Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 1337–1354
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Behavior analysis for a size-structured population model with Logistic term and periodic vital rates

Rong Liu, Guirong Liu*

School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi, 030006, China.

Communicated by P. Olejnik

Abstract

In this paper, we investigate the large-time behavior of a nonlinear size-structured population model with logistic term and T-periodic vital rates. We establish the existence of a unique non-negative solution of the given model with the given initial distribution. We prove that there exists at most two T-periodic non-negative solutions (one of them being the trivial one) of the periodic model associated with the given model. We show that for any initial distribution of population the solution of the given model tends to the nontrivial non-negative T-periodic solution of the associated model. At last, we give the numerical tests, which are used to demonstrate the effectiveness of the theoretical results in our paper.

Keywords: Behavior analysis, logistic term, periodic vital rates, size-structure.

2010 MSC: 35L60, 74H40, 92D25.

1. Introduction

Most engineering, physical, and biological problems are governed by functional equations, for example, ordinary differential equations, partial differential equations, integral equations, integro-differential equations, fractional-order differential equations, and stochastic differential equations. As a result, many authors have discussed the numerical solutions ([1]), approximate solutions ([3, 4, 9–11, 31]) and asymptotic behaviors ([7, 14, 19]) of functional equations.

In the past few decades, functional equations are widely used in population dynamics. In [2], the authors investigated a mathematical model of the generalized biological population model and got a new exact solution with a conformable derivative operator. In [33], the authors studied the boundedness and the asymptotic behavior of positive solutions for the difference equation in a new population model. It is worth pointing out that populations consist of individuals, with many structural differences such as age, size, gender, and gene. In the last century, a number of studies appeared on the topic of dynamical population models with individual structure. Especially, age-structured first-order partial differential equations provide a main tool for modeling population systems and are recently employed in economics.

*Corresponding author
Email address: lgr5791@sxu.edu.cn (Guirong Liu)

doi: 10.22436/jnsa.011.12.05

Received: 2018-03-09 Revised: 2018-08-07 Accepted: 2018-08-10

To name a few, see [5–8, 14–17, 23, 25, 30] and their references therein. Book [5] focuses on the well-posedness and optimal control problems of the age-dependent population dynamics, while [23] is mainly on the mathematical theory analysis of age-structured population dynamics. In [16], the authors studied the single-species population dynamics with delayed argument and age-structure. In [7], Aniţa et al. investigated the global behavior for the following age-dependent population model with logistic term and periodic vital rates.

$$\begin{cases} \frac{\partial u(a,t)}{\partial t} + \frac{\partial u(a,t)}{\partial a} + \mu(a,t)u(a,t) + \Phi(I(t))u(a,t) = 0, & (a,t) \in Q, \\ u(0,t) = \int_0^{a_\dagger} \beta(a,t)u(a,t)\, da, & t \in (0,+\infty), \\ u(a,0) = u_0(a), & x \in (0,a_\dagger), \\ I(t) = \int_0^{a_\dagger} u(a,t)\, da, & t \in (0,+\infty), \end{cases} \tag{1.1}$$

where $Q = (0, a_\dagger) \times (0, +\infty)$ and $a_\dagger \in (0, +\infty)$ is the maximal age for the population species.

Long-term ecological researches show that, for many populations especially in many ectothermic animal species, body size is a more significant parameter and has a strong influence upon dynamical processes like its feeding, growth and reproduction [34]. In [13], Caswell said: "size-dependent demography is probably the rule rather than the exception". Here by size we mean some indices displaying the physiological or statistical characteristics of population individuals. Sizes can be mass, length, diameter, volume, maturity, and so on. Body size determines the predation ability, survival, mortality, birth and other vital rates of individuals, and it is more intuitive and easier to measure [29]. For example, for plants, an individual's size is important to capture light to grow [32]. As a result, modelling population dynamics with size structure has been an active and fruitful theme in mathematical biology. There have been many investigations on population models where the growth rate depends on body size (to name a few, see [12, 18–22, 24, 26–28, 34]). In [22], the authors studied the theoretical aspects for an optimal harvesting problem of the following nonlinear size-structured population model in a periodic environment.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial (V(x,t)u(x,t))}{\partial x} = f(x,t) - \mu(x,t,I(t))u(x,t) + \alpha(x,t)u(x,t), & (x,t) \in Q, \\ V(0,t)u(0,t) = \int_0^{l} \beta(x,t,J(t))u(x,t)\, dx, & t \in [0,+\infty), \\ u(x,t) = u(x,t+T), & (x,t) \in Q, \\ I(t) = \int_0^{l} m(x)u(x,t)\, dx, & t \in [0,+\infty), \\ J(t) = \int_0^{l} b(x)u(x,t)\, dx, & t \in [0,+\infty), \end{cases}$$

where $Q = [0, l) \times [0, +\infty)$ and $l \in (0, +\infty)$ is the maximum size for the population species.

To the best of our knowledge, so far only a few papers deal with large time behavior analysis [19, 26] for the population models with size-structure. Moreover, natural populations are actually subject to seasonal fluctuations, which leads to the vital rates of individuals vary periodically. Unfortunately, there is no investigation on the behavior analysis of the size-structured population models with periodic vital rates. The purpose of this paper is to make some contribution in this direction.

Motivated by the above discussion, in this paper, we consider the following nonlinear size structured model with logistic term and T-periodic vital rates which describes the dynamics of a single population species in periodic environments.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial (g(x)u(x,t))}{\partial x} + \mu(x,t)u(x,t) + \Phi(I(t))u(x,t) = 0, & (x,t) \in Q, \\ g(0)u(0,t) = \int_0^{l} \beta(x,t)u(x,t)\, dx, & t \in [0,+\infty), \\ u(x,0) = u_0(x), & x \in [0,l), \\ I(t) = \int_0^{l} u(x,t)\, dx, & t \in [0,+\infty), \end{cases} \tag{1.2}$$

where $Q = [0, l) \times [0, +\infty)$ and $l \in (0, +\infty)$ is the maximum size for the population species. The state variable $u(x, t)$ denotes the population density of size x at time t and by $u_0(x)$ we have denoted the initial distribution of densities. $g(x)$ stands for the growth rate of individual's size, that is $dx/dt = g(x)$. $\mu(x, t)$ and $\beta(x, t)$ are, respectively, the natural mortality and fertility. $\Phi(I(t))u$ is the logistic term, where $I(t) = \int_0^l u(x, t)\, dx$ is the density of the total population at the moment t. The term $\Phi(I(t))$ stands for an external mortality rate which depends on the total population $I(t)$ possibly due to the intra-competition such as the limitation of the resources.

Compared with (1.1), our model generalizes it from age-structure to size-distribution. Note that

$$\frac{da}{dt} = 1 \quad \text{and} \quad \frac{dx}{dt} = g(x).$$

If we take $g(x) \equiv 1$ for all $(x, t) \in Q$, then we can get (1.1). To study the limit behavior of the solution of (1.2), it is necessary to consider the following periodic model.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial (g(x)u(x,t))}{\partial x} + \mu(x, t)u(x, t) + \Phi(I(t))u(x, t) = 0, & (x, t) \in Q, \\ g(0)u(0, t) = \displaystyle\int_0^l \beta(x, t)u(x, t)\, dx, & t \in [0, +\infty), \\ u(x, t) = u(x, t + T), & (x, t) \in Q, \\ I(t) = \displaystyle\int_0^l u(x, t)\, dx, & t \in [0, +\infty), \end{cases} \tag{1.3}$$

which is associated with the given model (1.2). That is, the vital rates in model (1.3) are also T-periodic.

The purpose of this paper is to prove system (1.2) has a unique non-negative solution u, and the periodic system (1.3) admits at most two nonnegative solutions and one of them being the trivial one. Moreover, we will show

$$\lim_{t\to\infty} \|u(\cdot, t) - \hat{u}(\cdot, t)\|_{L^\infty(0,l)} = 0,$$

where $\hat{u}$ is one of the nonnegative solutions to system (1.3).

Take $R_+ = [0, \infty)$. Let $C^1([0, l); R_+)$ be the family of C^1-class functions $\phi : [0, l) \to R_+$. Let $C(R_+; L^\infty[0, l))$ be the family of continuous functions $\psi(\cdot, t)$ with respect to $t \in R_+$ and $\psi(\cdot, t) \in L^\infty[0, l)$ for each $t \in R_+$. Throughout this paper we assume that the following hypotheses hold.

(A_1) $g \in C^1([0, l); R_+)$ is a bounded function. $g(0) = 1$, $\lim_{x\uparrow l} g(x) = 0$ and $g(x) > 0$, $g'(x) \leqslant 0$ for $x \in [0, l)$. Moreover, there is a positive constant L_V such that for $x_1,\ x_2 \in [0, l)$

$$|g(x_1) - g(x_2)| \leqslant L_V|x_1 - x_2|.$$

(A_2) $\beta \in C(R_+; L^\infty[0, l))$ and there exists $\bar{\beta} \in R_+$ such that $0 < \beta(x, t) = \beta(x, t + T) \leqslant \bar{\beta}$ for $(x, t) \in Q$.

(A_3) $\mu \in C(R_+; L^\infty[0, l))$ and $0 \leqslant \mu(x, t) = \mu(x, t + T)$ for all $(x, t) \in Q$. Moreover, we have $\mu(x, t) + g'(x) \geqslant 0$ for all $(x, t) \in Q$.

(A_4) $u_0(\cdot) \in L^\infty[0, l)$ and there exists $\check{u} \in R_+$ such that $0 < u_0(x) \leqslant \check{u}$ for all $x \in [0, l)$.

(A_5) $\Phi : R_+ \to R_+$ is an increasing continuously differentiable function satisfying $\Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = +\infty$. Moreover, $\Phi(r)$ has bounded derivatives with respect to r.

The remaining part of this paper is organized as follows. First, we give some definitions and preliminaries in Section 2. Well-posedness and large time behavior of the state system are studied in Section 3 and Section 4, respectively. Section 5 contains some examples and numerical results, which are used to demonstrate the effectiveness of the theoretical results in our paper. The paper ends with a conclusion section.

2. Preliminaries

In this section, we give some preliminaries. As in [7, 22], we first introduce some definitions.

Definition 2.1. The unique solution $x = \varphi(t; t_0, x_0)$ of the initial-valued problem $x'(t) = g(x),\ x(t_0) = x_0$ is said to be a characteristic curve of system (1.2). Let $z(t) = \varphi(t; 0, 0)$ be the characteristic curve through $(0, 0)$ in the $x - t$ plane.

Definition 2.2. The derivative of the function $u(x, t)$ at (x, t) along the characteristic curve φ is given by

$$D_\varphi u(x,t) = \lim_{h\to 0} \frac{u(\varphi(t+h; t, x), t+h) - u(x,t)}{h}.$$

Lemma 2.3. *Let assumptions* (A_1)-(A_3) *hold. Then the following system*

$$\begin{cases} \frac{\partial u^*(x,t)}{\partial t} + \frac{\partial (g(x)u^*(x,t))}{\partial x} + \mu(x,t)u^*(x,t) + \alpha u^*(x,t) = 0, & (x,t) \in Q, \\ g(0)u^*(0,t) = \int_0^l \beta(x,t)u^*(x,t)\,dx, & t \in R_+, \\ u^*(x,0) = u^*(x, t+T), & (x,t) \in Q, \end{cases} \tag{2.1}$$

has at most two nonnegative solutions, the trivial one and a nontrivial one, u^*.

Proof. For an arbitrary point (x, t) in the first quadrant of the $x - t$ plane such that $x \leqslant z(t)$, that is, $\varphi(t; t, x) \leqslant z(t)$, define the initial time $\tau \triangleq \tau(x, t)$ implicitly by the relation $\varphi(t; \tau, 0) = x$ if and only if $\varphi(\tau; t, x) = 0$. Due to the periodicity of the state variables, $u^*(x, t)$, we consider the case where $t > z^{-1}(l)$. Utilizing the characteristic curve technique and $\tau = \varphi^{-1}(0; t, x)$, we can get $u^*(x, t)$ satisfies

$$u^*(x,t) = u^*(0, \varphi^{-1}(0;t,x)) \exp\left\{-\int_\tau^t [\mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x)) + \alpha]\,ds\right\}. \tag{2.2}$$

Denote $b^*(t) \triangleq g(0)u^*(0, t)$. It follows from $g(0) = 1$ and (2.2) that

$$\begin{aligned} b^*(t) = g(0)u^*(0,t) &= \int_0^l \beta(x,t)u^*(x,t)\,dx \\ &= \int_0^l \beta(x,t)u^*(0,\varphi^{-1}(0;t,x))e^{\{-\int_\tau^t[\mu(\varphi(s;t,x),s)+g'(\varphi(s;t,x))+\alpha]\,ds\}}\,dx \\ &= \int_0^l \beta(x,t)g(0)u^*(0,\varphi^{-1}(0;t,x))e^{\{-\int_\tau^t[\mu(\varphi(s;t,x),s)+g'(\varphi(s;t,x))+\alpha]\,ds\}}\,dx \\ &= \int_0^l \beta(x,t)b^*(\varphi^{-1}(0;t,x))e^{\{-\int_\tau^t[\mu(\varphi(s;t,x),s)+g'(\varphi(s;t,x))+\alpha]\,ds\}}\,dx \\ &= \int_0^l e^{-\alpha(t-\tau)}\beta(x,t)b^*(\varphi^{-1}(0;t,x))e^{\{-\int_\tau^t[\mu(\varphi(s;t,x),s)+g'(\varphi(s;t,x))]\,ds\}}\,dx. \end{aligned} \tag{2.3}$$

For (2.3), let $\sigma = \varphi^{-1}(0; t, x)$. Then by Definition 2.1, $\sigma = \varphi^{-1}(0; t, 0) = t$ when $x = 0$ while $\sigma = \varphi^{-1}(0; t, l) = 0$ when $x = l$. It follows from $\sigma = \varphi^{-1}(0; t, x)$ that $x = \varphi(t; \sigma, 0)$. Note that $dx/d\sigma$ is the solution of the initial-valued problem

$$\begin{cases} \frac{dz}{dt} = g'(\varphi(t;\sigma,0))z, \\ z(\sigma) = -g(0) = -1. \end{cases} \tag{2.4}$$

It is clear that the solution of (2.4) is $z(t) = -\exp\left\{\int_\sigma^t g'(\varphi(r;\sigma,0))\,dr\right\}$, which implies

$$dx = -\exp\left\{\int_\sigma^t g'(\varphi(r;\sigma,0))\,dr\right\} d\sigma. \tag{2.5}$$

Then (2.3) can be transformed into

$$b^*(t) = \int_0^t e^{-\alpha(t-\sigma)} K(\varphi(t;\sigma,0),t) b^*(\sigma)\, d\sigma, \tag{2.6}$$

which is equivalent to the following equation

$$e^{\alpha t} b^*(t) = \int_0^t K(\varphi(t;\sigma,0),t) e^{\alpha\sigma} b^*(\sigma)\, d\sigma, \tag{2.7}$$

with

$$K(\varphi(t;\sigma,0),t) = \beta(\varphi(t;\sigma,0),t) e^{\int_\sigma^t g'(\varphi(r;\sigma,0))\,dr} e^{-\int_\sigma^t [\mu(\varphi(s;t,\varphi(t;\sigma,0)),s)+g'(\varphi(s;t,\varphi(t;\sigma,0)))]\,ds}. \tag{2.8}$$

Assume that

$$\|K(\cdot,t)\|_{L^1[0,l)} = \int_0^l K(s,t)\, ds < 1. \tag{2.9}$$

Denote $\bar{b}^*(t) = e^{\alpha t} b^*(t)$. It follows from the Banach fixed-point theorem that the solution of (2.7) can be obtained via the standard iteration procedure

$$\begin{cases} \bar{b}_0^*(t) = \bar{b}_0^*, \\ \bar{b}_{n+1}^*(t) = \displaystyle\int_0^t K(\varphi(t;\sigma,0),t) \bar{b}_n^*(\sigma)\, d\sigma. \end{cases}$$

Due to the periodicity of b^*, we need only to consider the case $t \in [z^{-1}(l), z^{-1}(l)+T]$. Then by (2.8) and assumptions we have $\bar{b}_n^* \in \mathcal{C}_\mathcal{T} \triangleq C([z^{-1}(l), z^{-1}(l)+T])$ and $\bar{b}_n^*(t) \geqslant 0$. Moreover

$$\begin{aligned} \left|\bar{b}_{n+1}^*(t) - \bar{b}_n^*(t)\right| &= \left| \int_0^t K(\varphi(t;\sigma,0),t) \bar{b}_n^*(\sigma)\, d\sigma - \int_0^t K(\varphi(t;\sigma,0),t) \bar{b}_{n-1}^*(\sigma)\, d\sigma \right| \\ &\leqslant \int_0^t K(\varphi(t;\sigma,0),t) \left|\bar{b}_n^*(\sigma) - \bar{b}_{n-1}^*(\sigma)\right|\, d\sigma, \end{aligned}$$

and

$$\begin{aligned} \left\|\bar{b}_{n+1}^* - \bar{b}_n^*\right\|_{\mathcal{C}_\mathcal{T}} &\leqslant \left\|\bar{b}_n^* - \bar{b}_{n-1}^*\right\|_{\mathcal{C}_\mathcal{T}} \int_0^t K(\varphi(t;\sigma,0),t)\, d\sigma \\ &\leqslant \left\|\bar{b}_n^* - \bar{b}_{n-1}^*\right\|_{\mathcal{C}_\mathcal{T}} \int_0^l K(s,t)\, ds \\ &\leqslant \|K(\cdot,t)\|_{L^1[0,l)} \left\|\bar{b}_n^* - \bar{b}_{n-1}^*\right\|_{\mathcal{C}_\mathcal{T}}. \end{aligned}$$

Thus by (2.9), the sequence $\bar{b}_n^*(t)$ converges, uniformly on $[z^{-1}(l), z^{-1}(l)+T]$, to a solution $\bar{b}^*(t)$, such that $\bar{b}^* \in \mathcal{C}_\mathcal{T}$. Concerning uniqueness of this solution we see that if $\bar{b}_1^*(t)$ and $\bar{b}_2^*(t)$ are two solutions of (2.7) we must have

$$\left\|\bar{b}_1^* - \bar{b}_2^*\right\|_{\mathcal{C}_\mathcal{T}} \leqslant \|K(\cdot,t)\|_{L^1[0,l)} \left\|\bar{b}_1^* - \bar{b}_2^*\right\|_{\mathcal{C}_\mathcal{T}}.$$

So that, by (2.9), $\bar{b}_1^*(t) = \bar{b}_2^*(t)$. Moreover, if $\bar{b}_0^* = 0$, then $\bar{b}^*(t) \equiv 0$. If $\bar{b}_0^*$ is a positive constant, then $\bar{b}^*(t) > 0$. Then by the periodicity of b^* we know Eq. (2.6) admits at most non-negative solutions the trivial one $b^* \equiv 0$ and a nontrivial one $b^* \geqslant 0$ in R_+. Thus, system (2.1) has at most two nonnegative solutions, the trivial one $u^* \equiv 0$ and a nontrivial one, u^*. The proof is therefore complete. □

Lemma 2.4. *Let assumptions* (A_1)-(A_4) *hold. Then the following system*

$$\begin{cases} \frac{\partial \bar{u}(x,t)}{\partial t} + \frac{\partial(g(x)\bar{u}(x,t))}{\partial x} + \mu(x,t)\bar{u}(x,t) = 0, & (x,t) \in Q, \\ g(0)\bar{u}(0,t) = \int_0^l \beta(x,t)\bar{u}(x,t)\,dx, & t \in R_+, \\ \bar{u}(x,0) = u_0(x), & x \in [0,l), \end{cases} \tag{2.10}$$

has a unique nonnegative solution $\bar{u} \in L^\infty(\overline{Q})$.

Proof. Utilizing the characteristic curve technique and $\tau = \varphi^{-1}(0;t,x)$, we can get $\bar{u}(x,t)$ satisfies

$$\bar{u}(x,t) = \begin{cases} \bar{u}(0,\varphi^{-1}(0;t,x)) \exp\left\{-\int_\tau^t \mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x))\,ds\right\}, & x \leqslant z(t), \\ u_0(\varphi(0;t,x)) \exp\left\{-\int_0^t \mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x))\,ds\right\}, & x > z(t). \end{cases} \tag{2.11}$$

Let $\overline{M} \triangleq \check{u}l + \bar{\beta}\check{u}l\overline{T}e^{\bar{\beta}\overline{T}}$ and $\mathbf{X} = L^\infty(\overline{Q})$. Define a new norm in $\mathbf{X}$ by

$$\|u\|_* = \operatorname{Ess}\sup_{t\in[0,\overline{T}]} \left\{ e^{-\lambda t} \int_0^l |u(x,s)|\,dx \right\}$$

for some $\lambda > 0$, which is equivalent to the usual norm. It is clear that $(\mathbf{X}, \|\cdot\|_*)$ is a Banach space. Let

$$\mathcal{X} = \left\{ u \in \mathbf{X} \,\middle|\, u(x,t) \geqslant 0 \text{ a.e. } (x,t) \in \overline{Q} \text{ and } \int_0^l u(x,t)\,dx \leqslant \overline{M} \right\}.$$

It is easy to show that $\mathcal{X}$ is a nonempty closed subset in $\mathbf{X}$. Define $\mathcal{A} : \mathcal{X} \to \mathbf{X}$ by

$$(\mathcal{A}\bar{u})(x,t) = \begin{cases} \bar{u}(0,\varphi^{-1}(0;t,x)) \exp\left\{-\int_\tau^t \mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x))\,ds\right\}, & x \leqslant z(t), \\ u_0(\varphi(0;t,x)) \exp\left\{-\int_0^t \mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x))\,ds\right\}, & x > z(t). \end{cases}$$

In the following, we verify that mapping $\mathcal{A}$ satisfies the conditions of the Banach fixed point theorem.

(i) It is easy to show that $(\mathcal{X}, d)$ is a complete metric space. Here $d(u,v) = \|u - v\|_*$ for any $u, v \in \mathbf{X}$.

(ii) The mapping $\mathcal{A}$ maps $\mathcal{X}$ into $\mathcal{X}$. Denote $\bar{b}(t) \triangleq g(0)\bar{u}(0,t)$. From $g(0) = 1$, (A_2), and (A_4), it follows that

$$\begin{aligned} \bar{b}(t) &= \int_0^{z(t)} \beta(x,t)\bar{u}(x,t)\,dx + \int_{z(t)}^l \beta(x,t)\bar{u}(x,t)\,dx \\ &= \int_0^{z(t)} \beta(x,t)\bar{b}(\varphi^{-1}(0;t,x)) \exp\left\{-\int_\tau^t \mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x))\,ds\right\} dx \\ &\quad + \int_{z(t)}^l \beta(x,t)u_0(\varphi(0;t,x)) \exp\left\{-\int_0^t \left[\mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x))\right] ds\right\} dx \\ &\leqslant \int_0^{z(t)} \beta(x,t)\bar{b}(\varphi^{-1}(0;t,x))\,dx + \int_{z(t)}^l \beta(x,t)u_0(\varphi(0;t,x))\,dx \\ &\leqslant \bar{\beta}\check{u}l + \bar{\beta}\int_0^{z(t)} \bar{b}(\varphi^{-1}(0;t,x))\,dx. \end{aligned} \tag{2.12}$$

For (2.12), let $\sigma = \varphi^{-1}(0;t,x)$. Then by Definition 2.1, $\sigma = \varphi^{-1}(0;t,0) = t$ when $x = 0$, while $\sigma = \varphi^{-1}(0;t,z(t)) = 0$ when $x = z(t)$. It follows from (2.5) and (2.12) that

$$\bar{b}(t) \leqslant \bar{\beta}\check{u}l + \bar{\beta}\int_0^t \bar{b}(\sigma)\,d\sigma.$$

It follows from Gronwall's inequality that $\bar{b}(t) \leqslant \bar{\beta}\check{u}le^{\bar{\beta}t} \leqslant \bar{\beta}\check{u}le^{\bar{\beta}\overline{T}}$. Now, we consider $(\mathcal{A}\bar{u})(x,t)$. It follows from (2.5) and (2.11) that

$$\begin{aligned}\int_0^l |\mathcal{A}\bar{u}|(x,t)\,dx &= \int_0^{z(t)} \bar{u}(x,t)\,dx + \int_{z(t)}^l \bar{u}(x,t)\,dx\\ &\leqslant \int_0^{z(t)} \bar{b}(\varphi(0;t,x))\,dx + \int_0^l u_0(\varphi(0;t,x))\,dx \leqslant \check{u}l + \int_0^t \bar{b}(\sigma)\,d\sigma \leqslant \check{u}l + \bar{\beta}\check{u}l\bar{T}e^{\bar{\beta}\bar{T}}.\end{aligned}$$

It follows that $\mathcal{A}$ is a mapping from $\mathcal{X}$ to $\mathcal{X}$.

(iii) $\mathcal{A}$ is a contraction mapping on the complete metric space $(\mathcal{X}, d)$. It follows from (2.5) and the definition of mapping $\mathcal{A}$ that

$$\begin{aligned}\int_0^l |\mathcal{A}\bar{u} - \mathcal{A}\bar{u}'|(x,t)\,dx &= \int_0^{z(t)} |\mathcal{A}\bar{u} - \mathcal{A}\bar{u}'|(x,t)\,dx + \int_{z(t)}^l |\mathcal{A}\bar{u} - \mathcal{A}\bar{u}'|(x,t)\,dx\\ &\leqslant \int_0^{z(t)} |\bar{b}(\varphi^{-1}(0;t,x)) - \bar{b}'(\varphi^{-1}(0;t,x))|\,dx\\ &\leqslant \int_0^t |\bar{b}(s) - \bar{b}'(s)|\,ds\\ &\leqslant \bar{\beta}\int_0^t\int_0^l |\bar{u}(x,s) - \bar{u}'(x,s)|\,dx\,ds\\ &= \bar{\beta}\int_0^t \|\bar{u}(\cdot,s) - \bar{u}'(\cdot,s)\|_{L^1[0,l)}\,ds.\end{aligned}$$

Then

$$\begin{aligned}d(\mathcal{A}\bar{u}, \mathcal{A}\bar{u}') &= \mathrm{Ess}\sup_{t\in[0,\bar{T}]}\left\{e^{-\lambda t}\int_0^l |(\mathcal{A}\bar{u})(x,t) - (\mathcal{A}\bar{u}')(x,t)|\,dx\right\}\\ &\leqslant \bar{\beta}\,\mathrm{Ess}\sup_{t\in[0,\bar{T}]}\left\{e^{-\lambda t}\int_0^t e^{\lambda s}\left[e^{-\lambda s}\int_0^l |\bar{u}(x,s) - \bar{u}'(x,s)|\,dx\right]ds\right\} = \frac{\bar{\beta}}{\lambda}d(\bar{u}, \bar{u}').\end{aligned}$$

Choose λ such that $\lambda > \bar{\beta}$. Then $\mathcal{A}$ becomes a contraction on the complete metric space $(\mathcal{X}, d)$.

Therefore, by Banach fixed point theorem the mapping $\mathcal{A}$ has a unique fixed point, which is the solution of system (2.10). The proof is therefore complete. □

By [23, Theorem 2.3, P36], we have the following results.

Lemma 2.5. *Let assumptions* (A_1)-(A_4) *hold. Then there exists a unique* $\alpha^* \in R$ *such that*

$$\bar{b}(t) = e^{\alpha^* t}b^*(t)(1 + \Omega(t)),$$

where $\lim_{t\to\infty}\Omega(t) = 0$.

Lemma 2.6. *Let assumptions* (A_1)-(A_4) *hold. Then there exists a unique* $c_0 \in R_+$ *such that*

$$\lim_{t\to+\infty}\|\bar{u}(\cdot,t)e^{-\alpha^* t} - c_0 u^*(\cdot,t)\|_{L^\infty[0,l)} = 0, \tag{2.13}$$

where $\bar{u}$ *is a solution to* (2.10) *and* u^* *is a solution to* (2.1) *with* $\alpha \triangleq \alpha^*$.

3. Well-posedness of the state system

This section is devoted to the well-posedness of system (1.2).

3.1. Special cases

We fix function $I(\cdot) \in L^\infty(R_+)$ and for $\forall\, t \in R_+$, $I(t) \geqslant 0$. Then by Lemma 2.4, we have the following result.

Proposition 3.1. *Let assumptions* (A_1)-(A_5) *hold. For fixed* $I(\cdot) \in L^\infty(R_+)$, *system* (1.2) *has one and only one non-negative solution* $u(x,t,I) \in L^\infty(Q)$.

3.2. General cases

By assumption (A_5), we have $|\Phi'(r)| \leqslant M$. Utilizing the characteristic curve technique, the solution of system (1.2) can be given as

$$u(x,t,I) = \begin{cases} u(0,\varphi^{-1}(0;t,x),I)E(t;x,I), & x \leqslant z(t), \\ u_0(\varphi(0;t,x))\Pi(t;x,I), & x > z(t), \end{cases} \tag{3.1}$$

where

$$E(t;x,I) = \exp\left\{-\int_\tau^t \left[\mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x)) + \Phi(I(s))\right] \, ds\right\},$$
$$\Pi(t;x,I) = \exp\left\{-\int_0^t \left[\mu(\varphi(s;t,x),s) + g'(\varphi(s;t,x)) + \Phi(I(s))\right] \, ds\right\}.$$

Denote $b(t;I) \triangleq g(0)u(0,t;I)$. It follows from $g(0) = 1$ that

$$\begin{aligned} b(t;I) &= \int_0^{z(t)} \beta(x,t)u(x,t;I)\,dx + \int_{z(t)}^{l} \beta(x,t)u(x,t;I)\,dx \\ &= \int_0^{z(t)} \beta(x,t)E(t;x,I)b(\varphi^{-1}(0;t,x),I)\,dx + \int_{z(t)}^{l} \beta(x,t)u_0(\varphi(0;t,x))\Pi(t;x,I)\,dx \\ &\triangleq S_1 + S_2. \end{aligned} \tag{3.2}$$

For S_1, let $\sigma = \varphi^{-1}(0;t,x)$. It follows from (2.5) that $dx = -\exp\{\int_\sigma^t g'(\varphi(r;\sigma,0))\,dr\}d\sigma$. Then (3.2) can be transformed into

$$b(t;I) = \int_0^t K(\varphi(t;\sigma,0),t;I)b(\sigma;I)\,d\sigma + F(t;I), \tag{3.3}$$

with

$$\begin{aligned} K(\varphi(t;\sigma,0),t;I) &= \beta(\varphi(t;\sigma,0),t)\exp\left\{\int_\sigma^t g'(\varphi(r;\sigma,0))\,dr\right\} \\ &\quad \times \exp\left\{-\int_\sigma^t \left[\mu(\varphi(s;t,\varphi(t;\sigma,0)),s) + g'(\varphi(s;t,\varphi(t;\sigma,0))) + \Phi(I(s))\right] \, ds\right\}, \end{aligned} \tag{3.4}$$

$$F(t;I) = \int_{z(t)}^{l} \beta(x,t)u_0(\varphi(0;t,x))\Pi(t;x,I)\,dx. \tag{3.5}$$

For $\forall\, I_1, I_2 \in L^\infty(R_+)$, using (3.4) and the fundamental inequality $|e^{-a} - e^{-b}| \leqslant |a-b|$ for any $a, b \geqslant 0$, yields

$$\begin{aligned} &\left|K(\varphi(t;\sigma,0),t;I_1) - K(\varphi(t;\sigma,0),t;I_2)\right| \\ &= \beta(\varphi(t;\sigma,0),t)\exp\left\{\int_\sigma^t g'(\varphi(r;\sigma,0))\,dr\right\} \\ &\quad \times \left|\exp\left\{-\int_\sigma^t \left[\mu(\varphi(s;t,\varphi(t;\sigma,0)),s) + g'(\varphi(s;t,\varphi(t;\sigma,0))) + \Phi(I_1(s))\right] \, ds\right\}\right. \\ &\quad \left. - \exp\left\{-\int_\sigma^t \left[\mu(\varphi(s;t,\varphi(t;\sigma,0)),s) + g'(\varphi(s;t,\varphi(t;\sigma,0))) + \Phi(I_2(s))\right] \, ds\right\}\right| \\ &\leqslant \bar\beta \left|\exp\left\{-\int_\sigma^t \left[\mu(\varphi(s;t,\varphi(t;\sigma,0)),s) + g'(\varphi(s;t,\varphi(t;\sigma,0))) + \Phi(I_1(s))\right] \, ds\right\}\right. \\ &\quad \left. - \exp\left\{-\int_\sigma^t \left[\mu(\varphi(s;t,\varphi(t;\sigma,0)),s) + g'(\varphi(s;t,\varphi(t;\sigma,0))) + \Phi(I_2(s))\right] \, ds\right\}\right| \\ &\leqslant \bar\beta \int_\sigma^t |\Phi(I_1(s)) - \Phi(I_2(s)|\,ds \\ &\leqslant M\bar\beta \int_0^t |I_1(s) - I_2(s)|\,ds. \end{aligned} \tag{3.6}$$

By (3.5), we obtain

$$\begin{aligned}|F(t;I_1)-F(t;I_2)| &= \left|\int_{z(t)}^{l}\beta(x,t)u_0(\varphi(0;t,x))\Pi(t;x,I_1)\,dx-\int_{z(t)}^{l}\beta(x,t)u_0(\varphi(0;t,x))\Pi(t;x,I_2)\,dx\right|\\ &\leqslant \bar{\beta}\check{u}\int_{z(t)}^{l}|\Pi(t;x,I_1)-\Pi(t;x,I_2)|\,dx\\ &\leqslant \bar{\beta}\check{u}\int_{z(t)}^{l}\int_0^t|\Phi(I_1(s))-\Phi(I_2(s))|\,ds\,dx\\ &\leqslant M\bar{\beta}\check{u}l\int_0^t|I_1(s)-I_2(s)|\,ds.\end{aligned}\tag{3.7}$$

By (3.3)-(3.5), we get $b(t;I)\leqslant\bar{\beta}\check{u}l+\bar{\beta}\int_0^t b(\sigma;I)\,d\sigma$. From Gronwall's inequality, it follows that

$$b(t;I)\leqslant l\bar{\beta}\check{u}e^{\bar{\beta}t}.\tag{3.8}$$

From (3.3) and (3.6)–(3.8), it follows that

$$\begin{aligned}|b(t;I_1)-b(t;I_2)| &\leqslant |F(t;I_1)-F(t;I_2)|+\left|\int_0^t K(\varphi(t;\sigma,0),t;I_1)b(\sigma;I_1)\,d\sigma-\int_0^t K(\varphi(t;\sigma,0),t;I_2)b(\sigma;I_2)\,d\sigma\right|\\ &\leqslant |F(t;I_1)-F(t;I_2)|+\int_0^t|K(\varphi(t;\sigma,0),t;I_1)-K(\varphi(t;\sigma,0),t;I_2)|b(\sigma;I_1)\,d\sigma\\ &\quad+\int_0^t K(\varphi(t;\sigma,0),t;I_2)|b(\sigma;I_1)-b(\sigma;I_2)|\,d\sigma\\ &\leqslant\left(M\bar{\beta}\check{u}l+lM\bar{\beta}^2\check{u}e^{\bar{\beta}t}t\right)\int_0^t|I_1(s)-I_2(s)|\,ds+\bar{\beta}\int_0^t|b(\sigma;I_1)-b(\sigma;I_2)|\,d\sigma\\ &\triangleq\left(M\bar{\beta}\check{u}l+lM\bar{\beta}^2\check{u}e^{\bar{\beta}t}t\right)W(t)+\bar{\beta}\int_0^t|b(\sigma;I_1)-b(\sigma;I_2)|\,d\sigma,\end{aligned}\tag{3.9}$$

where $W(t)=\int_0^t|I_1(s)-I_2(s)|\,ds$. For the given $\overline{T}>0$, we consider the case $t\in[0,\overline{T}]$. It follows from (3.8) that

$$b(t;I)\leqslant l\bar{\beta}\check{u}e^{\bar{\beta}\overline{T}}\triangleq M_1.\tag{3.10}$$

From (3.9), we have

$$\begin{aligned}|b(t;I_1)-b(t;I_2)| &\leqslant\left(M\bar{\beta}\check{u}l+lM\bar{\beta}^2\check{u}e^{\bar{\beta}\overline{T}}\overline{T}\right)W(t)+\bar{\beta}\int_0^t|b(s;I_1)-b(s;I_2)|\,ds\\ &\triangleq M_2W(t)+\bar{\beta}\int_0^t|b(s;I_1)-b(s;I_2)|\,ds,\end{aligned}\tag{3.11}$$

where $M_2=M\bar{\beta}\check{u}l+lM\bar{\beta}^2\check{u}e^{\bar{\beta}\overline{T}}\overline{T}$. Thus, use of Gronwall's inequality in (3.11) gives

$$|b(t;I_1)-b(t;I_2)|\leqslant M_2W(t)+M_2\bar{\beta}e^{\bar{\beta}\overline{T}}\int_0^t W(s)\,ds\leqslant M_3W(t),\tag{3.12}$$

where M_3 is a positive constant independent of u.

Let $M_4\triangleq M_1\overline{T}+\check{u}l$ and $\mathbf{Y}=L^\infty([0,\overline{T}],L^1[0,l))$. Define a new norm by

$$\|v\|_\lambda=\text{Ess}\sup_{t\in[0,\overline{T}]}\left\{e^{-\lambda t}\int_0^l v(x,s)\,dx\right\}=\text{Ess}\sup_{t\in[0,\overline{T}]}e^{-\lambda t}\|v(\cdot,t)\|_{L^1[0,l)}$$

for any $v \in \mathbf{Y}$ and some $\lambda > 0$, which is equivalent to the usual norm. It is clear that $(\mathbf{Y}, \|\cdot\|_\lambda)$ is a Banach space. Let

$$\mathcal{Y} = \left\{ v \in \mathbf{Y} \,\middle|\, v(x,t) \geqslant 0 \text{ a.e. } (x,t) \in Q \ \text{ and } \int_0^l v(x,t)\,dx \leqslant M_4 \right\}.$$

Define $\mathcal{B} : \mathcal{Y} \to \mathbf{Y}$ by

$$(\mathcal{B}v)(x,t) = u(x,t;X), \quad X(t) = \int_0^l v(x,t)\,dx,$$

where $u(x,t;X)$ is a solution of system (1.2) corresponding to $I(t) = X(t)$ and given by the right hand side of (3.1).

In the following, we verify that mapping $\mathcal{B}$ satisfies the conditions of the Banach fixed point theorem.
(i) It is easy to show that $(\mathcal{Y}, d)$ is a complete metric space. Here $d(u,v) = \|u - v\|_\lambda$ for any $u, v \in \mathbf{Y}$.
(ii) The mapping $\mathcal{B}$ maps $\mathcal{Y}$ into $\mathcal{Y}$. For any $v \in \mathcal{Y}$, by (2.5), (3.10), and the definition of $\mathcal{B}$, we have

$$\begin{aligned}
\int_0^l u(x,t;X)\,dx &= \int_0^{z(t)} b(\varphi^{-1}(0;t,x);X)E(t,x;X)\,dx + \int_{z(t)}^l u_0(\varphi(0;t,x))\Pi(t,x;X)\,dx \\
&\leqslant \int_0^{z(t)} b(\varphi^{-1}(0;t,x);X)\,dx + \int_0^l u_0(\varphi(0;t,x))\,dx \\
&\leqslant \int_0^t b(s;X)\,ds + \int_0^l u_0(\varphi(0;t,x))\,dx \leqslant M_1\overline{T} + \check{u}l,
\end{aligned}$$

thus $\mathcal{B}v \in \mathcal{Y}$, which means $\mathcal{B}$ is a mapping from $\mathcal{Y}$ to $\mathcal{Y}$.
(iii) $\mathcal{B}$ is a contraction mapping on the complete metric space $(\mathcal{Y}, d)$. For arbitrary $v_1, v_2 \in \mathcal{Y}$, it follows from (3.10) and (3.12) that

$$\begin{aligned}
&\|(\mathcal{B}v_1)(\cdot,t) - (\mathcal{B}v_2)(\cdot,t)\|_{L^1[0,l)} \\
&= \int_0^{z(t)} |u(x,t;X_1) - u(x,t;X_2)|\,dx + \int_{z(t)}^l |u(x,t;X_1) - u(x,t;X_2)|\,dx \\
&\leqslant \int_0^{z(t)} b(\varphi^{-1}(0;t,x);X_1)|E(t,x;X_1) - E(t,x;X_2)|\,dx \\
&\quad + \int_0^{z(t)} |b(\varphi^{-1}(0;t,x);X_1) - b(\varphi^{-1}(0;t,x);X_2)|E(t,x;X_2)\,dx \\
&\quad + \int_{z(t)}^l u_0(\varphi(0;t,x))|\Pi(t,x;X_1) - \Pi(t,x;X_1)|\,dx \\
&\leqslant M\int_0^{z(t)} b(\varphi^{-1}(0;t,x);X_1)\int_0^t |X_1(s) - X_2(s)|\,ds\,dx \\
&\quad + \int_0^{z(t)} |b(\varphi^{-1}(0;t,x);X_1) - b(\varphi^{-1}(0;t,x);X_2)|\,dx + M\check{u}\int_0^l\int_0^t |X_1(s) - X_2(s)|\,ds\,dx \\
&\leqslant M\int_0^t b(\sigma;X_1)\int_0^t |X_1(s) - X_2(s)|\,ds\,d\sigma + \int_0^t |b(s;X_1) - b(s;X_2)|\,ds + M\check{u}l\int_0^t |X_1(s) - X_2(s)|\,ds \\
&\leqslant MM_1\overline{T}\int_0^t |X_1(s) - X_2(s)|\,ds + M_3\overline{T}\int_0^t |X_1(s) - X_2(s)|\,ds + M\check{u}l\int_0^t |X_1(s) - X_2(s)|\,ds \\
&\triangleq M_5\int_0^t |X_1(s) - X_2(s)|\,ds \\
&\leqslant M_5\int_0^t \|v_1(\cdot,s) - v_2(\cdot,s)\|_{L^1[0,l)}\,ds,
\end{aligned}$$

where $M_5 = MM_1\overline{T} + M_3\overline{T} + M\breve{u}l$. Then

$$\begin{aligned} d(\mathcal{B}v_1, \mathcal{B}v_2) &= \operatorname{Ess}\sup_{t\in[0,\overline{T}]} \left\{ e^{-\lambda t} \int_0^l |(\mathcal{B}v_1)(x,t) - (\mathcal{B}v_2)(x,t)|\, dx \right\} \\ &\leqslant M_5 \operatorname{Ess}\sup_{t\in[0,\overline{T}]} \left\{ e^{-\lambda t} \int_0^t \int_0^l |v_1(x,s) - v_2(x,s)|\, dx\, ds \right\} = \frac{M_5}{\lambda} d(v_1, v_2). \end{aligned}$$

Choose λ such that $\lambda > M_5$. Then $\mathcal{B}$ becomes a contraction on the complete metric space $(\mathcal{Y}, d)$.

Thus, by Banach fixed point theorem the mapping $\mathcal{B}$ has a unique fixed point, which is the solution of system (1.2).

We summarize the above analysis as follows.

Theorem 3.2. *Let assumptions* (A_1)-(A_5) *hold. Then system* (1.2) *has a unique solution* $u \in L^\infty(\overline{Q})$, *which is non-negative and bounded.*

By the analysis as that in Lemma 2.3 and Theorem 3.2, we can get the following result.

Theorem 3.3. *Let assumptions* (A_1)-(A_5) *hold. Then system* (1.3) *admits at most two non-negative solutions one of them being the trivial one.*

Note that $\Phi(r) > 0$ for any $r \in [0, \infty)$. It is easy to get the following result.

Theorem 3.4. *If* u *and* $\bar{u}$ *are solutions of* (1.2) *and* (2.10), *respectively, then we have* $u(x,t) \leqslant \bar{u}(x,t)$, *a.e. in* Q.

4. Large time behavior of the solution

In this section, we discuss the large time behavior of the solution to system (1.2). It is clear that system (1.2) can be equivalently written as

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial(g(x)u)}{\partial x} + \mu(x,t)u(x,t) + \alpha^* u(x,t) + (\Phi(I(t)) - \alpha^*)u(x,t) = 0, & (x,t) \in Q, \\ g(0)u(0,t) = \displaystyle\int_0^l \beta(x,t)u(x,t)\, dx, & t \in [0, +\infty), \\ u(x,0) = u_0(x), & x \in [0, l), \\ I(t) = \displaystyle\int_0^l u(x,t)\, dx, & t \in [0, +\infty). \end{cases}$$

Denote by $\tilde{u}$, the solution to

$$\begin{cases} \frac{\partial \tilde{u}(x,t)}{\partial t} + \frac{\partial(g(x)\tilde{u}(x,t))}{\partial x} + \mu(x,t)\tilde{u}(x,t) + \alpha^* \tilde{u}(x,t) = 0, & (x,t) \in Q, \\ g(0)\tilde{u}(0,t) = \displaystyle\int_0^l \beta(x,t)\tilde{u}(x,t)\, dx, & t \in [0, +\infty), \\ \tilde{u}(x,0) = u_0(x), & x \in [0, l), \end{cases} \tag{4.1}$$

for the existence and uniqueness of the solution we refer to Lemma 2.4.

By Theorem 3.2, we know that system (1.2) has a unique non-negative solution and the solution $u(x,t)$ to (1.2) has the form (see [24]) $u(x,t) = y(t)\tilde{u}(x,t)$, where $y(t)$ is the Carathéodory solution to

$$\begin{cases} y'(t) + \Phi(\tilde{I}(t)y(t))y(t) - \alpha^* y(t) = 0, & t \in R_+, \\ y(0) = 1. \end{cases} \tag{4.2}$$

Here

$$\tilde{I}(t) = \int_0^l \tilde{u}(x,t)\, dx, \quad t \in R_+.$$

Theorem 4.1. *Let assumptions* (A_1)-(A_5) *hold. If* $\alpha^* < 0$*, then* $\lim_{t\to\infty} \|u(\cdot,t)\|_{L^\infty[0,l)} = 0$*, where* $u(x,t)$ *is a solution to system* (1.2).

Proof. By assumption (A_5), we know that $\Phi : R_+ \to R_+$. Then using Theorem 3.4 we get that

$$0 \leqslant u(x,t) \leqslant \bar{u}(x,t), \quad \forall\, t \geqslant 0, \text{ a.e. } x \in [0,l),$$

where $\bar{u}$ is a solution to (2.10). Since $\alpha^* < 0$, by (2.13), we can conclude that $\lim_{t\to\infty} \|\bar{u}(\cdot,t)\|_{L^\infty[0,l)} = 0$ and we get the conclusion. The proof is therefore complete. □

Next, we consider the case $\alpha^* \geqslant 0$. We will see that the logistic term play an important role in the large-time behavior of the solution to (1.2). It is easy for us to show that

$$\tilde{u}(x,t) = e^{-\alpha^* t}\bar{u}(x,t), \quad \forall\, t \in R_+, \text{ a.e. } x \in [0,l),$$

where $\tilde{u}(x,t)$ is a solution to system (4.1) and $\bar{u}(x,t)$ is a solution to system (2.10). Then by (2.13) in Lemma 2.6, we may infer that

$$\lim_{t\to+\infty} \|\tilde{u}(\cdot,t) - c_0 u^*(\cdot,t)\|_{L^\infty[0,l)} = 0, \tag{4.3}$$

and consequently

$$\lim_{t\to+\infty} \left| \tilde{I}(t) - c_0 \int_0^l u^*(x,t)\,dx \right| = 0.$$

Denote

$$g(t) = c_0 \int_0^l u^*(x,t)\,dx, \quad t \geqslant 0.$$

Therefore, by Lemma 2.3, we can conclude that g is a T-periodic and continuous function. First, we consider system (1.3), which can be equivalently written as

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial (g(x)u(x,t))}{\partial x} + \mu(x,t)u(x,t) + \alpha^* u(x,t) + (\Phi(I(t)) - \alpha^*)u(x,t) = 0, & (x,t) \in Q, \\ g(0)u(0,t) = \int_0^l \beta(x,t)u(x,t)\,dx, & t \in [0,+\infty), \\ u(x,t) = u(x,t+T), & (x,t) \in Q, \\ I(t) = \int_0^l u(x,t)\,dx, & t \in [0,+\infty). \end{cases} \tag{4.4}$$

By Theorem 3.3, system (4.4) admits at most two non-negative solutions one of them being the trivial one. Moreover, the solution $\hat{u}$ to system (4.4) has the form

$$\hat{u}(x,t) = h^*(t)u_1(x,t), \quad \forall\, t \in R_+, \text{ a.e. } x \in [0,l),$$

where $u_1(x,t)$ is a solution to (2.1) corresponding to $\alpha \triangleq \alpha^*$. Moreover, $u_1(x,t) = cu^*(x,t)$, where c is a positive constant. And $h^*(t)$ is a solution to the following system

$$\begin{cases} (h^*)'(t) = -\Phi\left(ch^*(t)\int_0^l u^*(x,t)\,dx\right)h^*(t) + \alpha^* h^*(t), & t \in R_+, \\ h^*(t) = h^*(t+T), & t \in R_+. \end{cases} \tag{4.5}$$

By [7, Lemma 3.1], we know that system (4.2) has one and only one non-negative solution $y \in C^1(R_+)$. Moreover, we have the following result.

Lemma 4.2.

$$\lim_{t\to\infty} |y(t) - h(t)| = 0,$$

where h *is the unique nontrivial and non-negative solution to*

$$\begin{cases} h'(t) = -\Phi(g(t)h(t))h(t) + \alpha^* h(t), & t \in R_+, \\ h(t) = h(t+T), & t \in R_+. \end{cases} \tag{4.6}$$

Proof. See the proof of the Lemma 3.1 in [7, P371]. □

Theorem 4.3. *Let assumptions* (A_1)-(A_5) *hold. If* $c_0 > 0$ *and* $\alpha^* > 0$, *then*

$$\lim_{t\to\infty} \|u(\cdot,t) - \hat{u}(\cdot,t)\|_{L^\infty[0,l)} = 0,$$

where $u(x,t)$ *is the solution to system* (1.2) *and* $\hat{u}(x,t)$ *is the nontrivial nonnegative solution to system* (4.3).

Proof. Note that $u(x,t) = y(t)\tilde{u}(x,t)$ is the solution to system (1.2) and

$$\begin{aligned} |y(t)\tilde{u}(x,t) - c_0 h(t)u^*(x,t)| &= |y(t)\tilde{u}(x,t) - c_0 y(t)u^*(x,t) + c_0 y(t)u^*(x,t) - c_0 h(t)u^*(x,t)| \\ &\leqslant |y(t)|\,|\tilde{u}(x,t) - c_0 u^*(x,t)| + |c_0 u^*(x,t)|\,|y(t) - h(t)|, \end{aligned} \tag{4.7}$$

where u^* is a solution to system (2.1) corresponding to $\alpha \triangleq \alpha^*$. Then, by Lemma 2.6, we have

$$\lim_{t\to\infty} \|\tilde{u}(\cdot,t) - c_0 u^*(\cdot,t)\|_{L^\infty[0,l)} = 0, \tag{4.8}$$

which, together with Lemma 4.2, yields

$$\lim_{t\to\infty} |y(t) - h(t)| = 0. \tag{4.9}$$

Thus, from (4.7)-(4.9), yields

$$\lim_{t\to\infty} \|u(\cdot,t) - c_0 h(t)u^*(\cdot,t)\|_{L^\infty[0,l)} = \lim_{t\to\infty} \|y(t)\tilde{u}(\cdot,t) - c_0 h(t)u^*(\cdot,t)\|_{L^\infty[0,l)} = 0. \tag{4.10}$$

Next, we show that $c_0 h(t)u^*(x,t)$ is the solution to system (1.3). Note that $\hat{u}(x,t) = h^*(t)u_1(x,t)$ and $u_1(x,t) = cu^*(x,t)$. Now we show that

$$h^*(t) = \frac{c_0}{c} h(t),$$

where h^* is a solution to system (4.5) and h is a solution to system (4.6).

If $h^*(t)$ is a solution to system (4.5), then we have

$$\frac{c_0}{c}\cdot h'(t) = -\Phi\left(c\cdot\frac{c_0}{c}\cdot h(t)\int_0^l u^*(x,t)\,dx\right)\frac{c_0}{c}\cdot h(t) + \alpha^*\cdot\frac{c_0}{c}\cdot h(t),$$

which means h is a solution to system (4.6).

If $h(t)$ is a solution to system (4.6), then we have

$$\frac{c}{c_0}\cdot (h^*)'(t) = -\Phi\left(c_0\cdot\frac{c}{c_0}\cdot h^*(t)\int_0^l u^*(x,t)\,dx\right)\frac{c}{c_0}\cdot h^*(t) + \alpha^*\cdot\frac{c}{c_0}\cdot h^*(t),$$

which means h is a solution to system (4.5).

Therefore, $\hat{u}(x,t) = h^*(t)u_1(x,t) = c_0 h(t)u^*(x,t)$. From (4.10), we get the result. The proof is therefore complete. □

5. Numerical tests

In this section, we give the numerical tests, which are used to demonstrate the effectiveness of the theoretical results in our paper.

Example 5.1. Consider the problem (1.2) with

$$\begin{cases} \beta(x,t) = 20x^2(1-x)(1+\sin \pi t), \quad \mu(x,t) = e^{-4x}(1-x)^{-1.4}(2+\cos \pi t), \\ g(x) = e^{-5x}, \quad \Phi(I(t)) = 0.5I(t), \quad u_0(x) = 5(1-x). \end{cases}$$

By the algorithm, we take $t \in [0, 5T]$, the graphics of β and μ are given in Figure 1. The graphic of $u(x,t)$ and the comparison of the shape of population size $I(t)$ are given in Figure 2.

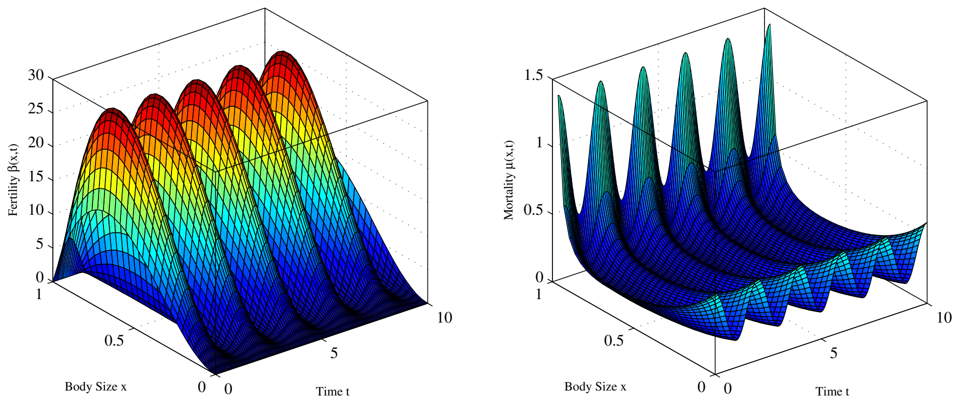

Figure 1: Left: the fertility of individuals of size x at time t, Right: the mortality of individuals of size x at time t.

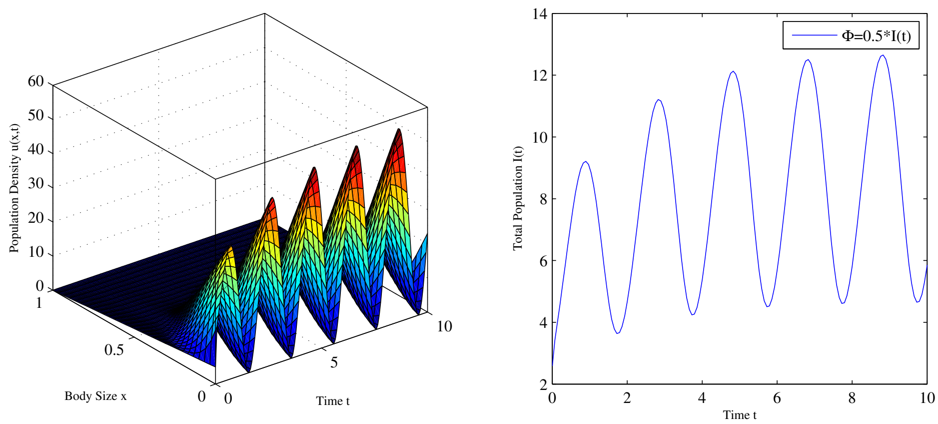

Figure 2: Left: the population density with $\Phi(I(t)) = 0.5I(t)$, Right: the population size with $\Phi(I(t)) = 0.5I(t)$.

It can be seen from Example 5.1 that the fertility and mortality of individuals are 2-time periodic. As can be seen from Fig. 2, when the time increases to a certain moment, the solution of model (1.2) will show a periodic change.

Example 5.2. This paper consider the large time behavior of the system. Next we give the numerical tests for $t \in [0, 10T]$. Consider the problem (1.2) with the parameters are exact to Example 5.1. The left and right in Figure 3 are the population density curves for the system (1.2) corresponding to $\Phi(I(t)) = 0.5I(t)$ and $\Phi(I(t)) = 2I(t)$, respectively. Figure 4 shows the comparison of the shape of population size $I(t)$ for $\Phi(I(t)) = 0.5I(t)$ and $\Phi(I(t)) = 2I(t)$.

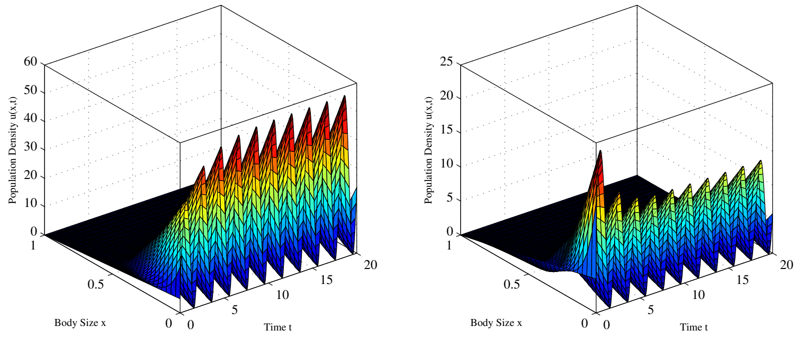

Figure 3: Left: the population density with $\Phi(I(t)) = 0.5I(t)$, Right: the population density with $\Phi(I(t)) = 2I(t)$.

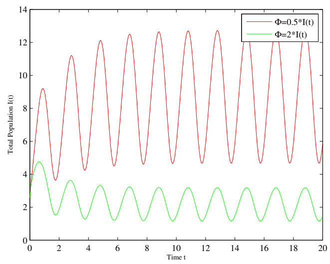

Figure 4: the representation of $I(t)$ for different values of Φ.

Comparing the left figure and right figure of Figure 3, we know that population density u is monotonically decreasing with respect to Φ, which is consistent with the comparison principle in Theorem 3.4. Furthermore, with the increase of time, the population size changed periodically.

Example 5.3. Consider the problem (1.2) with

$$\begin{cases} \beta(x,t) = 20x^2(1-x)(1+\sin\pi t), \\ \mu(x,t) = e^{-4x}(1-x)^{-1.4}(2+\cos\pi t), \\ g(x) = e^{-5x}, \\ \Phi(I(t)) = 1.5I(t). \end{cases}$$

The graphics of population density u corresponding to $u_0(x) = 5(1-x)$, $u_0(x) = 3$, and $u_0(x) = 5e^{-0.5x^2}$ are given in Figures 5 and 6. The graphics of population size I are given in Figure 7.

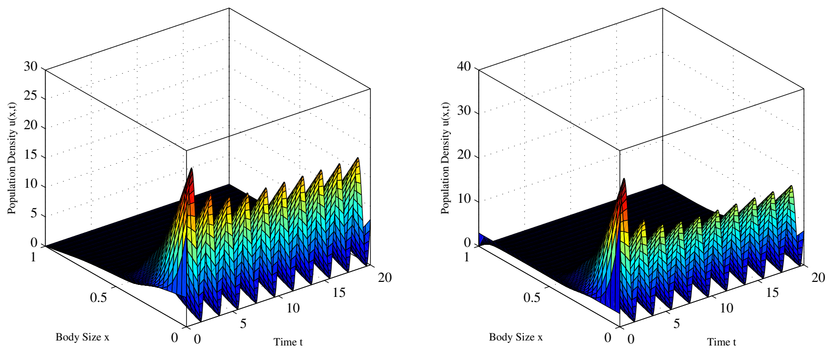

Figure 5: Left: the population density with $u_0(x) = 5(1-x)$, Right: the population density with $u_0(x) = 3$.

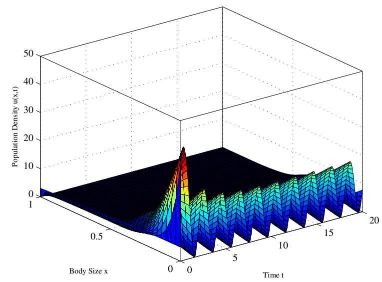

Figure 6: the population density with $u_0(x) = 5e^{-0.5x^2}$.

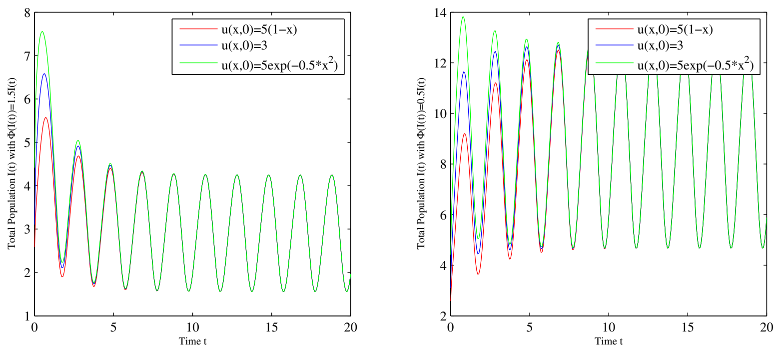

Figure 7: Left: the population size with $\Phi(I(t)) = 1.5I(t)$, Right: the population size with $\Phi(I(t)) = 0.5I(t)$.

From Figures 5, 6 and the left figure of 7, with the evolution of time, the solutions of system (1.2) tend to the same period solution for different initial datum u_0. From the Figure 7, the larger the value of Φ, the shorter time of solution of system (1.2) tends to the solution of system (1.3).

6. Conclusion

This paper investigates the behavior analysis for a nonlinear size-structured population model with logistic term and T-periodic vital rates. In the previous sections, the existence of a unique non-negative solution of (1.2) is proved by means of frozen coefficients and fixed point reasoning. At the same time, we prove that there exists at most two T-periodic non-negative solutions (one of them being the trivial one) of the periodic model (1.3). Moreover, the large-time behavior of the solution of (1.2) is investigated. The study show that for any initial distribution of population the solution of (1.2) tends to the nontrivial non-negative T-periodic solution of the associated model (1.3).

From [11], we know that fractional derivatives provide more accurate models of realism problems than integer-order derivatives. And fractional derivatives are actually found to be a suitable tool to describe certain physical, engineering and biological problems including reaction diffusion models, dynamical mathematical models, and so on. As done in [11], one can introduce time-fractional derivatives into systems (1.2) and (1.3) and get time-fractional partial differential equations. We think that this is a significant problem and leave it to future consideration.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (Nos. 11471197, 11571210, 11501339) and the Natural Science Foundation of Shanxi Province (No. 201601D202002).

References

[1] Z. Abo-Hammour, O. A. Arqub, S. Momani, N. Shawagfeh, *Optimization solution of Troesch's and Bratu's problems of ordinary type using novel continuous genetic algorithm*, Discrete Dyn. Nat. Soc., **2014** (2014), 15 pages. 1

[2] O. Acana, M. M. A. Qurashib, D. Baleanu, *New exact solution of generalized biological population model*, J. Nonlinear Sci. Appl., **10** (2017), 3916–3929. 1

[3] M. Al-Smadi, O. A. Arqub, N. Shawagfeh, S. Momani, *Numerical investigations for systems of second-order periodic boundary value problems using reproducing kernel method*, Appl. Math. Comput., **291** (2016), 137–148. 1

[4] M. Al-Smadi, A. Freihat, O. A. Arqub, N. Shawagfeh, *A novel multistep generalized differential transform method for solving fractional-order Lü chaotic and hyperchaotic systems*, J. Computat. Anal. Appl., **19** (2015), 713–724. 1

[5] S. Aniţa, *Analysis and control of age-dependent population dynamics*, Kluwer Academic Publishers, Dordrecht, (2000). 1

[6] L.-I. Aniţa, S. Aniţa, *Note on some periodic optimal harvesting problems for age-structured population dynamics*, Appl. Math. Comput., **276** (2016), 21–30.

[7] L.-I. Aniţa, S. Aniţa, V. Arnăutu, *Global behavior for age-dependent population model with logistic term and periodic vital rates*, Appl. Math. Comput., **206** (2008), 368–379. 1, 2, 4, 4

[8] L.-I. Aniţa, S. Aniţa, V. Arnăutu, *Optimal harvesting for periodic age-dependent population dynamics with logistic term*, Appl. Math. Comput., **215** (2009), 2701–2715. 1

[9] O. A. Arqub, *An iterative method for solving fourth-order boundary value problems of mixed type integro-differential equations*, J. Comput. Anal. Appl., **8** (2015), 857–874. 1

[10] O. A. Arqub, *Approximate solutions of DASs with nonclassical boundary conditions using novel reproducing kernel algorithm*, Fund. Inform., **146** (2016), 231–254.

[11] O. A. Arqub, *Fitted reproducing kernel Hilbert space method for the solutions of some certain classes of time-fractional partial differential equations subject to initial and Neumann boundary conditions*, Compu. Math. Appl., **73** (2017) 1243–1261. 1, 6

[12] S. Bhattacharya, M. Martcheva, *Oscillation in a size-structured prey-predator model*, Math. Biosci., **228** (2010), 31–44. 1

[13] H. Caswell, *Matrix population models: construction, analysis and interpretation*, Sinauer Associations, Sunderland, (2001). 1

[14] W. L. Chan, B. Z. Guo, *Global behaviour of age-dependent logistic population models*, J. Math. Biol., **28** (1990), 225–235. 1

[15] W. L. Chan, B. Z. Guo, *Optimal birth control of population dynamics II. Problem with free final time, phase constraints, and min-max costs*, J. Math. Anal. Appl., **146** (1990), 523–539.

[16] A. L. Dawidowicz, A. Poskrobko, *Age-dependent single-species population dynamics with delayed argument*, Math. Methods Appl. Sci., **33** (2010), 1122–1135. 1

[17] A. L. Dawidowicz, A. Poskrobko, *Stability of the age-dependent system with delayed dependence of the structure*, Math. Methods Appl. Sci., **37** (2013), 1240–1248. 1

[18] B. Ebenman, L. Persson, *Size-structured populations: ecology and evolution*, Springer-Verlag, Berlin, (1988). 1

[19] J. Z. Farkas, T. Hagen, *Asymptotic behavior of size-structured populations via juvenile-adult interaction*, Discrete Contin. Dyn. Syst. Ser. B, **9** (2008), 249–266. 1, 1

[20] X. L. Fu, D. M. Zhu, *Stability results for a size-structured population model with delayed birth process*, Discrete Contin. Dyn. Syst. Ser. B, **18** (2013), 109–131.

[21] Z.-R. He, Y. Liu, *An optimal birth control problem for a dynamical population model with size-structure*, Nonlinear Anal. Real World Appl., **13** (2012), 1369–1378.

[22] Z.-R. He, R. Liu, *Theory of optimal harvesting for a nonlinear size-structured population in periodic environments*, Int. J. Biomath., **7** (2014), 18 pages. 1, 2

[23] M. Iannelli, *Mathematical theory of age-structured population dynamics*, Giardini Editori, Pisa, (1995). 1, 2

[24] N. Kato, *Optimal harvesting for nonlinear size-structured population dynamics*, J. Math. Anal. Appl., **342** (2008), 1388–1398. 1, 4

[25] M. Kubo, M. Langlais, *Periodic solutions for nonlinear population dynamics models with age-dependence and spatial structure*, J. Differential Equations., **109** (1994), 274–294. 1

[26] Y. Liu, Z.-R. He, *Behavioral analysis of a nonlinear three-staged population model with age-size-structure*, Appl. Math. Comput., **227** (2014), 437–448. 1, 1

[27] R. Liu, G. R. Liu, *Optimal birth control problems for a nonlinear vermin population model with size-structure*, J. Math. Anal. Appl., **449** (2017), 265–291.

[28] P. Magal, S. Ruan, *Structured-population models in biology and epidemiology*, Springer-Verlag, Berlin, (2008). 1

[29] T. McMahon, *Size and shape in biology: elastic criteria impose limits on biological proportions, and consequently on metabolic rates*, Science, **179** (1973), 1201–1204. 1

[30] S. Piazzera, *An age-dependent population equation with delayed birth process*, Math. Meth. Appl. Sci., **27** (2004), 427–439. 1

[31] N. Shawagfeh, O. A. Arqub, S. Momani, *Analytical solution of nonlinear second-order periodic boundary value problem using reproducing kernel method*, J. Comput. Anal. Appl., **16** (2014) 750–762. 1

[32] H. R. Thieme, *Mathematics in population biology*, Princeton University Press, New Jersey, (2003). 1

[33] W. J. Wang, H. Feng, *On the dynamics of positive solutions for the difference equation in a new population model*, J. Nonlinear Sci. Appl., **9** (2016), 1748–1754. 1

[34] E. E. Werner, J. F. Gillian, *The ontogenetic niche and species interactions in size-structured populations*, Ann. Rev. Ecol. Syst., **15** (1984), 393–425. 1