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Nonlinear perturbed difference equations

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Abstract

The paper reports on an iteration algorithm to compute asymptotic solutions at any order for a wide class of nonlinear singularly perturbed difference equations.

Keywords: Perturbed difference equations, computational methods, boundary value problem, asymptotic expansions, iterative method.

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1. Introduction

There is a growing interest in difference equations because of their applications in modern sciences and computer control. Various researchers have been interested in analyzing the asymptotic behavior of perturbed difference equations, they proposed methods that can be subdivided into two aspects. The multiple scales perturbation method for ordinary difference equations introduced in 1977 by Hoppensteadt and Miranker [2], a formal technique which has been improved by Horssen et al. [8, 12]. The second method is the singular perturbation method proposed in 1976 by Comstock and Hsiao [1] for a linear second order equation defined over a finite interval, these authors presented the zeroth order solution as a composition of an *outer solution* and *a boundary layer correction solution*; their method followed the theory of singular perturbations for ODES. Naidu and Rao [7] have heuristically extended this procedure to higher order equations and have given a lot of applications to discrete control problems while Jodar and Morera [3] studied linear systems of second order difference equations. Some particular cases of nonlinear second order equations have been examined by Reinhardt [9] and Suzuki [11]. Kelley [4] studied a general second order model with a *right-end perturbation*; his method is given for zeroth and first order of approximation but higher order approximations have not been made explicitly. In [10, 13], we have elucidated that we could define homogeneous asymptotic expansions for singularly perturbed difference

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equations bypassing the correction terms and we gave applications to control problems in [14–17]. Recently in [18, 19], we used this homogeneous method for a class of nonlinear equations giving explicitly asymptotic solutions up to any order. The plan of this paper is to extend this procedure based on Faa Di Bruno formula [6] and contraction mapping principle, to a wide class of nonlinear singularly-perturbed difference equations. In the next Section, we introduce a nonlinear model with *left-end perturbation*, we describe an iterative algorithm to compute asymptotic solutions of a boundary value problem. Section 3, is reserved to another kind of perturbation said *right-end perturbation*, an analogous method is briefly exposed. Finally the paper ends in Section 4 with a concise conclusion.

2. Left end perturbation

Let $I_N = \{0, 1, \cdots, N\}$, $N \in \mathbf{N}^*$; $\mathbf{N}^*$ denote the set of positive integers. We assume that $(X, \|.\|)$ is a Banach space, $F(I_N, X)$ is the space of all mappings of I_N into X, and the mapping $f : F(I_N, X)^{r+1} \times (-1, 1) \times I_N \to X$, is *n*-differentiable in its arguments. We consider the difference equation

$$f\left(x(t), x(t+1), \ldots, x(t+r-1), \varepsilon x(t+r), \varepsilon, t\right) = 0, \quad t \in I_{N-r}, \tag{2.1}$$

subject to the multipoint boundary-value conditions

$$x(0) = \alpha_0(\varepsilon),\ x(1) = \alpha_1(\varepsilon), \ldots, x(r-2) = \alpha_{r-2}(\varepsilon),\ x(N) = \beta(\varepsilon). \tag{2.2}$$

We assume that for $|\varepsilon| < \delta \leqslant 1$, $\alpha_k(\varepsilon)$ and $\beta(\varepsilon)$ have the asymptotic representations

$$\alpha_k(\varepsilon) = \alpha_k^{(0)} + \varepsilon\alpha_k^{(1)} + \cdots + \varepsilon^n\alpha_k^{(n)}, \quad \beta(\varepsilon) = \beta^{(0)} + \varepsilon\beta^{(1)} + \cdots + \varepsilon^n\beta^{(n)}.$$

In the singularly perturbed model (2.1) the small parameter ε is assigned to the term of highest order, i.e., $x(t+r)$, in the literature it is called a left-end perturbation. In this section, for the BVP (2.1)-(2.2), we study the existence and uniqueness of a solution $x(t, \varepsilon), t \in I_N$, and we show how to compute recursively the coefficients of its asymptotic development

$$x(t, \varepsilon) = x(t)^{(0)} + \varepsilon x(t)^{(1)} + \varepsilon^2 x(t)^{(2)} + \cdots + \varepsilon^n x(t)^{(n)} + \mathcal{O}(\varepsilon^{n+1}). \tag{2.3}$$

In what follows, we will denote the partial derivative $\frac{\partial^{k_0+k_1+\cdots+k_p} f(x_0,x_1,\ldots,x_p)}{\partial x_0^{k_0}\partial x_1^{k_1}\cdots\partial x_p^{k_p}}$ by $D_0^{k_0}D_1^{k_1}\cdots D_p^{k_p} f$.

2.1. Reduced Problem

As for the singular perturbation theory of differential equations, it is assumptive that the singular nature of difference equations containing a small parameter is simply caused by a reduction of its order when the parameter is canceled. In the *reduced problem*

$$\begin{gathered} f\left(x(t), x(t+1), \ldots, x(t+r-1), 0, 0, t\right), \quad t = 0, 1, \ldots, N-r, \\ x(0) = \alpha_0^{(0)}, \quad x(1) = \alpha_1^{(0)}, \cdots, \quad x(r-2) = \alpha_{r-2}^{(0)}, \end{gathered} \tag{2.4}$$

$$x(N) = \beta^{(0)}, \tag{2.5}$$

the order of the difference equation is equal to $r-1$, the problem displays a *boundary layer behavior* at the endpoint, i.e., the values $x(0)$, $x(1), \ldots,$ $x(N-1)$, can be calculated regardless of the final condition (2.5). Seeing that the boundary conditions are uncoupled, we will only have to solve recursively the initial value problem (2.4). The following hypothesis guarantee that the reduced problem (2.4)-(2.5) has a unique solution.

H1. Suppose the range of f contains the value 0, and $\forall x \in X$,

$$D_{r-1} f\left(x(t), x(t+1), \ldots, x(t+r-1), 0, 0, t\right) \neq 0, \quad t = 0, 1, \ldots, N-r.$$

Proposition 2.1. *If* H1 *holds, then problem* (2.4)-(2.5) *has a unique solution.*

2.2. *Preliminary results*

To establish the existence of a solution as well to find approximate solutions, the BVP (2.1)-(2.2) is converted to a system of equations depending on a parameter by denoting $\chi = (x(0), x(1), \ldots, x(N))$, for $x \in X$, and writing the system (2.1)-(2.2) in the form

$$\mathcal{F} : (-1,1) \times X^{N+1} \longrightarrow X^{N+1}, \quad \mathcal{F}(\varepsilon, \chi) = 0, \tag{2.6}$$

where $\mathcal{F}(\varepsilon,\chi) = (\mathcal{F}_0(\varepsilon,\chi), \mathcal{F}_1(\varepsilon,\chi), \ldots, \mathcal{F}_N(\varepsilon,\chi))$,

$$\begin{aligned} \mathcal{F}_t(\varepsilon,\chi) &= x(t) - \alpha_t(\varepsilon), \quad t = 0, \ldots, r-2, \\ \mathcal{F}_{t+r-1}(\varepsilon,\chi) &= f(x(t), x(t+1), \ldots, x(t+r-1), \varepsilon x(t+r), \varepsilon, t), \\ t &= 0, \ldots, N-r, \\ \mathcal{F}_N(\varepsilon,\chi) &= x(N) - \beta(\varepsilon). \end{aligned}$$

Under suitable assumptions, we can apply the classical Implicit Function Theorem [5] to (2.6). If the hypothesis H1 is satisfied, then for a small parameter ε, we can determine a function

$$\phi(\varepsilon) = (\phi_0(\varepsilon), \phi_1(\varepsilon), \ldots, \phi_N(\varepsilon)),$$

with same regularity as $\mathcal{F}$, i.e., of class C^n, such that $\mathcal{F}(\varepsilon, \phi(\varepsilon)) = 0$. Therefore,

$$\begin{aligned} & f(\phi_t(\varepsilon), \phi_{t+1}(\varepsilon), \ldots, \phi_{t+r-1}(\varepsilon), \varepsilon\phi_{t+r}(\varepsilon), \varepsilon, t) = 0, \quad t = 0, 1, \ldots, N-r, \\ & \phi_0(\varepsilon) = \alpha_0(\varepsilon),\ \phi_1(\varepsilon) = \alpha_1(\varepsilon), \ldots,\ \phi_{r-2}(\varepsilon) = \alpha_{r-2}(\varepsilon),\ \phi_N(\varepsilon) = \beta(\varepsilon). \end{aligned} \tag{2.7}$$

For small $|\varepsilon|$, to get an approximate value for the function $\phi_t(\varepsilon)$, we use the Taylor/Maclaurin polynomial expansion

$$\phi_t(\varepsilon) = \phi_t(0) + \frac{\varepsilon}{1!}\frac{d\phi_t}{d\varepsilon}(0) + \frac{\varepsilon^2}{2!}\frac{d^2\phi_t}{d\varepsilon^2}(0) + \cdots + \frac{\varepsilon^n}{n!}\frac{d^n\phi_t}{d\varepsilon^n}(0) + \mathcal{O}(\varepsilon^{n+1}), \tag{2.8}$$

and we apply Faa di Bruno formula [6] to explicitly find the sequential differentiation of (2.7) to be able to calculate the coefficients of (2.8). For making writing concise, we drop the arguments for f in the following lemma.

Lemma 2.2. *Assume that* f *satisfies* (2.7), *and that all the necessary derivatives are defined. Then we have for* $n \geqslant 2$,

$$\begin{aligned} \sum_{l=0}^{r-1} D_l f \frac{\partial^n \phi(\varepsilon)}{\partial \varepsilon^n} + n D_r f \frac{\partial^{n-1}\phi_{t+r}(\varepsilon)}{\partial \varepsilon^{n-1}} &= -\underline{\sum_0 \sum_1 \cdots \sum_n} D_0^{p_0} D_1^{p_1} \cdots D_{r+1}^{p_{r+1}} f \\ \times \frac{n! \prod_{i=1}^n \left(\frac{\partial^i \phi_t(\varepsilon)}{\partial \varepsilon^i}\right)^{q_{i0}} \cdots \left(\frac{\partial^i \phi_{t+r-1}(\varepsilon)}{\partial \varepsilon^i}\right)^{q_{ir-1}} \left(i\frac{d^{i-1}\phi_{t+r}(\varepsilon)}{\partial \varepsilon^{i-1}} + \varepsilon \frac{\partial^i \phi_{t+r}(\varepsilon)}{\partial \varepsilon^i}\right)^{q_{ir}} \left(\varepsilon^{(i)}\right)^{q_{ir+1}}}{\prod_{i=1}^n (i!)^{k_i} \prod_{i=1}^n \prod_{j=0}^{r+1} q_{ij}!}&, \end{aligned} \tag{2.9}$$

where the coefficients k_i, q_{ij}, *and* p_j, $i = 0, \ldots, 1$, $i = 0, \ldots, 1$, *are all nonnegative integer solutions of the Diophantine equations*

$$\begin{aligned} & \sum_0 \rightarrow k_1 + 2k_2 + \cdots + nk_n = n, \\ & \sum_i \rightarrow q_{i0} + q_{i1} + \cdots + q_{ir} + q_{ir+1} = k_i, \quad i = 1, \ldots, n, \\ & p_j = q_{1j} + q_{2j} + \cdots + q_{nj}, \quad j = 0, 1, \ldots, r+1, \\ & p_0 + p_1 + \cdots + p_{r+1} = k_1 + k_2 + \cdots + k_n, \end{aligned} \tag{2.10}$$

in $\underline{\sum_0 \cdots \sum_n}$ *we fix* $k_n = 0$; *the case* $k_n = 1$ *is omitted and corresponds to the left side of equation* (2.9), *and*

$$\left(\varepsilon^{(i)}\right)^{q_{ir+1}} := \begin{cases} 1, & i = 1 \quad \vee \quad q_{ir+1} = 0, \\ 0, & i \geqslant 2 \quad \wedge \quad q_{ir+1} \neq 0. \end{cases}$$

Proof. Define $\phi_{r+1}(\varepsilon) := \varepsilon$, we have

$$\frac{d^i\phi_{r+1}(\varepsilon)}{d\varepsilon^i} = \varepsilon^{(i)} = \begin{cases} 1, & i = 1, \\ 0, & i \geqslant 2. \end{cases}$$

Expanding Faa Di Bruno's Formula into (2.8) and noticing that for $i \geqslant 1$,

$$\frac{d^i\left(\varepsilon\phi_{k+r}(\varepsilon)\right)}{d\varepsilon^i} = i\frac{d^{i-1}\phi_{k+r}(\varepsilon)}{d\varepsilon^{i-1}} + \varepsilon\frac{d^i\phi_{k+r}(\varepsilon)}{d\varepsilon^i},$$

we obtain (2.9) by arranging the equation (2.10) so that on the left hand side, we have the terms corresponding to $k_n^{(0)} = 1$. □

2.3. Description of the method

In this section we describe an iterative process giving the coefficients of (2.3). Knowing that the coefficients of 0^{th}-order approximation are the solution sequence of the reduced problem (2.4)-(2.5), substituting

$$x^{(i)}(t+l) := \frac{1}{i!}\frac{\partial^i\phi_{t+l}}{\partial\varepsilon^i}(0), \quad t = 0, 1, \dots, N, \quad l = 0, \dots, r \tag{2.11}$$

into (2.9), we obtain the following. For 1^{st}-order approximation,

$$\begin{gathered} D_{r-1}f_0x^{(1)}(t+r-1) = -\sum_{l=0}^{r-2} D_lf_0x^{(1)}(t+l) - D_rf_0x^{(0)}(t+r) - D_{r+1}f_0, \quad t = 0, \dots, N-r, \\ x^{(1)}(0) = \alpha_0^{(1)},\ x^{(1)}(1) = \alpha_1^{(1)}, \dots,\ x^{(1)}(r-2) = \alpha_{r-2}^{(1)},\ x^{(1)}(N) = \beta^{(1)}, \end{gathered} \tag{2.12}$$

where f_0 shortly denotes tha value

$$f(x^{(0)}(t), x^{(0)}(t+1), \dots, x^{(0)}(t+r-1), 0, 0, t).$$

To calculate the coefficients $x^{(1)}(0), x^{(1)}(1), \dots, x^{(1)}(N-1)$, only the initial values are needed, the final value $x^{(1)}(N)$ does not serve in this recurrence, but the 0^{th}-order solution found from (2.4)-(2.5) are also needed. For the 2^{th}-order development, and with the same calculation method as for the previous step, we have

$$\begin{aligned} &D_{r-1}f_0x^{(2)}(t+r-1) \\ &\quad = -\sum_{l=0}^{r-2} D_lf_0x^{(2)}(t+l) - 2D_rf_0x^{(1)}(t+r) - D_{r+1}^2f_0 \\ &\quad\quad -2\sum_{0\leqslant l\leqslant r-1} D_lD_rf_0x^{(1)}(t+l)x^{(0)}(t+r) - D_r^2f_0\left(x^{(0)}(t+r)\right)^2 \\ &\quad\quad -\sum_{l=0}^{r-1} D_l^2f_0\left(x^{(1)}(t+l)\right)^2 - 2\sum_{0\leqslant l\leqslant r-1} D_lD_{r+1}f_0x^{(1)}(t+l) \\ &\quad\quad -2D_rD_{r+1}f_0x^{(0)}(t+r) - 2\sum_{0\leqslant l<m\leqslant r-1} D_lD_mf_0x^{(1)}(t+l)x^{(1)}(t+m), \quad t = 0, \dots, N-r, \\ &x^{(2)}(0) = \alpha_0^{(2)},\ x^{(2)}(1) = \alpha_1^{(2)}, \dots,\ x^{(2)}(r-2) = \alpha_{r-2}^{(2)},\ x^{(2)}(N) = \beta^{(2)}. \end{aligned} \tag{2.13}$$

For the n^{th}-order development, using the initial values

$$x^{(n)}(0) = \alpha_0^{(n)},\ x^{(n)}(1) = \alpha_1^{(n)}, \dots,\ x^{(n)}(r-2) = \alpha_{r-2}^{(n)}, \tag{2.14}$$

we have the recurrence formula

$$\begin{aligned} D_{r-1}f_0x^{(n)}(t+r-1) &= -\sum_{l=0}^{r-2} D_l f_0 x^{(n)}(t+l) - nD_r f_0 x^{(n-1)}(t+r) \\ &- \underline{\sum_0 \cdots \sum_n} \frac{D_0^{p_0}\cdots D_{r+1}^{p_{r+1}} f_0 \prod_{i=1}^n (x^{(i)}(t))^{q_{i0}} \cdots (x^{(i)}(t+r-1))^{q_{ir-1}} (x^{(i)}(t+r))^{q_{ir}} (\delta_i)^{q_{ir+1}}}{\prod_{i=1}^n \prod_{j=0}^{r+1} q_{ij}!}, \end{aligned} \tag{2.15}$$

where in $\underline{\sum_0 \cdots \sum_n}$ we fix $k_n^{(0)} = 0$, and the final value remains fixed,

$$x^{(n)}(N) = \beta^{(n)}. \tag{2.16}$$

Above are the consecutive steps to compute the coeffients that make up the approximate solutions defined by (2.3). This computation process is confirmed in the following theorem.

Theorem 2.3. *If* H1 *holds, there exists* $\epsilon > 0$, *such that for all* $|\varepsilon| < \epsilon$, *the boundary value problem* (2.1)-(2.2) *has a unique solution which satisfies* (2.3)*; where* $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, *and* $x_k^{(n)}$, *are the solutions of* (2.4)-(2.5), (2.12), (2.13), *and* (2.15)-(2.14)-(2.16), *respectively.*

Proof. Let $\tilde{\chi} = (\varepsilon, \chi)$, $|\varepsilon| \leqslant \delta < 1$, we introduce the functional F defined by $F(\tilde{\chi}) = (\varepsilon, \mathcal{F}(\tilde{\chi}))$, and we denote by $\mathcal{D}F$ its jacobian matrix. Obviously $\mathcal{D}F$ is invertible at $\tilde{\chi}^{(0)} = \left(0, x(0)^{(0)}, x(1)^{(0)}, \dots, x(N)^{(0)}\right)$, where $x(t)^{(0)}, t \in I_N$, is the solution of the reduced problem (2.4)-(2.5), since from H1 we have

$$\det \mathcal{D}F\left(\tilde{\chi}^{(0)}\right) = \prod_{i=0}^{N-r} \mathcal{D}_{r-1} f\left(x_i^{(0)}, x_{i+1}^{(0)}, \dots, x_{i+r-1}^{(0)}, 0, 0, i\right) \neq 0.$$

We can choose $\xi > 0$ such that, if $\|\tilde{\chi} - \tilde{\chi}^{(0)}\| < \xi$, we have

$$\|\mathcal{D}F(\tilde{\chi}) - \mathcal{D}F\left(\tilde{\chi}^{(0)}\right)\| < \frac{1}{2}\|\left(\mathcal{D}F\left(\tilde{\chi}^{(0)}\right)\right)^{-1}\|^{-1},$$

resulting from the continuity of $\mathcal{D}F$. We denote $\epsilon = \frac{\xi}{2}\|\left(\mathcal{D}F\left(\tilde{\chi}^{(0)}\right)\right)^{-1}\|^{-1}$, we can easily verify that the mapping $\Phi_\tau(\tilde{\chi}) = \tilde{\chi} - \left(\mathcal{D}F\left(\tilde{\chi}^{(0)}\right)\right)^{-1}(F(\tilde{\chi}) - \tau)$ is a contraction that maps $B\left(\tilde{\chi}^{(0)}, \xi\right)$ to itself, when $|\varepsilon| < \epsilon$ and $\|\tau\| < \epsilon$. Therefore Φ_τ has a unique fixed point $\tilde{\chi}$, which means that for τ fixed, $\|\tau\| < \epsilon$, there exists a unique $\tilde{\chi}$ such that $\|\tilde{\chi} - \tilde{\chi}^{(0)}\| < \xi$, and $\tau = F(\tilde{\chi})$, i.e., F is one-to-one from $F^{-1}(B(0,\epsilon))$ into $B(0,\epsilon)$.

If $|\varepsilon| < \epsilon$, obviously $(\varepsilon, 0, \dots, 0) \in B(0,\epsilon)$, there exists a unique $(\varepsilon, \phi(\varepsilon))$ in $B\left(\tilde{\chi}^{(0)}, \xi\right)$, such that $(\varepsilon, 0, \dots, 0) = F(\varepsilon, \phi(\varepsilon))$, $\phi(\varepsilon) = (\phi_0(\varepsilon), \dots, \phi_N(\varepsilon))$. We proved that $|\varepsilon| < \epsilon$, there exists a unique $\phi(\varepsilon)$ such that $\mathcal{F}(\varepsilon, \phi(\varepsilon)) = 0$, then the boundary value problem (2.1)-(2.2) has a unique solution. Moreover, the function ϕ is $C^n(-\epsilon, \epsilon)$, as are F and F^{-1}, and its derivatives are given in Lemma 2.2. □

The iterative problems given above are defined for any order provided f being a smooth function and the asymptotic developments for the boundary conditions are convergent.

H2. Assume that $\|\alpha_k^{(i)}\| \leqslant \frac{A}{\delta^i}$, $\|\beta^{(i)}\| \leqslant \frac{B}{\delta^i}$, A and B are constants.

Theorem 2.4. *If assumptions* H1 *and* H2 *hold, and* f *is a smooth function, then there exists* $\epsilon > 0$, *for all* $|\varepsilon| < \epsilon$, *that the boundary value problem* (2.1)-(2.2) *has a unique solution* $x_k(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n x_k^{(n)}$, *where* $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, *and* $x_k^{(n)}$, *are solutions of* (2.4)-(2.5), (2.12), (2.13), *and* (2.15)-(2.14)-(2.16), *respectively.*

3. Right end perturbation

In this section, we report on a model said right-end perturbation, the small parameter ε is assigned to the term of lowest order, i.e., $x(t)$,

$$f(\varepsilon x(t), x(t+1), \dots, x(t+r-1), x(t+r), \varepsilon, t) = 0, \quad t = 0, \dots, N-r, \tag{3.1}$$

combined with the boundary conditions

$$x(0)=\alpha(\varepsilon),\ x(N-r+2)=\beta_{r-2}(\varepsilon),\dots,\ x(N)=\beta_0(\varepsilon), \tag{3.2}$$

where $|\varepsilon|<\delta\leqslant 1$, and $\alpha(\varepsilon)$ and $\beta_k(\varepsilon)$ have the asymptotic representations

$$\alpha(\varepsilon)=\alpha^{(0)}+\varepsilon\alpha^{(1)}+\cdots+\varepsilon^n\alpha^{(n)},\quad \beta_k(\varepsilon)=\beta_k^{(0)}+\varepsilon\beta_k^{(1)}+\cdots+\varepsilon^n\beta_k^{(n)}.$$

We indicate directly the results, which are found by a similar technique as in previous section, the modifications required are obvious and are left to the reader. In the reduced problem

$$x_0=\alpha^{(0)}, \tag{3.3}$$

$$\begin{gathered} f\left(0,x(t+1),\dots,x(t+r-1),x(t+r),0,t\right)=0,\ t=0,1,\dots,N-r,\\ x(N-r+2)=\beta_{r-2}^{(0)},\cdots,\ x(N-1)=\beta_1^{(0)},\ x(N)=\beta_0^{(0)}, \end{gathered} \tag{3.4}$$

involving only the final values, we compute backward the values $x_1,\dots,x_{N-r+1}$, without using (3.3); the boundary layer behavior is located at the initial value.

H3. Suppose that f has range containing zero, and $\forall x\in X$, we have

$$D_1f(0,x(t+1),x(t+2),\dots,x(t+r),0,t)\neq 0.$$

Proposition 3.1. *If* H3 *holds, then* (3.3)-(3.4) *has a unique solution.*

Under some conditions, there exists an open neighborhood $V(0)$ where a function

$$\phi(\varepsilon)=(\phi_0(\varepsilon),\dots,\phi_N(\varepsilon))\in C^n(V)$$

satisfies

$$\begin{gathered} f(\varepsilon\phi(t,\varepsilon),\phi(t+1,\varepsilon),\dots,\phi(t+r,\varepsilon),\varepsilon,t)=0,\quad t=0,\dots,N-r,\\ \phi_0(\varepsilon)=\alpha(\varepsilon),\ \phi_{N-r+2}(\varepsilon)=\beta_{r-2}(\varepsilon),\dots,\ \phi_N(\varepsilon)=\beta_0(\varepsilon). \end{gathered} \tag{3.5}$$

The consecutive derivatives of the equation in (3.5) are given in the following Lemma.

Lemma 3.2. *Assume that the functions ϕ_k and f satisfy* (3.5)*, and that all the necessary derivatives are defined. Then we have for $n\geqslant 2$,*

$$\begin{aligned} nD_0f\frac{\partial^{n-1}\phi_t(\varepsilon)}{\partial\varepsilon^{n-1}}+\sum_{l=1}^{r}D_lf\frac{\partial^n\phi_{t+l}(\varepsilon)}{\partial\varepsilon^n}&=-\underline{\sum_0\sum_1\cdots\sum_n}D_0^{p_0}D_1^{p_1}\cdots D_{r+1}^{p_{r+1}}f\\ \times\frac{n!\prod_{i=1}^n\left(i\frac{d^{i-1}\phi_t(\varepsilon)}{\partial\varepsilon^{i-1}}+\varepsilon\frac{\partial^i\phi_t(\varepsilon)}{\partial\varepsilon^i}\right)^{q_{i0}}\cdots\left(\frac{\partial^i\phi_{t+r-1}(\varepsilon)}{\partial\varepsilon^i}\right)^{q_{ir-1}}\left(\frac{\partial^i\phi_{t+r}(\varepsilon)}{\partial\varepsilon^i}\right)^{q_{ir}}\left(\varepsilon^{(i)}\right)^{q_{ir+1}}}{\prod_{i=1}^n(i!)^{k_i}\prod_{i=1}^n\prod_{j=0}^{r+1}q_{ij}!}&, \end{aligned} \tag{3.6}$$

where the coefficients in (3.6) *are the solutions of the Diophantine* (2.10)*; in $\underline{\sum_0\cdots\sum_n}$ we fix $k_n^{(0)}=0$, the case $k_n^{(0)}=1$ corresponds to the left side of equation* (3.6).

Therefore, from (2.3), (2.11), and (3.6), we find for the following. For 1^{st}-order development,

$$\begin{aligned} &x^{(1)}(a)=\alpha^{(1)},\\ &D_1f_0x^{(1)}(t+1)=-\sum_{l=2}^{r}D_lf_0x^{(1)}(t+l)-D_0f_0x^{(0)}(t)-D_{r+1}f_0,\quad t=0,\dots,N-r,\\ &x^{(1)}(N-r+2)=\beta_{r-2}^{(1)},\cdots,\ x^{(1)}(N-1)=\beta_1^{(1)},\ x^{(1)}(N)=\beta_0^{(1)}. \end{aligned} \tag{3.7}$$

For 2^{nd}-order development, we have

$$
\begin{aligned}
x^{(2)}(0) &= \alpha^{(2)},\\
D_1 f_0 x^{(2)}(k+1) &= -\sum_{l=2}^{r} D_l f_0 x^{(2)}(t+l) - 2D_0 f_0 x^{(1)}(t)\\
&\quad - D_{r+1}^2 f_0 - 2\sum_{1\leqslant l\leqslant r} D_0 D_l f_0 x^{(0)}(t) x^{(1)}(t+l) - D_0^2 f_0 (x^{(0)}(t))^2\\
&\quad - \sum_{l=1}^{r} D_l^2 f_0 (x^{(1)}(t+l))^2 - 2\sum_{1\leqslant l\leqslant r} D_l D_{r+1} f_0 x^{(1)}(t+l)\\
&\quad - 2D_0 D_{r+1} f_0 x^{(0)}(t)\\
&\quad - 2\sum_{1\leqslant l<m\leqslant r} D_l D_m f_0 x^{(1)}(t+l) x^{(1)}(t+m), \quad k=0,\ldots,N-r,\\
x^{(2)}(N-r+2) &= \beta_{r-2}^{(2)},\ldots, x^{(2)}(N-1)=\beta_1^{(2)}, x^{(2)}(N)=\beta_0^{(2)}.
\end{aligned}
\tag{3.8}
$$

For n^{th}-order order development, $n \geqslant 2$, we have

$$
\begin{aligned}
D_1 f_0 x^{(n)}(t+1) &= -\sum_{l=2}^{r} D_l f_0 x^{(n)}(t+l) - nD_0 f_0 x^{(n-1)}(t)\\
&\quad -\underline{\sum_0 \cdots \sum_n} \frac{D_0^{p_0}\cdots D_{r+1}^{p_{r+1}} f_0 \prod_{i=1}^{n} (x^{(i-1)}(t))^{q_{i0}} \cdots (x^{(i)}(t+r-1))^{q_{ir-1}} (x^{(i)}(t+r))^{q_{ir}} (\delta_i)^{q_{ir+1}}}{\prod_{i=1}^{n}\prod_{j=0}^{r+1} q_{ij}!},
\end{aligned}
\tag{3.9}
$$

in $\underline{\sum_0 \cdots \sum_n}$ we fix $k_n^{(0)} = 0$. In the above recurrence formulas we only use the final values, the initial values do not serve in the computation process. The iteration is done backward from the final values

$$
x^{(n)}(N-r+2) = \beta_{r-2}^{(n)},\ldots, x^{(n)}(N-1)=\beta_1^{(n)}, x^{(n)}(N)=\beta_0^{(n)}, \tag{3.10}
$$

while the initial value remains fixed,

$$
x^{(n)}(0) = \alpha^{(n)}. \tag{3.11}
$$

Theorem 3.3. *If* H3 *holds, there exists* $\epsilon > 0$*, such that for all* $|\varepsilon| < \epsilon$*, the boundary value problem* (3.1)-(3.2) *has a unique solution which satisfies* (2.3)*; the coefficients* $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, *and* $x_k^{(n)}$, *are the solutions of* (3.3)-(3.4), (3.7), *and* (3.8), (3.9)-(3.10)-(3.11), *respectively.*

For a development at any order, we need that the asymptotic developments for the boundary conditions are convergent.

H4. Assume that $\|\alpha^{(i)}\| \leqslant \frac{A}{\delta^i}$, $\|\beta_k^{(i)}\| \leqslant \frac{B}{\delta^i}$, A and B are constants.

Theorem 3.4. *If* H3 *and* H4 *hold, and* f *is a smooth function, then there exists* $\epsilon > 0$ *such that for all* $|\varepsilon| < \epsilon$*, the boundary value problem* (3.1)-(3.2) *has a unique solution which satisfy* $x_k(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n x_k^{(n)}$*, where the coefficients* $x_k^{(0)}$, $x_k^{(1)}$, $x_k^{(2)}$, *and* $x_k^{(n)}$ *are the solution of the problems* (3.3)-(3.4), (3.7), (3.8), *and* (3.9)-(3.10)-(3.11), *respectively.*

4. conclusion

In this paper, we consider some boundary value problems related to a large class of singular perturbation models of nonlinear difference equations. We study the existence and uniqueness of the solution and formulate an iterative process to find approximate asymptotic solutions up to any order by combining Faa

Di Bruno formula and the contraction mapping principle. The same procedure can be applied to initial value problems with a left-end perturbation, or final value problems with right-end perturbation. The singular perturbation theory developed herein shows a possibility to study models with multiple scale parameters or singularly perturbed discrete-time systems, which may be a future research topic.

References

[1] C. Comstock, G. C. Hsiao, *Singular Perturbations for Difference Equations*, Rocky Mountain J. Math., **6** (1976), 561–567. 1
[2] F. C. Hoppensteadt, W. L. Miranker, *Multitime methods for systems of difference equations*, Studies in Appl. Math., **56** (1977), 273–289. 1
[3] L. Jodar, J. L. Morera, *Singular Perturbations for Systems of Difference Equations*, Appl. Math. Lett., **3** (1990), 51–54. 1
[4] W. G. Kelley, *Singularly perturbed difference equations*, J. Differ. Equations Appl., **5** (1999), 97–110. 1
[5] S. G. Krantz, H. R. Parks, *The Implicit Function Theorem: History, Theory, and Applications*, Birkhäuser, Boston, (2003). 2.2
[6] R. L. Mishkov, *Generalization of the Formula of Faa Di Bruno for a Composite Function with a Vector Argument*, Int. J. Math. Math. Sci., **24** (2000), 481–491. 1, 2.2
[7] D. S. Naidu, A. K. Rao, *Singular Perturbation Analysis of Discrete Control Systems*, Springer-Verlag, Berlin, (1985). 1
[8] M. Rafei, W. T. Van Horssen, *Solving systems of nonlinear difference equations by the multiple scales perturbation method*, Nonlinear Dynam., **69** (2012), 1509–1516. 1
[9] H. J. Reinhardt, *On asymptotic expansions in nonlinear, singularly perturbed difference equations*, Numer. Funct. Anal. Optim., **1** (1979), 565–587. 1
[10] T. Sari, T. Zerizer, *Perturbations for Linear Difference Equations*, J. Math. Anal. Appl., **305** (2005), 43–52. 1
[11] M. Suzuki, *Singular perturbation for difference equations*, Nonlinear Anal. Theory Methods Appl., **47** (2001), 4083–4093. 1
[12] W. T. Van Horssen, M. C. ter Brake, *On the multiple scales perturbation method for difference equations*, Nonlinear Dynam., **55** (2009), 401–418. 1
[13] T. Zerizer, *Perturbation Method for Linear Difference Equations with Small Parameters*, Adv. Differ. Equ., **11** (2006), 1–12. 1
[14] T. Zerizer, *Perturbation Method for a Class of Singularly Perturbed Systems*, Adv. Dyn. Syst. Appl., **9** (2014), 239–248. 1
[15] T. Zerizer, *Boundary Value Problems for Linear Singularly Perturbed Discrete Systems*, Adv. Dyn. Syst. Appl., **10** (2015), 215–224.
[16] T. Zerizer, *Problems for a Linear Two-Time-Scale Discrete Model*, Adv. Dyn. Syst. Appl., **10** (2015), 85–96.
[17] T. Zerizer, *Boundary Value Problem for a Three-Time-Scale Singularly Perturbed Discrete System*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **23** (2016), 263–272. 1
[18] T. Zerizer, *A Class of Multi-Scales Nonlinear Difference Equations*, Appl. Math. Sci., **12** (2018), 911–919. 1
[19] T. Zerizer, *A Class of Nonlinear Perturbed Difference Equations*, Int. J. Math. Anal., **12** (2018), 235–243. 1