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The endpoint Fefferman-Stein inequality for the strong maximal function with respect to nondoubling measure

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Abstract

Let $d\mu(x_1,\dots,x_n) = d\mu_1(x_1)\cdots d\mu_n(x_n)$ be a product measure which is not necessarily doubling in $\mathbb{R}^n$ (only assuming $d\mu_i$ is doubling on $\mathbb{R}$ for $i = 2,\dots,n$), and $M^n_{d\mu}$ be the strong maximal function defined by

$$M^n_{d\mu}f(x) = \sup_{x\in R\in\mathcal{R}} \frac{1}{\mu(R)} \int_R |f(y)|d\mu(y),$$

where $\mathcal{R}$ is the collection of rectangles with sides parallel to the coordinate axes in $\mathbb{R}^n$, and ω,ν are two nonnegative functions. We give a sufficient condition on ω,ν for which the operator $M^n_{d\mu}$ is bounded from $L(1+(\log^+)^{n-1})(\nu d\mu)$ to $L^{1,\infty}(\omega d\mu)$. By interpolation, $M^n_{d\mu}$ is bounded from $L^p(\nu d\mu)$ to $L^p(\omega d\mu)$, $1 < p < \infty$.

Keywords: Fefferman-Stein inequality, strong maximal function, nondoubling measure, A^∞ weights, reverse Hölder's inequality.

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1. Introduction

Since the classical theory of harmonic analysis may be described as centering around the Hardy-Littlewood maximal operator and its relationship with certain singular integral operators, the maximal function have attracted the attention of a lot of researchers, such as [1, 3–5, 8–10, 13, 16, 22, 30].

Let $\mathcal{B}_x$ be a collection of bounded sets containing $x \in \mathbb{R}^n$, and ν be a positive measure. Given a locally integrable function f, denote

$$\mathcal{M}f(x) = \sup_{R\in\mathcal{B}_x} \frac{1}{\nu(R)} \int_R |f(y)|d\nu(y).$$

If $\mathcal{B}_x$ is the collection of all the cubes containing $x \in \mathbb{R}^n$ and whose sides parallel to the coordinate axes, $d\nu(x) = dx$, then we obtain the usual Hardy-Littlewood maximal function $Mf(x)$. When $\mathcal{B}_x$ denotes the

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collection of all rectangles R containing $x \in \mathbb{R}^n$ whose sides parallel to the coordinate axes, $\mathcal{M} \triangleq M^n_{d\nu}$ is the strong maximal operator with respect to measure $d\nu$. If $d\nu(x) = dx$, denote $M^n = M^n_{d\nu}$.

For every non-negative, locally integrable weight ω, Fefferman and Stein in [11] proved the following well known inequality

$$\int_{\mathbb{R}^n} (Mf)^p(x)\omega(x)dx \lesssim \int_{\mathbb{R}^n} (|f(x)|^p M\omega(x)dx, \ 1 < p < \infty.$$

Inequalities of this type are important, for example, they can be used to derive the boundedness of vector-valued maximal operators. More details can be seen in [11–13]. Similar inequalities were also obtained for singular integral operators in [5]. In this situation, $M\omega(x)$ in the right hand was replaced by $M(\omega^r)^{1/r}$. The above inequality is also true for the strong maximal function M^n if $\omega \in A^n_\infty$ [19, 21].

For the usual Hardy-Littlewood maximal function $Mf(x)$, the form of the endpoint Fefferman-Stein inequality is the following

$$\omega(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| M\omega(x)dx, \ \lambda > 0.$$

For strong maximal function $M^n f(x)$, it is more complicated. When $n = 2$, Mitsis in [21] has obtained that

$$\omega(\{x : M^2(f)(x) > \lambda\}) \lesssim \int \frac{|f(x)|}{\lambda}\Big(1 + \log^+ \frac{|f(x)|}{\lambda}\Big) M^2\omega(x)dx, \ \lambda > 0,$$

if $\omega \in A^2_p$ for some $1 < p < \infty$. Recently, Luque and Parissis in [19] improved this result to any dimension $n \geqslant 2$ provided only $\omega \in A^n_\infty$. We want to point out that there is no assumption on the weight to establish Fefferman-Stein inequality for the Hardy-Littlewood maximal function.

The classical theory of one-parameter harmonic analysis for maximal functions and singular integrals on $(\mathbb{R}^n; \mu)$ has been developed under the assumption that the underlying measure μ satisfies the doubling property, i.e., there exists a constant $C > 0$ such that $\mu(B(x; 2r)) \leqslant C\mu(B(x; r))$ for every $x \in \mathbb{R}^n$ and $r > 0$. However, some recent results [20, 23, 25, 31] show that it is possible to dispense with the doubling condition for most of the classical theory. It is well known that the use of doubling measure has two main advantages. One is that we can work with nested property. Another one is that the faces of the cubes have measure zero. As in the paper [20, 25], we will only maintain the last property. If μ is a nonnegative Radon measure without mass-points, one can choose an orthonormal system in $\mathbb{R}^n$ so that any cube Q with sides parallel to the coordinate axes satisfies the property $\mu(\partial Q) = 0$ ([20, Theorem 2]). The profit of this property is the continuity of the measure μ on cubes which can ensure that there is a Calderón-Zygmund decomposition [20, 25]. For the development of multi-parameter harmonic analysis, we refer the readers to the works in [2, 9, 14, 15, 17].

Therefore, there is a nature question: can the Fefferman-Stein inequality be established with a general measure ν for the strong maximal operator $M^n_{d\nu}$?

Let $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ be a product measure, where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures without mass-points and complete. The assumption that μ_i are complete is just a technical requirement to allow exchange integral order. For a rectangle $R \subseteq \mathbb{R}^n$, we mean a rectangle whose sides parallel to the coordinate axes. Under this kind of product measure, in [7], we investigated the $L^p(\omega d\mu)$ boundedness of strong maximal functions $M^n_{\omega d\mu}$ and $M^n_{d\mu}$ when $\omega \in A^n_\infty$ defined by the following.

Definition 1.1. Let $1 < p < \infty$ and $p' = p/(p-1)$. We say that a weight ω satisfies the $A^n_p(\mu)$ condition if

$$[\omega]_{A^n_p(\mu)} = \sup_{R \in \mathcal{R}} \Big(\frac{1}{\mu(R)} \int_R \omega d\mu\Big)\Big(\frac{1}{\mu(R)} \int_R \omega^{1-p'} d\mu\Big)^{p-1} < \infty,$$

where $\mathcal{R}$ is a collection of all rectangles R whose sides parallel to the coordinate axes.

We say $\omega \in A_1^n(\mu)$ if there exists a constant $C > 0$ such that

$$M_{d\mu}^n \omega(x) \leqslant C\omega(x)$$

for almost every $x \in \mathbb{R}^n$.

Define $A_\infty^n(\mu)$ by

$$A_\infty^n(\mu) = \bigcup_{1 \leqslant p < \infty} A_p^n(\mu).$$

Notice that $A_r^n(\mu) \subseteq A_q^n(\mu)$ when $r \geqslant q$, and if $\omega \in A_\infty^n(\mu)$, then $\omega \in A_p^n(\mu)$ for some $1 < p < \infty$. It is easy to see that if $\omega \in A_p^n(\mu)$ for some $1 < p < \infty$, $\omega_i(x_i) = \omega(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n) \in A_p^1(\mu_i)$ uniformly with respect to $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. It has been proved in [7] that the behavior of $A_\infty^n(\mu)$ and the relationship between $A_p^n(\mu)$ weights and the strong maximal function $M_{d\mu}^n$ are very similar to the classical case when we add some conditions to the product measure μ.

There are also some interesting results about two weights. We refer the reader to the work in [6, 24, 26–28]. In [26, 27], Pérez provided a sufficient condition on weights ω, ν to ensure the boundedness of the general maximal functions $M_\wp$ including the boundedness of M_{dx}^n from $L^p(\omega)$ to $L^q(\nu)$. More precisely, if the couple of weights (ω, ν) satisfies the following condition: there are constants $0 < \lambda < 1$, $0 < c = c(\lambda) < \infty$ such that for all measurable sets E

$$\omega(\{x : M_\wp(x) > \lambda\}) \leqslant c\omega(E),$$

which is weaker than the A_∞^n condition, and

$$\sup_B \frac{1}{|B|} \int_B \omega(y) dy (\sup_B \frac{1}{|B|} \int_B \nu(y)^{(1-p')r} dy)^{(p-1)/r} < \infty$$

for some $1 < r < \infty$, then $M_\wp$ is bounded from $L^p(\omega)$ to $L^q(\nu)$.

This idea can be used to our strong maximal function $M_{d\mu}^n$. A couple of weights (ω, ν) is said to be satisfied condition (A), if

$$(\frac{1}{\mu(R)} \int_R \omega d\mu) \cdot \sup_{x \in R} \nu^{-1}(x) \leqslant c$$

for all rectangles R in $\mathbb{R}^n$. Then the main result of the current paper is the following.

Theorem 1.2. *Assume that $\mu(x) = \mu_1(x_1) \cdot \mu_2(x_2) \cdots \mu_n(x_n)$ is a product measure where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures in $\mathbb{R}$ without mass-points and complete. Assume also that each μ_i for $2 \leqslant i \leqslant n$ is doubling on $\mathbb{R}$. If (ω, ν) is a couple of weights such that $\omega \in A_\infty^n = A_\infty^n(\mu)$ and that the condition (A) holds, then*

$$\omega(\{x : M_{d\mu}^n(f)(x) > \lambda\}) \lesssim \int \frac{|f(x)|}{\lambda} \Big(1 + (\log^+ \frac{|f(x)|}{\lambda})^{n-1}\Big) \nu(x) d\mu(x), \tag{1.1}$$

where $\omega(E)$ denotes $\int_E \omega(x) d\mu(x)$ for every μ-measurable set E.

By interpolation, the above endpoint two weights Fefferman-Stein inequality implies the strong two weights Fefferman-Stein inequality.

Theorem 1.3. *Assume that $\mu(x) = \mu_1(x_1) \cdot \mu_2(x_2) \cdots \mu_n(x_n)$ is a product measure where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures in $\mathbb{R}$ without mass-points and complete. Assume also that each μ_i for $2 \leqslant i \leqslant n$ is doubling on $\mathbb{R}$. If (ω, ν) is a couple of weights such that $\omega \in A_\infty^n$ and that the condition (A) holds, then*

$$\|M_{d\mu}^n(f)\|_{L^p(\omega d\mu)} \lesssim \|M_{d\mu}^n(f)\|_{L^p(\nu d\mu)}, \ 1 < p < \infty.$$

It is easy to check that $(\omega, M_{d\mu}^n \omega)$ satisfies condition (A), and then we have.

Corollary 1.4. *Assume that $\mu(x) = \mu_1(x_1) \cdot \mu_2(x_2) \cdots \mu_n(x_n)$ is a product measure where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures in $\mathbb{R}$ without mass-points and complete. Assume also that each μ_i for $2 \leqslant i \leqslant n$ is doubling on $\mathbb{R}$, and $\omega \in A_\infty^n$, then*

$$\omega(\{x : M_{d\mu}^n(f)(x) > \lambda\}) \lesssim \int \frac{|f(x)|}{\lambda}\Big(1 + (\log^+ \frac{|f(x)|}{\lambda})^{n-1}\Big) M_{d\mu}^n \omega(x) d\mu(x),$$

and

$$\|M_{d\mu}^n(f)\|_{L^p(\omega d\mu)} \lesssim \|M_{d\mu}^n(f)\|_{L^p(M_{d\mu}^n \omega d\mu)},\ 1 < p < \infty.$$

Let $d\mu = dx$, then the above corollary is the main theorem of [19]. By changing variables, in the above results, the product measure μ can be assumed that $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures in $\mathbb{R}$ without mass-points and complete, permitted only one direction with non-doubling condition.

The organization of the paper is as follows. Section 2 gives some auxiliary lemmas, such as reverse Hölder's inequality of weights $A_\infty^n(\mu)$, and the asymptotic estimate of the $L^p(d\mu)$ norm of $M_{\omega d\mu}^n$ as $p \to 1^+$. In last section, we give the proof of Theorem 1.2.

Finally, we make some conventions. Throughout the paper, c denotes a positive constant that is independent of the main parameters involved, but whose value may vary from line to line. Constants with subscript, such as c_1, do not change in different occurrences. We denote $f \leqslant cg$ by $f \lesssim g$. If $f \lesssim g \lesssim f$, we write $f \approx g$. In order to indicate the dependence of the constant on some parameter n (say), we write $A \lesssim_n B$.

2. Auxiliary lemmas

In this section, firstly we give some lemmas about weights $A_\infty^n(\mu)$ obtained in [7].

Lemma 2.1. *Let μ be a nonnegative Radon measure. If $\omega \in A_\infty^n(\mu)$, then for $\forall\ 0 < \alpha < 1$, there is a positive constant $\beta < 1$ such that whenever F is a measurable set of a rectangle R, we have*

$$\frac{\mu(F)}{\mu(R)} \leqslant \alpha \ \textit{implies}\ \frac{\omega(F)}{\omega(R)} \leqslant \beta,$$

which is equivalent to say that for $\forall\ 0 < \alpha' < 1$, there is a positive constant $\beta' < 1$ such that whenever F is a measurable set of a rectangle R,

$$\frac{\mu(F)}{\mu(R)} \geqslant \alpha' \ \textit{implies}\ \frac{\omega(F)}{\omega(R)} \geqslant \beta'.$$

Lemma 2.2. *Assume that $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ is a product measure, where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures without mass-points and complete. If $\omega \in A_\infty^n(\mu)$, then ω satisfies a reverse Hölder's inequality, that is, there exist two positive constants c and δ such that for every rectangle R*

$$\Big(\frac{1}{\mu(R)}\int_R \omega^{1+\delta} d\mu\Big)^{1/(1+\delta)} \leqslant \frac{c}{\mu(R)}\int_R \omega d\mu,$$

and c may be taken as close to 1 as $\delta \to 0^+$.

All the proofs of above lemmas can be seen in [7], and we omit it. If $\omega \in A_p^n(\mu)$, $p > 1$, then $\omega^{1-p'} \in A_{p'}^n(\mu)$, where $1/p + 1/p' = 1$. Consequently, by Lemma 2.2, it is easy to deduce the following result.

Lemma 2.3. *Let $p > 1$, and $\omega \in A_p^n(\mu)$, then there is an $\varepsilon > 0$ such that $\omega \in A_{p-\varepsilon}^n(\mu)$.*

Let $x = (x_1, \ldots, x_n)$ and $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$. For convenience, fixed $x_n \in \mathbb{R}$, denote $d\omega' = \omega(x', x_n)d\mu'$, $x' = (x_1, \ldots, x_{n-1})$ and $d\mu' = d\mu_1 \cdots d\mu_{n-1}$. For some $0 < \epsilon < 1$, let $\{R_k\}$ be a sequence of rectangles in $\mathbb{R}^n$. $\{R_k\}$ is said to have property P_1: if

$$\mu(R_k \cap \bigcup_{i<k} R_i) \leqslant \epsilon\mu(R_k).$$

$\{R_k\}$ is said to satisfy property P_2: if its side lengths in the x_n direction are decreasing and

$$\mu(R_k \cap \bigcup_{i<k} \hat{R}_i) \leqslant \epsilon\mu(R_k),$$

where $\hat{R}$ is the rectangle with the same center as R and whose sides parallel to the first $n-1$ coordinate axes have the same lengths as the corresponding sides of R, and the side of $\hat{R}$ which is parallel to the n-th coordinate axis has length equal to three times the length of the corresponding side of R.

Lemma 2.4. *Assume that $\mu(x) = \mu_1(x_1)\mu_2(x_2)\cdots\mu_n(x_n)$ is a product measure where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures in $\mathbb{R}$. Let $\{R_k\}$ be a sequence of rectangles in $\mathbb{R}^n$ satisfying property P_2, and $S_k^{x_n}$ be the slice of R_k at x_n. Then $\{S_k^{x_n}\}$ satisfies property P_1 for any x_n, that is*

$$\mu'(S_k^{x_n} \cap \bigcup_{i<k} S_i^{x_n}) \leqslant \epsilon\mu'(S_k^{x_n}).$$

Proof. Since R_k is a rectangle in $\mathbb{R}^n$, we may assume $R_k = I_k \times J_k$ where J_k is the one-dimensional projection to the x_n axes and I_k is a rectangle in $\mathbb{R}^{n-1}$. The conclusion is obvious if $x_n \notin J_k$. When $x_n \in J_k$, setting $J = \{i < k, S_k^{x_n} \cap S_i^{x_n} \neq \emptyset\}$, one has

$$R_k \cap \bigcup_{i\in J} R_i \subseteq R_k \cap \bigcup_{i<k} R_i.$$

By the assumption that the side lengths of the x_n direction are decreasing in $\{R_k\}$, one has

$$R_k \cap \bigcup_{i\in J} \hat{R}_i = (\bigcup_{i\in J} S_k^{x_n} \cap S_i^{x_n}) \times J_k.$$

Hence by the property P_2, one has

$$\mu'(\bigcup_{i\in J} S_k^{x_n} \cap S_i^{x_n})\mu_n(J_k) = \mu(R_k \cap \bigcup_{i\in J} \hat{R}_i) \leqslant \mu(R_k \cap \bigcup_{i<k} \hat{R}_i) \leqslant \epsilon\mu(R_k),$$

which yields our desired result immediately. □

Lemma 2.5. *Assume that $\mu(x) = \mu_1(x_1)\mu_2(x_2)\cdots\mu_n(x_n)$ is a product measure where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures in $\mathbb{R}$ without mass-points and complete. Assume also that each μ_i for $2 \leqslant i \leqslant n$ is doubling on $\mathbb{R}$ and $\omega \in A_\infty^n$. Let $\{R_k\}$ be a sequence of rectangles in $\mathbb{R}^n$ satisfying property P_1. Then if $M_{\omega d\mu}^n$ is $L^p(\omega d\mu)$ bounded with norm at most $O((p-1)^{-r})$, $1 < p \leqslant 2$, for some $r > 0$, one has*

$$\|\sum \chi_{R_k}\|_{L^{p'}(\omega d\mu)} \leqslant C(p')^{r+1}\omega(\cup R_k)^{1/p'}, \qquad 1 < p \leqslant 2.$$

Proof. Setting $E_k = R_k \backslash \bigcup_{i<k} R_i$, one has $\mu(E_k) \geqslant (1-\epsilon)\mu(R_k)$ by property P_1 and the fact that the sets $\{E_k\}$ are pairwise disjoint. Since $\omega \in A_\infty^n(\mu)$, one has $\omega(E_k) \geqslant \beta\omega(R_k)$ for some $0 < \beta < 1$. Arguing by duality we assume that φ is a function satisfying $\|\varphi\|_{L^p(\omega d\mu)} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, then one has

$$\int \sum \chi_{R_k}\varphi\omega d\mu = \sum \int_{R_k} \varphi\omega d\mu = \sum (\frac{1}{\omega(R_k)} \int_{R_k} \varphi\omega d\mu)\omega(R_k)$$

$$\begin{aligned}
&\lesssim \sum \omega(E_k) \inf_{x\in R_k} M^n_{\omega d\mu}(\varphi)(x)\\
&\leqslant \int_{\cup E_k} M^n_{\omega d\mu}(\varphi)(x)\omega(x)d\mu(x)\\
&\leqslant \|M^n_{\omega d\mu}(\varphi)\|_{L^p(\omega d\mu)}\omega(\cup R_k)^{1/p'}\\
&\leqslant O((p'-1)^r)\omega(\cup R_k)^{1/p'} \leqslant O((p')^{r+1})\omega(\cup R_k)^{1/p'}
\end{aligned}$$

by the assumption of the $L^p(\omega d\mu)$ norm of $M^n_{\omega d\mu}$. □

Lemma 2.6. *Assume that* $\mu(x) = \mu_1(x_1)\mu_2(x_2)\cdots\mu_n(x_n)$ *is a product measure where* $\mu_i, i = 1,\dots,n$ *are all nonnegative Radon measures in* $\mathbb{R}$ *without mass-points and complete. Assume also that each* μ_i *for* $2 \leqslant i \leqslant n$ *is doubling on* $\mathbb{R}$ *and* $\omega \in A^n_\infty$*. Let* $\{R_k\}$ *be a sequence of rectangles in* $\mathbb{R}^n$ *satisfying property* P_2*. Suppose also that* $M^{n-1}_{\omega' d\mu'}$ *is bounded on* $L^p(\mathbb{R}^{n-1}, \omega' d\mu')$ *with norm at most* $O((p-1)^{-r})$, $1 < p \leqslant 2$, *uniformly in a.e.* x_n *for some* $r > 0$*. Then there exists a constant* C *independent of* $\{R_k\}$ *such that*

$$\|\sum \chi_{R_k}\|_{L^{p'}(\omega d\mu)} \leqslant C(p')^{r+1}\omega(\cup R_k)^{1/p'}, \qquad 1 < p \leqslant 2.$$

Proof. Let $S^{x_n}_k$ denote the slice of R_k by a hyperplane perpendicular to the x^n-axis, at height x_n. Using Lemma 2.4, we have

$$\mu'(S^{x_n}_k \cap \bigcup_{i<k} S^{x_n}_i) \leqslant \frac{1}{2}\mu'(S^{x_n}_k).$$

Since $\omega \in A^n_\infty$, one has $\omega' \in A^{n-1}_\infty$ uniformly in a.e. x_n. Then by Lemma 2.5,

$$\|\sum \chi_{S^{x_n}_k}\|_{L^{p'}(\omega' d\mu')} \leqslant C(p')^{r+1}\omega'(\cup S^{x_n}_k)^{1/p'}, \qquad 1 < p \leqslant 2,$$

uniformly in a.e. x_n, which follows the desired result by taking the p'-th power on both sides of above inequality and integrate in x_n. □

Lemma 2.7. *Assume that* $\mu(x) = \mu_1(x_1)\cdot\mu_2(x_2)\cdots\mu_n(x_n)$ *is a product measure where* $\mu_i, i = 1,\dots,n$ *are all nonnegative Radon measures in* $\mathbb{R}$ *without mass-points and complete. Assume also that each* μ_i *for* $2 \leqslant i \leqslant n$ *is doubling on* $\mathbb{R}$ *and* $\omega \in A^n_\infty$*. Then* $M^n_{\omega d\mu}$ *is* $L^p(\omega d\mu)$ *bounded with norm at most* $O((p-1)^{-n})$, $1 < p \leqslant 2$.

Remark 2.8. The asymptotic estimate of M^n was obtained by Long and Shen in [18].

Proof. The proof is by induction on n. For $n = 1$, $M^n_{\omega d\mu}$ is the classical Hardy-Littlewood maximal operator $M_{\omega d\mu}$ with respect to measure $\omega d\mu$. Let $\lambda > 0$ and $E_\lambda = \{x : M_{\omega d\mu}(f) > \lambda\}$, then the main result of [29] gives

$$\omega(E_\lambda) \leqslant 5\lambda^{-1}\int |f(x)|\omega(x)d\mu(x).$$

By interpolation since $M_{\omega d\mu}$ is $L^\infty(\omega d\mu)$ to $L^\infty(\omega d\mu)$ with norm 1, we obtain

$$\|M_{\omega d\mu}(f)\|_{L^p(\omega d\mu)} \leqslant c(p-1)^{-1}.$$

Suppose that $n > 1$ and the lemma holds for $n-1$. Since $\omega \in A^n_\infty$, one has $\omega' \in A^{n-1}_\infty$ uniformly in a.e. x_n. By the inductive hypothesis, $M^{n-1}_{\omega' d\mu'}$ is $L^p(\omega' d\mu')$ bounded with norm at most $O((p-1)^{1-n})$, $1 < p \leqslant 2$.

Let $\lambda > 0$ and and $\{R_k\}$ be a cover of $E_\lambda = \{x : M^n_{\omega d\mu}(f) > \lambda\}$ such that

$$\frac{1}{\omega(R_k)}\int_{R_k} |f(x)|\omega(x)d\mu(x) > \lambda.$$

With no loss of generality, we may assume that $\{R_k\}$ is a finite sequence, and that R_k are arranged so that the side length in x_n direction is decreasing. We now follow a well-known selecting procedure argument. We choose $R_1^* = R_1$, and assume $R_1^*, \dots, R_k^*$ have been selected. We obtain R_{k+1}^* as the first rectangle on the list of R_i after R_k^* such that

$$\mu(R \cap [\bigcup_{i \leqslant k} \hat{R}_i^*]) < \frac{1}{2}\mu(R).$$

That is $\{R_k^*\}$ satisfies the property P_2. By Lemma 2.6, we obtain

$$\| \sum \chi_{R_k} \|_{L^{p'}(\omega d\mu)} \leqslant C(p')^n \omega(\cup R_k)^{1/p'}, \qquad 1 < p \leqslant 2. \tag{2.1}$$

Moreover, arguing as in the proof of Theorem 1.2, one has

$$\omega(\cup R_k) \lesssim \omega(\cup R_k^*), \tag{2.2}$$

by the assumption that each μ_i for $2 \leqslant i \leqslant n$ is doubling on $\mathbb{R}$ and $\omega \in A_\infty^n$. Finally, by (2.1),

$$\begin{aligned}\omega(\cup R_k^*) \lesssim \sum \omega(R_k^*) &\leqslant \sum \frac{1}{\lambda} \int_{R_k^*} |f(x)|\omega(x)d\mu(x) \\ &\leqslant \frac{1}{\lambda} \| \sum \chi_{R_k^*} \|_{L^{p'}(\omega d\mu)} \|f\|_{L^p(\omega d\mu)} \leqslant C(p')^{n-1}\omega(\cup R_k^*)^{1/p'} \frac{1}{\lambda}\|f\|_{L^p(\omega d\mu)},\end{aligned}$$

from which it follows that

$$\omega(\cup R_k^*) \lesssim \left(C(p')^n \frac{1}{\lambda} \|f\|_{L^p(\omega d\mu)} \right)^p.$$

Hence, by (2.2),

$$\omega(E_\lambda) \lesssim \left(C(p')^n \frac{1}{\lambda} \|f\|_{L^p(\omega d\mu)} \right)^p. \tag{2.3}$$

From Lemma 2.3, since $\omega \in A_p^n(\mu)$, there is an $\varepsilon > 0$ such that $\omega \in A_{p-\varepsilon}^n(\mu)$. Then (2.3) holds for $p - \varepsilon$. It is also known that $\omega \in A_{p+\varepsilon}^n(\mu)$. By interpolation, we complete the proof. □

As a direct corollary of Lemma 2.6 and Lemma 2.7, we can obtain the following result.

Corollary 2.9. *Assume that $\omega \in A_\infty^n(\mu)$ and that $\{R_k\}$ is a sequence of rectangles in $\mathbb{R}^n$ satisfying property P_2. Then if p is big enough,*

$$\| \sum \chi_{R_k} \|_{L^p(\omega d\mu)} \lesssim p^n \omega(\cup R_k)^{1/p}.$$

3. Endpoint Fefferman-Stein inequality

Proof of Theorem 1.2. It suffices to prove the theorem for $\lambda = 1$. Denote $E = \{x : M_{d\mu}^n(f)(x) > 1\}$. Let $\{R_k\}$ be a cover of E such that

$$\frac{1}{\mu(R_k)} \int_{R_k} |f(x)| d\mu(x) > 1. \tag{3.1}$$

Since we only need to prove (1.1) for any compact subset K of E, without loss of generality, we may assume $\{R_k\}$ is a finite sequence, and R_k are arranged so that the side length in x_n direction is decreasing.

We now choose a subset $\{R_k^*\}$ of $\{R_k\}$ such that $\{R_k^*\}$ satisfies property P_2 and

$$\omega(\cup R_k) \lesssim \omega(\cup R_k^*). \tag{3.2}$$

Let $R_1^* = R_1$, and assume that $R_1^*, \dots, R_k^*$ have been selected. We obtain R_{k+1}^* as the first rectangle on the list of R_i after R_k^* such that

$$\mu(R \cap [\bigcup_{i \leqslant k} \hat{R}_i^*]) < \frac{1}{2}\mu(R).$$

This selection process will be end after a finite steps. It is obvious that $\{R_k^*\}$ satisfies the property P_2. Now assume that some $R \in \{R_k\}$ was not selected, then we can find some positive integer k such that

$$\mu(R \cap [\bigcup_{i \leqslant k} \hat{R}_i^*]) \geqslant \frac{1}{2}\mu(R),$$

which implies that for all $x \in R$,

$$M_{d\mu}^n(\chi_{\bigcup_{i \leqslant k} \hat{R}_i^*})(x) \geqslant \frac{1}{2}.$$

Hence

$$\cup R_k \subseteq \{x : M_{d\mu}^n(\chi_{\cup \hat{R}_i^*})(x) \geqslant \frac{1}{2}\}.$$

Since $\omega \in A_\infty^n$, then $\omega \in A_p^n(\mu)$ for some $1 < p < \infty$. By the result of $M_{d\mu}^n$ being bounded on $L^p(\omega d\mu)$ ([7, Theorem 1.6]), we conclude that

$$\omega(\cup R_k) \lesssim \omega(\cup \hat{R}_i^*). \tag{3.3}$$

Let $S_k^{x_n}$ denote the slice of R_k^* at x_n and then $R_k^* = S_k^{x_n} \times J_k$, $\hat{R}_k^* = S_k^{x_n} \times \hat{J}_k$ if $x_n \in J_k$. Using property P_2 of $\{R_k^*\}$, and Lemma 2.4 we have

$$\mu'(S_k^{x_n} \cap \bigcup_{i<k} S_i^{x_n}) \leqslant \frac{1}{2}\mu'(S_k^{x_n}).$$

Since $\omega \in A_\infty^n$, one has $\omega' \in A_\infty^{n-1}$ uniformly in a.e. x_n. Then

$$\omega'(S_k^{x_n} \cap \bigcup_{i<k} S_i^{x_n}) \leqslant \beta\omega'(S_k^{x_n}),$$

uniformly in a.e. x_n, for some $0 < \beta < 1$. Denote $F_k = S_k^{x_n} \backslash \bigcup_{i<k} S_i^{x_n}$. It is obvious that $\omega'(F_k) \geqslant (1-\beta)\omega'(S_k^{x_n})$. By classical result, when μ_n is doubling, $\omega(x', x_n)d\mu_n$ is also doubling uniformly for $x' \in \mathbb{R}^{n-1}$. Therefore using (3.3)

$$\begin{aligned}\omega(\cup R_k) &\lesssim \sum \omega(\hat{R}_i^*) = \sum_k \int_{S_k^{x_n}} (\int_{\hat{J}_k} \omega(x', x_n) d\mu_n(x_n)) d\mu'(x') \\ &\lesssim \sum_k \int_{J_k} (\int_{F_k} \omega(x_1, x_2) d\mu'(x')) d\mu_n = \int_{\cup J_k \times F_k} \omega(x) d\mu \leqslant \omega(\cup_k R_k^*),\end{aligned}$$

which gives (3.2).

Observe that property P_2 of $\{R_k^*\}$ also implies that

$$\mu(R_k^* \cap [\bigcup_{i<k} R_i^*]) < \frac{1}{2}\mu(R_k^*).$$

It follows that $\omega(R_k^* \cap [\bigcup_{i<k} R_i^*]) \leqslant \beta\omega(R_k^*)$ for some $0 < \beta < 1$, since $\omega \in A_\infty^n$. Then setting $E_k = R_k^* \backslash [\bigcup_{i<k} R_i^*]$, we have

$$\omega(R_k^*) \geqslant \omega(E_k) \geqslant (1-\beta)\omega(R_k^*), \qquad \mu(R_k^*) \geqslant \mu(E_k) \geqslant \frac{1}{2}\omega(R_k^*).$$

Using (3.1) and (3.2), we obtain

$$\omega(E) \lesssim \omega(\cup R_k^*) \leqslant \sum \omega(R_k^*) \leqslant \sum \frac{\omega(R_k^*)}{\mu(R_k^*)} \int_{R_k^*} |f(y)| d\mu(y) = \int |f(y)| \sum_k \frac{\omega(R_k^*)}{\mu(R_k^*)} \chi_{R_k^*}(y) d\mu(y).$$

For locally integrable functions f and g, define the linear operators

$$Tf(x) = \sum_k \frac{1}{\mu(R_k^*)} \int_{R_k^*} f(y)d\mu(y)\chi_{E_k}(x), \ T^*f(x) = \sum_k \frac{1}{\mu(R_k^*)} \int_{E_k} f(y)d\mu(y)\chi_{R_k^*}(x).$$

It is easy to check that

$$\int Tf(x)g(x)d\mu(x) = \int T^*g(x)f(x)d\mu(x),$$
$$T1(x) = \sum_k \chi_{E_k}(x), \quad T^*1(x) = \sum_k \frac{\mu(E_k)}{\mu(R_k^*)}\chi_{R_k^*}(x) \approx \sum_k \chi_{R_k^*}(x),$$

and

$$T^*\omega(x) = \sum_k \frac{\omega(E_k)}{\mu(R_k^*)}\chi_{R_k^*}(x) \approx \sum_k \frac{\omega(R_k)}{\mu(R_k^*)}\chi_{R_k^*}(x).$$

Hence

$$\begin{aligned}\omega(\cup R_k^*) \lesssim \int |f(y)|T^*\omega(y)d\mu(y) &= \Big(\int_{\{y:T^*\omega(y)\leqslant\nu(y)\}} + \int_{\{y:T^*\omega(y)>\nu(y)\}}\Big)|f(y)|T^*\omega(y)d\mu(y)\\ &\leqslant \int f(y)\nu(y)d\mu(y) + \int_{\{y:T^*\omega(y)>\nu(y)\}} |f(y)|\frac{T^*\omega(y)}{\nu(y)}\nu(y)d\mu(y).\end{aligned}$$

Recall a known result from[1]: For any $\theta > 0$, there exists a constant $c_\theta > 0$ such that for all $s, t > 0$ we have

$$st \leqslant c_\theta s[1 + (\log^+ s)^{n-1}] + \exp(\theta t^{1/(n-1)}) - 1, \ n \geqslant 2.$$

Applying the pointwise estimate above we get for any $\theta > 0$,

$$\begin{aligned}\int_{\{y:T^*\omega(y)>\nu(y)\}} |f(y)|\frac{T^*\omega(y)}{\nu(y)}\nu(y)d\mu(y) &\leqslant c_\theta \int_{\{y:T^*\omega(y)>\nu(y)\}} |f(y)|[1 + (\log^+ |f(y)|)^{n-1}]\nu(y)d\mu(y)\\ &\quad + \int_{\{y:T^*\omega(y)>\nu(y)\}} \left(\exp\left(\theta\Big(\frac{T^*\omega(y)}{\nu(y)}\Big)^{1/(n-1)}\right) - 1\right)\nu(y)d\mu(y).\end{aligned}$$

Therefore

$$\omega(\cup R_k^*) \lesssim (1 + c_\theta) \int |f(y)|[1 + (\log^+ |f(y)|)^{n-1}]\nu(y)d\mu(y) + I,$$

where

$$I = \int_{\{y:T^*\omega(y)>\nu(y)\}} \left(\exp\left(\theta\Big(\frac{T^*\omega(y)}{\nu(y)}\Big)^{1/(n-1)}\right) - 1\right)\nu(y)d\mu(y).$$

Using the Taylor expansion of e^t we can write

$$I = \sum_{j=1}^{\infty} \frac{\theta^j}{j!} \int_{\{y:T^*\omega(y)>\nu(y)\}} \Big(\frac{T^*\omega(y)}{\nu(y)}\Big)^{j/(n-1)}\nu(y)d\mu(y) = \sum_{1\leqslant j\leqslant n-1} + \sum_{j>n-1} = I_1 + I_2.$$

For I_1, one can easily get

$$\Big(\frac{T^*\omega(y)}{\nu(y)}\Big)^{j/(n-1)} = \Big(\frac{T^*\omega(y)}{\nu(y)}\Big)^{\frac{j}{n-1}-1}\frac{T^*\omega(y)}{\nu(y)} \leqslant \frac{T^*\omega(y)}{\nu(y)},$$

since $\frac{j}{n-1} \leqslant 1$ and $\frac{T^*\omega(y)}{\nu(y)} > 1$. Hence

$$I_1 \leqslant \sum_{1\leqslant j\leqslant n-1} \frac{\theta^j}{j!} \int \theta T^*\omega(y)d\mu(y) \lesssim_n \theta \int T1(x)\omega(y)d\mu(y) = \theta \sum_k \int_{E_k} \omega(y)d\mu = \theta\omega(\cup R_k^*)$$

from the definition of T and E_k provided $\theta < 1$.

For item I_2, since weights ω, ν satisfy condition (A), one has

$$\frac{1}{\mu(R_k^*)}\omega(R_k^*) \leqslant c \inf_{x\in R_k^*} \nu(x),$$

then

$$T^*\omega(x) \approx \sum_k \frac{\omega(R_k)}{\mu(R_k^*)}\chi_{R_k^*}(x) \leqslant \sum_k \nu(x)\chi_{R_k^*}(x) \approx \nu(x)T^*1(x).$$

Therefore

$$\begin{aligned}
I_2 &= \sum_{j>n-1} \frac{\theta^j}{j!} \int (\frac{T^*\omega(y)}{\nu(y)})^{j/(n-1)-1} \frac{T^*\omega(y)}{\nu(y)} \nu(y)d\mu(y) \\
&\lesssim \sum_{j>n-1} \frac{\theta^j}{j!} \int (T^*1(y))^{j/(n-1)-1} T^*\omega(y)d\mu(y) \\
&\lesssim \sum_{j>n-1} \frac{\theta^j}{j!} \int (T^*1(y))^{j/(n-1)} T^*\omega(y)d\mu(y) \\
&= \sum_{j>n-1} \frac{\theta^j}{j!} \int_{\cup R_k^*} T(T^*1(y))^{j/(n-1)} \omega(y)d\mu(y).
\end{aligned}$$

Observing that $Tf(x) \leqslant \sum_k \chi_{E_k}(x) \inf_{y\in R_k^*} M^n_{d\mu}(f)(y) \leqslant M^n_{d\mu}(f)(x)$, so we have

$$\|Tf\|_{L^{p_0}(\omega d\mu)} \lesssim \|f\|_{L^{p_0}(\omega d\mu)}$$

for some $1 < p_0 < \infty$, since $\omega \in A^n_\infty$. This together with Corollary 2.9 and Hölder's inequality yield

$$\begin{aligned}
\int_{\cup R_k^*} T(T^*1(y))^{j/(n-1)} \omega(y)d\mu(y) &\lesssim \omega(\cup R_k^*)^{\frac{1}{p_0'}} \|(T^*1)^{j/(n-1)}\|_{L^{p_0}(\omega d\mu)} \\
&= \omega(\cup R_k^*)^{\frac{1}{p_0'}} \Big(\int (T^*1)^{jp_0/(n-1)} \omega(y)d\mu(y)\Big)^{\frac{1}{p_0}} \lesssim (\frac{jp_0}{n-1})^j \omega(\cup R_k^*),
\end{aligned}$$

which follows that

$$I_2 \lesssim \sum_{j>n-1} \frac{\theta^j}{j!} (\frac{jp_0}{n-1})^j \omega(\cup R_k^*) \lesssim_n \sum_{j>n-1} \frac{(e\theta p_0/(n-1))^j}{\sqrt{j}} \omega(\cup R_k^*) \lesssim_n \frac{(e\theta p_0/(n-1))^n}{\sqrt{n}} \omega(\cup R_k^*),$$

by choosing θ small enough such that $e\theta p_0/(n-1) < 1$.

At last, we obtain that

$$\omega(\cup R_k^*) \lesssim_{\omega,n} (1+c_\theta) \int |f(y)|[1 + (\log^+ |f(y)|)^{n-1}]\nu(y)d\mu(y) + (\theta + \frac{(e\theta p_0/(n-1))^n}{\sqrt{n}})\omega(\cup R_k^*),$$

which yields that

$$\omega(\cup R_k^*) \lesssim \int |f(y)|[1 + (\log^+ |f(y)|)^{n-1}]\nu(y)d\mu(y),$$

by letting θ sufficiently small.

Thus we complete the proof using (3.2). □

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