Lemma 2.3. *Let m and q be positive integers and let $f \in C^q[1,\infty)$ satisfies the following three conditions:*

1. $\int_1^{\infty} \left|f^{(q)}(x)\right|\,dx < \infty$;
2. $\lim_{n\in\mathbb{N},\, n\mapsto\infty} f(2n) = 0$;
3. *Finite* $\lambda_{2i-1} := \lim_{n\in\mathbb{N},\, n\mapsto\infty} f^{(2i-1)}(2n)$ *exists for every integer* i, $1 \leqslant i \leqslant \lfloor \frac{q}{2} \rfloor$.

Then $\sum_{i=1}^{\infty}(-1)^{i+1}f(i)$ converges and the equality[3]

$$\sum_{i=1}^{\infty}(-1)^{i+1}f(i) = \sum_{i=1}^{2m-1}(-1)^{i+1}f(i) - \frac{f(2m)}{2} - \sum_{i=1}^{\lfloor q/2 \rfloor}\left(4^i - 1\right)\frac{B_{2i}}{(2i)!}\Big(\lambda_{2i-1} - f^{(2i-1)}(2m)\Big) + r_q(m)$$

holds with the remainder estimated as

$$|r_q(m)| \leqslant \frac{1}{3\pi^{q-2}}\int_{2m}^{\infty}\left|f^{(q)}(t)\right|\,dt.$$

3. Approximating the alternating pseries

For $x \in \mathbb{R}$ and $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, we define the upper (rising) Pochhammer [10] product $x^{(k)}$ as

$$x^{(0)} := 1 \quad \text{and} \quad x^{(k)} := \prod_{i=0}^{k-1}(x+i) = x(x+1)\cdot\ldots\cdot(x+k-1) \quad (k \in \mathbb{N}). \tag{3.1}$$

For $p \in \mathbb{R}$, $q \in \mathbb{N}$ and $x \in \mathbb{R}^+$ we also define the auxiliary sums[4]

$$\sigma_q^*(x,p) := \sum_{i=1}^{\lfloor q/2 \rfloor}\left(4^i - 1\right)\frac{B_{2i}\cdot p^{(2i-1)}}{x^{p+2i-1}\cdot(2i)!}. \tag{3.2}$$

Lemma 3.1. *For $p \in \mathbb{R}$ and for positive integers k, l, and q, where $k < l$, we have*

$$\sum_{i=2k}^{2l-1}(-1)^{i+1}\frac{1}{i^p} = \frac{1}{2}\left(\frac{1}{(2l)^p} - \frac{1}{(2k)^p}\right) + \sigma_q^*(2l,p) - \sigma_q^*(2k,p) + r_q(k,l,p),$$

where

[2]Here $\lfloor x \rfloor$ denotes the floor of x, i.e. the greatest integer less than or equal to x.
[3]See footnote 1.
[4]See footnote 1.

$$\left|r_q(k,l,p)\right| \leqslant \begin{cases} \dfrac{\left|p^{(q)}\right|}{3|p+q-1|\pi^{q-2}}\left((2l)^{|p+q-1|}-(2k)^{|p+q-1|}\right), & \textit{if } q<-p+1, \\ \dfrac{\left|p^{(q)}\right|}{3(p+q-1)\pi^{q-2}}\left(\dfrac{1}{(2k)^{p+q-1}}-\dfrac{1}{(2l)^{p+q-1}}\right), & \textit{if } q>-p+1. \end{cases}$$

Additionally, we have, as the consequence of Lemma 2.2, $r_q(k,l,p)=0$ *for all* $p \in \mathbb{Z}^- \cup \{0\}$ *and* $q \geqslant \max\{1,|p|\}$.

Proof. The derivatives of the function $f_p(t) := t^{-p}$ are given as

$$\frac{d^q f_p}{dt^q}(t) \equiv (-1)^q \frac{p^{(q)}}{t^{p+q}} \qquad (p \in \mathbb{R},\ q \in \mathbb{N},\ x \in \mathbb{R}^+),$$

where $p^{(q)}$ is the upper Pochhammer product defined in (3.1). Consequently, invoking Lemmas 2.1 and 2.2, and referring to the definition (3.2), the proof completes, for $q \neq -p+1$, the estimates

$$\left|\int_{2k}^{2l} \frac{d^q f_p}{dt^q}(t)\right| \leqslant \int_{2k}^{2l}\left|\frac{d^q f_p}{dt^q}(t)\right| dt \leqslant \int_{2k}^{2l} \frac{\left|p^{(q)}\right|}{t^{p+q}}\, dt$$
$$= \frac{-\left|p^{(q)}\right|}{p+q-1}\left[t^{-p-q+1}\right]_{2k}^{2l} = \frac{\left|p^{(q)}\right|}{p+q-1}\left(\frac{1}{(2k)^{p+q-1}}-\frac{1}{(2l)^{p+q-1}}\right). \square$$

Referring to the obvious identity[5]

$$\sum_{i=1}^{n} \varphi(i) = \sum_{i=1}^{2k-1} \varphi(i) + \underline{\underline{\sum_{i=2k}^{2\lfloor (n+1)/2 \rfloor -1}}} \varphi(i) + \frac{1}{2}\left[1+(-1)^n\right]\varphi(n),$$

true for a function φ and $n \geqslant 2k+1 \geqslant 3$, we derive from Lemma 3.1, considering also the clear identity $\frac{1}{2}\varphi\left(2\lfloor \frac{n+1}{2} \rfloor\right) - \frac{1}{2}\left[1+(-1)^n\right]\varphi(n) \equiv \frac{1}{2}(-1)^{n+1}\varphi\left(2\lfloor \frac{n+1}{2} \rfloor\right)$, the following theorem.

Theorem 3.2 (Estimating the alternating generalized harmonics)**.** *Considering* (3.2)*, for* $p \in \mathbb{R}$ *and for any* $k,n,q \in \mathbb{N}$*, where* $n \geqslant 2k+1 \geqslant 3$*, we have*

$$H^*(n,p) = H_q^*(k,n,p) + r_q^*(k,n,p),$$

where

$$\begin{aligned} H_q^*(k,n,p) &:= A_q^*(k,p) + B_q^*(n,p), \\ A_q^*(k,p) &:= H^*(2k-1,p) - \frac{1}{2(2k)^p} - \sigma_q^*(2k,p), \\ B_q^*(n,p) &:= \frac{(-1)^{n+1}}{2\left(2\lfloor \frac{n+1}{2} \rfloor\right)^p} + \sigma_q^*\left(2\lfloor \tfrac{n+1}{2} \rfloor, p\right), \end{aligned}$$

and $r_q^*(k,n,p) := r_q\left(k, \lfloor \frac{n+1}{2} \rfloor, p\right)$ *is bounded in Lemma 3.1, consequently we roughly estimate*

$$\left|r_q^*(k,n,p)\right| \leqslant \begin{cases} \dfrac{\left|p^{(q)}\right| \cdot \left(2\pi\lfloor \frac{n+1}{2} \rfloor\right)^{|p|+1-q}}{3(|p|+1-q)\pi^{|p|-1}}, & \textit{for } p \in \mathbb{R}^- \textit{ and } q<|p|+1, \\ \dfrac{\left|p^{(q)}\right| \cdot \pi^{1-|p|}}{3(q-|p|-1)(2k\pi)^{q-|p|-1}}, & \textit{for } p \in \mathbb{R}^- \textit{and } q>|p|+1, \\ \dfrac{p^{(q-1)} \cdot \pi^{p+1}}{3(2k\pi)^{p+q-1}}, & \textit{for } p \geqslant 0 \textit{ and } q \geqslant 1, \end{cases}$$

where $r_q^*(k,n,p) = 0$ *for integer* $p \leqslant 0$ *and* $q \geqslant \max\{1,|p|\}$.

[5]See footnote 2.

Corollary 3.3. *For* $p \in \mathbb{R}$ *and for every* $k, n, q \in \mathbb{N}$ *with* $n \geqslant 2k+1$, *the approximation* $H^*(n,p) \approx H_q^*(k,n,p) = A_q^*(k,p) + B_q^*(n,p)$ *has the relative error*

$$\rho_q^*(k,n,p) := \frac{H^*(n,p) - H_q^*(k,n,p)}{H_q^*(k,n,p)},$$

bounded as

$$\left|\rho_q^*(k,n,p)\right| \leqslant \begin{cases} \dfrac{\left|p^{(q)}\right|}{3(|p|+1-q)\pi^{|p|-1}} \cdot \dfrac{\left(2\pi\left\lfloor\frac{n+1}{2}\right\rfloor\right)^{|p|+1-q}}{\left|H_q^*(k,n,p)\right|} & (p<0,\ q<|p|+1), \\ \dfrac{\left|p^{(q)}\right|\,\pi^{1-|p|}}{3\left|H_q^*(k,n,p)\right|(q-|p|-1)\,(2k\pi)^{q-|p|-1}} & (p<0,\ q>|p|+1), \\ \dfrac{p^{(q-1)}\,\pi^{p+1}}{3\left|H_q^*(k,n,p)\right|(2k\pi)^{p+q-1}} & (p\geqslant 0,\ q\geqslant 1). \end{cases}$$

According to the equality $H^*(n,p) = H_q^*(k,n-1,p) + r_q^*(k,n-1,p) + (-1)^{n+1}\frac{1}{n^p}$ we obtain from Theorem 3.2 its different form presented as the next corollary.

Corollary 3.4. *For* $p \in \mathbb{R}$ *and for any* $k, n, q \in \mathbb{N}$ *with* $n \geqslant 2k+1 \geqslant 3$ *the sum* $H^*(n,p)$ *is given also by the formula*

$$H^*(n,p) = H_q^{**}(k,n,p) + r_q^{**}(k,n,p)$$

with

$$\begin{aligned} H_q^{**}(k,n,p) :=& A_q^*(k,p) + \left((-1)^n\left(\frac{1}{2(2\lfloor\frac{n}{2}\rfloor)^p} - \frac{1}{n^p}\right) + \sigma_q^*(2\lfloor\tfrac{n}{2}\rfloor, p)\right) \\ =& A_q^*(k,p) + \left(\frac{-1}{2(2\lfloor\frac{n}{2}\rfloor)^p} + \sigma_q^*(2\lfloor\tfrac{n}{2}\rfloor, p) + \left(1+(-1)^{n+1}\right)\frac{1}{2n^p}\right) \end{aligned} \tag{3.3}$$

and

$$r_q^{**}(k,n,p) := r_q^*(k,n-1,p).$$

Setting $q = 2$ in Corollary 3.4 we obtain the following corollary.

Corollary 3.5. *For* $p \in \mathbb{R}$ *and for any* $k, n \in \mathbb{N}$ *with* $n \geqslant 2k+1$ *we have*

$$\begin{aligned} H^*(n,p) =& H^*(2k-1,p) - \frac{1}{2(2k)^p} - \frac{p}{4(2k)^{p+1}} \\ &+ \left((-1)^n\left(\frac{1}{2(2\lfloor\frac{n}{2}\rfloor)^p} - \frac{1}{n^p}\right) + \frac{p}{4\left(2\lfloor n/2\rfloor\right)^{p+1}}\right) + r_2^{**}(k,n,p), \end{aligned} \tag{3.4}$$

where

$$|r_2^{**}(k,n,p)| \leqslant \begin{cases} \dfrac{|p||p+1| \cdot \left(2\lfloor n/2\rfloor\right)^{|p|-1}}{3(|p|-1)}, & \textit{for } p<-1, \\ \dfrac{|p||p+1|}{3(1-|p|)\,(2k)^{1-|p|}}, & \textit{for } -1<p<0, \\ \dfrac{p}{3(2k)^{p+1}}, & \textit{for } p\geqslant 0. \end{cases} \tag{3.5}$$

The approximation $H^*(n,p) \approx H_q^{**}(k,n,p)$, see (3.3), has the relative error

$$\rho_q^{**}(k,n,p) := \frac{H^*(n,p) - H_q^{**}(k,n,p)}{H_q^{**}(k,n,p)},$$

bounded in the following two examples.

Example 3.6. For $p \in \mathbb{R}$ and for any $k, n \in \mathbb{N}$ with $n \geqslant 2k+1$, the approximation

$$\begin{aligned} H^*(n,p) \approx H_2^{**}(k,n,p) =& H^*(2k-1,p) - \frac{1}{2(2k)^p} - \frac{p}{4(2k)^{p+1}} \\ & - \frac{1}{2\big(2\lfloor n/2 \rfloor\big)^p} + \frac{p}{4\big(2\lfloor n/2 \rfloor\big)^{p+1}} + \Big(1 + (-1)^{n+1}\Big)\frac{1}{2n^p}, \end{aligned}$$

has

$$\left|\rho_2^{**}(k,n,p)\right| \leqslant \begin{cases} \dfrac{|p||p+1|}{3(|p|-1)} \cdot \dfrac{\big(2\lfloor n/2 \rfloor\big)^{|p|-1}}{\left|H_2^{**}(k,n,p)\right|}, & \text{for } p < -1, \\ \dfrac{|p||p+1|}{3\left|H_2^{**}(k,n,p)\right|(1-|p|)\,(2k)^{1-|p|}}, & \text{for } -1 < p < 0, \\ \dfrac{p}{3\left|H_2^{**}(k,n,p)\right|(2k)^{p+1}}, & \text{for } p \geqslant 0. \end{cases}$$

Example 3.7. For $p \in \mathbb{R}$ and $k, n \in \mathbb{N}$ satisfying $n \geqslant 2k+1$ the approximation

$$\begin{aligned} H^*(n,p) \approx H_4^{**}(k,n,p) = H^*(2k-1,p) - \frac{1}{2(2k)^p} - \frac{p}{4(2k)^{p+1}} + \frac{p(p+1)(p+2)}{48(2k)^{p+3}} \\ - \frac{1}{2\big(2\lfloor n/2 \rfloor\big)^p} + \frac{p}{4\big(2\lfloor n/2 \rfloor\big)^{p+1}} - \frac{p(p+1)(p+2)}{48\big(2\lfloor n/2 \rfloor\big)^{p+3}} + \Big(1 + (-1)^{n+1}\Big)\frac{1}{2n^p}, \end{aligned}$$

has

$$\left|\rho_4^{**}(k,n,p)\right| \leqslant \begin{cases} \dfrac{\left|p(p+1)(p+2)(p+3)\right|}{3\,\pi^2\,(|p|-3)} \cdot \dfrac{|\big(2\lfloor n/2 \rfloor\big)^{|p|-3}}{\left|H_4^{**}(k,n,p)\right|}, & \text{for } p < -3, \\ \dfrac{\left|p(p+1)(p+2)\right|}{3\,\pi^2\,(3-|p|)\left|H_4^{**}(k,n,p)\right|(2k)^{3-|p|}}, & \text{for } -3 < p < 0, \\ \dfrac{\left|p(p+1)(p+2)(p+3)\right|}{3\,\pi^2\left|H_4^{**}(k,n,p)\right|(2k)^{p+3}}, & \text{for } p \geqslant 0. \end{cases}$$

For example, for $n \geqslant 2 \cdot 5 + 1 = 11$ and $p > 0$, we estimate

$$\left|\rho_4^{**}(5,n,p)\right| < \frac{(p+2)^4}{3\,\pi^2\,10^{p+3}} \cdot \frac{1}{\left|H_4^{**}(k,n,p)\right|} < \frac{6 \times 10^{-4}}{\left|H_4^{**}(k,n,p)\right|}.$$

Thus, for $n \geqslant 11$ and $p > 0$, we have

$$\big(1 - 6 \times 10^{-4}\big)H_4^{**}(3,n,p) < H^{**}(n,p) < \big(1 + 6 \times 10^{-4}\big)H_4^{**}(3,n,p). \tag{3.6}$$

4. Estimating $H^*(n,p)$ for $p < 0$

If $p < 0$, the sequence of generalized alternating harmonics alternate in sign. Indeed, we have the next proposition.

Proposition 4.1. *For every $p \in \mathbb{R}^-$ and every integer $n \geqslant 2$ there holds the double inequality*

$$0 < (-1)^{n+1}H^*(n,p) < n^{|p|}. \tag{4.1}$$

Proof by induction. We have $(-1)^{2+1}H^*(2,p) = -(1-2^{|p|}) = -1+2^{|p|}$. Hence $0 < (-1)^{2+1}H^*(2,p) < 2^{|p|}$.
Say that the estimate (4.1) is true for some $n \geqslant 2$. Then

$$\begin{aligned}(-1)^{n+2}H^*(n+1,p) &= (-1)^n\left(H^*(n,p)+(-1)^{n+2}(n+1)^{|p|}\right)\\ &= -(-1)^{n+1}H^*(n,p)+(n+1)^{|p|} > -(-1)^{n+1}H^*(n,p)+n^{|p|} > 0\end{aligned}$$

and also

$$\begin{aligned}-(-1)^{n+2}H^*(n+1,p)+(n+1)^{|p|} &= (-1)^{n+1}\left(H^*(n,p)+(-1)^{n+2}(n+1)^{|p|}\right)+(n+1)^{|p|}\\ &= (-1)^{n+1}H^*(n,p)\cancel{-(n+1)^{|p|}}+\cancel{(n+1)^{|p|}} > 0.\end{aligned}$$

□

Referring to Corollary 3.5, using $k=2$ in (3.4) and (3.5), we obtain the next proposition.

Proposition 4.2. *For any $p \in \mathbb{R}^-$ and for every integer $n \geqslant 5$ we have*

$$\begin{aligned}H^*(n,p) =&1-2^{|p|}+3^{|p|}-\frac{1}{2}\cdot 4^{|p|}+\frac{|p|}{16}\cdot 4^{|p|}\\ &+\left((-1)^n\left(\tfrac{1}{2}\left(2\lfloor n/2\rfloor\right)^{|p|}-n^{|p|}\right)-\frac{|p|}{4}\left(2\lfloor n/2\rfloor\right)^{|p|-1}\right)+r_2^{**}(2,n,p),\end{aligned}$$

where

$$|r_2^{**}(2,n,p)| < \begin{cases}\dfrac{|p||p+1|}{3(|p|-1)}\cdot\left(2\lfloor n/2\rfloor\right)^{|p|-1}, & \text{if } p<-1,\\ \dfrac{|p||p+1|}{12(1-|p|)}\,4^{|p|}, & \text{if } -1<p<0.\end{cases}$$

Corollary 4.3. *For any even $n \geqslant 6$ and for every $p < -1$ there holds the double inequality*

$$a_1(n,p) < H^*(n,p) < b_1(n,p), \tag{4.2}$$

where

$$\begin{aligned}a_1(n,p) :=&H^*(3,p)+\frac{1}{2}\left(\frac{|p|}{8}-1\right)4^{|p|}-\frac{1}{2}n^{|p|}-\frac{|p|}{12}\left(3+4\frac{|p+1|}{|p|-1}\right)n^{|p|-1},\\ b_1(n,p) :=&H^*(3,p)+\frac{1}{2}\left(\frac{|p|}{8}-1\right)4^{|p|}-\frac{1}{2}n^{|p|}-\frac{|p|}{12}\left(3-4\frac{|p+1|}{|p|-1}\right)n^{|p|-1}.\end{aligned}$$

Figure 1 illustrates the estimate (4.2) by plotting[6], for $p = -3\pi$ and $p = -\pi/3$, the graphs of the functions (n even) $n \mapsto a_1(n,p)$, $n \mapsto b_1(n,p)$ (continuous lines) and the graphs of the sequences (n even) $n \mapsto H^*(n,p)$.

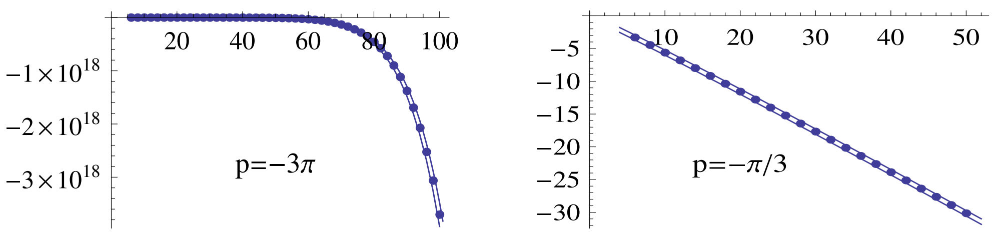

Figure 1: The graphs of the functions (n even) $n \mapsto a_1(n,p)$, $n \mapsto b_1(n,p)$ (continuous lines) and the graphs of the sequences (n even) $n \mapsto H^*(n,p)$.

[6]In this contribution all figures are produced by Mathematica [24].

Corollary 4.4. *For any even $n \geqslant 6$ and for every $p \in (-1,0)$ we have the estimate*

$$a_2(n,p) < H^*(n,p) < b_2(n,p) \tag{4.3}$$

with

$$a_2(n,p) := H^*(3,p) - \frac{1}{2}n^{|p|} - \frac{|p|}{4}n^{|p|-1} + \frac{1}{2}\left(\frac{|p|}{8} - 1 - \frac{|p||p+1|}{6(1-|p|)}\right)4^{|p|},$$
$$b_2(n,p) := H^*(3,p) - \frac{1}{2}n^{|p|} - \frac{|p|}{4}n^{|p|-1} + \frac{1}{2}\left(\frac{|p|}{8} - 1 + \frac{|p||p+1|}{6(1-|p|)}\right)4^{|p|}.$$

Figure 2 illustrates the double inequality (4.3) by plotting, for $p = -e/\pi$ and $p = -1/\pi$, the graphs of the functions (n even) $n \mapsto a_2(n,p)$, $n \mapsto b_2(n,p)$ (using continuous lines) and the graphs of the sequences (n even) $n \mapsto H^*(n,p)$.

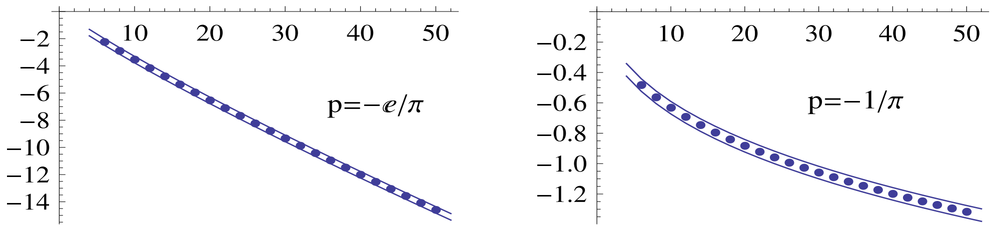

Figure 2: The graphs of the functions (n even) $n \mapsto a_2(n,p)$, $n \mapsto b_2(n,p)$ (continuous lines) and the graphs of the sequences (n even) $n \mapsto H^*(n,p)$.

If p is negative integer and $q \geqslant |p|$, the error $r^*_q(k,n,p)$ is equal to zero. More precisely, we have the next theorem.

Theorem 4.5 (Computing the alternating sums of powers with positive integer exponents)**.** *For any $p \in \mathbb{Z}^-$ the following equalities, involving Bernoulli's polynomials $B_m(x)$, hold:*

$$\begin{aligned} H^*(n,p) &= H^*_q(k,n,p) \quad (k,n,q \in \mathbb{N},\ k < \lfloor n/2 \rfloor,\ q \geqslant |p|) && (4.4)\\ &= A^*_q(k,p) + B^*_q(n,p)\\ &= \frac{1}{|p|+1}\left(B_{|p|+1}(2\lfloor n/2 \rfloor + 1) - 2^{|p|+1}B_{|p|+1}(\lfloor n/2 \rfloor + 1) + \left(2^{|p|+1} - 1\right)B_{|p|+1}\right)\\ &\quad + \frac{1}{2}\left(1 + (-1)^{n+1}\right)n^{|p|} \quad (n \in \mathbb{N}). && (4.5) \end{aligned}$$

Hence, the estimate (2.1) in Lemma 2.1 is sharp, because $R_q(k,l) = 0$ for $q \geqslant |p|$.

Proof. The equality (4.4) follows from Lemma 2.2. Moreover, for any $m \in \mathbb{N}$ ($m = \lfloor n/2 \rfloor$) we transform

$$\begin{aligned} H^*(2m,p) &= \sum_{i=1}^{m}(2i-1)^{-p} - \sum_{i=1}^{m}(2i)^{-p} = \left(H(2m,p) - \sum_{i=1}^{m}(2i)^{-p}\right) - 2^{-p}H(m,p)\\ &= H(2m,p) - 2^{-p+1}H(m,p), \end{aligned} \tag{4.6}$$

where, according to [2, 23.1.4], we have

$$H(k,p) = \frac{1}{|p|+1}\left(B_{|p|+1}(k+1) - B_{|p|+1}\right) \qquad (k \in \mathbb{N},\ p \in \mathbb{Z}^-). \tag{4.7}$$

Now, referring to (4.6)-(4.7) we obtain

$$H^*(2m,p) = \frac{1}{|p|+1}\left(B_{|p|+1}(2m+1) - 2^{|p|+1}B_{|p|+1}(m+1) + \left(2^{|p|+1} - 1\right)B_{|p|+1}\right).$$

Consequently, using the identity $H^*(2m+1,p) = H^*(2m,p) + (2m+1)^{|p|}$, the equality (4.5) follows for $n \geqslant 2$. But, since $B_k(1) = B_k(0) = B_k$ for $k \geqslant 2$, the relation (4.5) is true also for $n = 1$. □

5. Estimating $H^*(n, p)$ for $p > 0$

Using Corollary 3.5 we derive several simple estimates for $H^*(n, p)$ presented in the next theorem.

Theorem 5.1. *For every integer $n \geqslant 5$ and any $p \in \mathbb{R}^+$, the following inequalities hold:*

$$\begin{aligned} H^*(n,p) &> 1 - \frac{1}{2^p} + \frac{1}{3^p} - \left(1 + \frac{7p}{24}\right)\frac{1}{2\cdot 4^p} + \delta(n,p) && (5.1)\\ &> 1 - \frac{1}{2^p} + \frac{1}{3^p} - \left(1 + \frac{7p}{48}\right)\frac{1}{4^p} && (5.2)\\ &> 1 - \frac{1}{2^p} > 0, \end{aligned}$$

and

$$\begin{aligned} H^*(n,p) &< 1 - \frac{1}{2^p} + \frac{1}{3^p} - \left(1 - \frac{p}{24}\right)\frac{1}{2\cdot 4^p} + \delta(n,p) && (5.3)\\ &< 1 - \frac{1}{2^p} + \frac{1}{3^p} + \frac{p}{12\cdot 4^p} < 1, && (5.4) \end{aligned}$$

where

$$\delta(n,p) := (-1)^n\left(\frac{1}{2\left(2\lfloor\frac{n}{2}\rfloor\right)^p} - \frac{1}{n^p}\right) + \frac{p}{4\left(2\lfloor n/2\rfloor\right)^{p+1}}.$$

Proof. Setting $k = 2$ in (3.4) we get

$$H_2^*(2,n,p) = H^*(3,p) - \frac{1}{2\cdot 4^p} - \frac{p}{16\cdot 4^p} + \delta(n,p) + r_2^{**}(k,n,p) \quad (5.5)$$

with

$$\delta(n,p) = \begin{cases} -\dfrac{1}{2n^p} + \dfrac{p}{4n^{p+1}}, & \text{for } n \text{ even}\\[2ex] -\underset{\sim\sim\sim}{\dfrac{1}{2(n-1)^p}} + \dfrac{p}{4(n-1)^{p+1}} + \underset{\sim\sim}{\dfrac{1}{n^p}}, & \text{for } n \text{ odd.} \end{cases}$$

Therefore, for $n \geqslant 5$, we roughly estimate

$$-\frac{1}{2\cdot 4^p} < \delta(n,p) < \frac{p}{16\cdot 4^p} + \frac{1}{2\cdot 4^p}. \quad (5.6)$$

Additionally, considering (3.5), we have

$$-\frac{p}{12\cdot 4^p} < r_2^*(2,n,p) < \frac{p}{12\cdot 4^p}. \quad (5.7)$$

Now, using (5.5)-(5.7), we estimate as follows:

$$\begin{aligned} H^*(n,p) &> H^*(3,p) - \frac{1}{2\cdot 4^p} - \underset{\sim\sim}{\frac{p}{16\cdot 4^p}} + \delta(n,p) - \underset{\sim\sim}{\frac{p}{12\cdot 4^p}}\\ &> H^*(3,p) - \underset{\sim\sim}{\frac{1}{2\cdot 4^p}} - \frac{7p}{3\cdot 4^{p+2}} - \underset{\sim\sim}{\frac{1}{2\cdot 4^p}}\\ &= 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p}\left(1 + \frac{7p}{48}\right)\\ &= 1 - \frac{1}{2^p} + \frac{1}{3^p}\underbrace{\left[1 - \left(1 + \frac{7p}{48}\right)\left(\frac{4}{3}\right)^{-p}\right]}_{>0}, \end{aligned}$$

where, substituting $p = 24t > 0$, we have

$$\left(1+\frac{7p}{48}\right)\left(\frac{4}{3}\right)^{-p} < \left(1+\frac{8p}{48}\right)\left(e^{1/4}\right)^{-p} = (1+4t)\,e^{-6t} < 1.$$

From (5.5)-(5.7) we are also estimating, for $n > 4$, in the following way:

$$\begin{aligned} H^*(n,p) &< H^*(3,p) - \frac{1}{2\cdot 4^p} - \frac{p}{16\cdot 4^p} + \delta(n,p) + \frac{p}{12\cdot 4^p} \\ &< H^*(3,p) - \cancel{\frac{1}{2\cdot 4^p}} + \frac{p}{48\cdot 4^p} + \left(\frac{p}{16\cdot 4^p} + \cancel{\frac{1}{2\cdot 4^p}}\right) \\ &= 1 - \frac{1}{2^p} + \frac{1}{3^p} + \frac{p}{12\cdot 4^p} < 1 - \frac{1}{2^p} + \frac{1}{3^p} + \frac{p}{12\cdot 3^p} \\ &= 1 - \frac{1}{2^p}\underbrace{\left[1-\left(1+\frac{p}{12}\right)\left(\frac{3}{2}\right)^{-p}\right]}_{>0} < 1, \end{aligned}$$

where, substituting $p = 12t > 0$, we have

$$\left(1+\frac{p}{12}\right)\left(\frac{3}{2}\right)^{-p} < \left(1+\frac{p}{12}\right)\left(e^{1/3}\right)^{-p} = (1+t)\,e^{-4t} < 1. \qquad \square$$

Figure 3 illustrates the estimates (5.1) and (5.3), where the graphs of the lower/upper bounds are plotted using dashed curves and the graph of the function $p \mapsto H^*(n,p)$ is presented by continuous line.

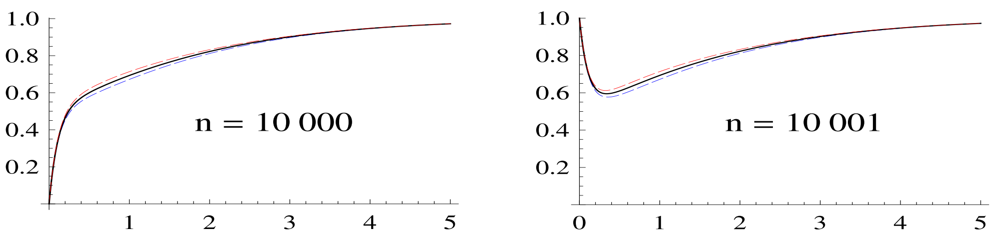

Figure 3: The graphs of the lower/upper bounds (5.1) and (5.3) are represented by dashed curves and the graphs of the functions $p \mapsto H^*(n,p)$ are continuous lines.

Figure 4 shows the estimates (5.2) and (5.4), where the graphs of the lower/upper bounds are plotted using dashed curves and the graph of the function $p \mapsto H^*(n,p)$ is continuous line.

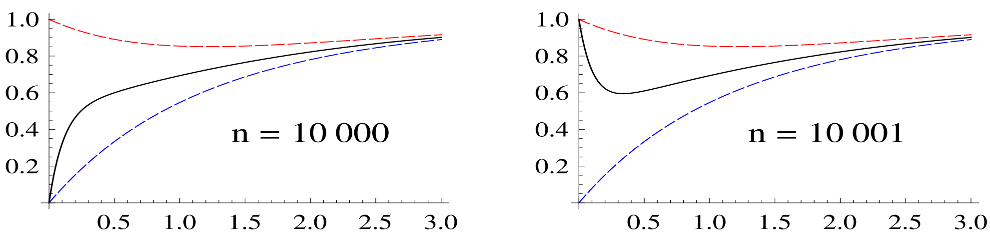

Figure 4: The graphs of the lower/upper bounds (5.2) and (5.4) are represented by dashed curves and the graphs of the functions $p \mapsto H^*(n,p)$ as continuous lines.

Using Lemma 2.1 or directly Theorem 3.2, together with the Leibniz alternating series convergence criterion, we deduce the next theorem.

Theorem 5.2 (Estimating the alternating Riemann's zeta function[7]). *For* $p \in \mathbb{R}^+$ *and for any* $k, q \in \mathbb{N}$ *we have (see* (3.2)*),*

$$\eta(p) := H^*(\infty, p) = H^*(2k-1, p) - \frac{1}{2(2k)^p} - \sigma_q^*(2k, p) + \varepsilon_q(k, p),$$

where

$$\left|\varepsilon_q(k, p)\right| < \frac{p^{(q-1)}}{3\pi^{q-2}(2k)^{p+q-1}}.$$

From Theorem 5.1 we obtain the next proposition.

Proposition 5.3. *For every* $p \in \mathbb{R}^+$ *there holds the following double inequality:*

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \left(1 + \frac{7p}{24}\right)\frac{1}{2 \cdot 4^p} < \eta(p) < 1 - \frac{1}{2^p} + \frac{1}{3^p} - \left(1 - \frac{p}{24}\right)\frac{1}{2 \cdot 4^p}. \tag{5.8}$$

This proposition estimates the speed of convergence for $\lim_{p \downarrow 0} \eta(p) = \frac{1}{2}$ and $\lim_{p \to \infty} \eta(p) = 1$.

Figure 5 shows the estimates (5.8), where the graphs of the lower/upper bounds are plotted using dashed curves and the graphs of the eta function as continuous lines.

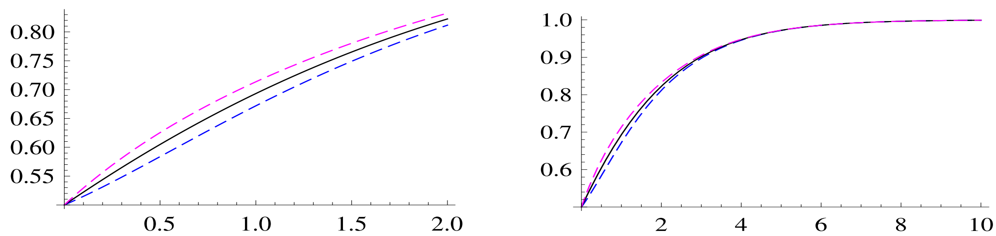

Figure 5: The graphs of the lower/upper bounds (5.8) are plotted using dashed curves and the graphs of the eta function as continuous lines.

Using the estimate (3.6) in Example 3.7, we plot Figure 6 presenting the graphs of the functions $p \mapsto H^*(n, p)$ for $n = 50$, $n = 51$, $n = 1\,000\,000$ and $n = 1\,000\,001$ (dashed curves), and for $n = \infty$ (continuous line, starting at the point $(0, \frac{1}{2})$.

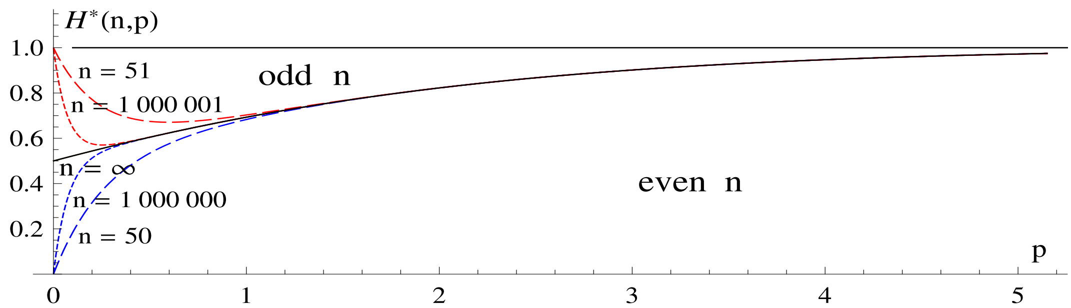

Figure 6: The graphs of the functions $p \mapsto H^*(n, p)$ (dashed curves, for finite n, and continuous one, starting at the point $(0, \frac{1}{2})$, for $n = \infty$).

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