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Global existence and blow-up behavior for a degenerate and singular parabolic equation with nonlocal boundary condition

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Abstract

The aim of this article is to investigate the global existence and blow-up behavior of the nonnegative solution to a degenerate and singular parabolic equation with nonlocal boundary condition. The conditions on the existence and non-existence of the global solution are given. Furthermore, under some appropriate hypotheses, the precise blow-up rate estimate and the uniform blow-up profile of the blow-up solutions are discussed.

Keywords: Degenerate and singular parabolic equation, global existence, blow-up, blow-up rate, uniform blow-up profile.

2010 MSC: 35K50, 35K55, 35K65.

1. Introduction

We are interested in the blow-up behavior of the following degenerate and singular parabolic equation subject to weighted nonlocal boundary condition

$$\begin{cases} u_t = (x^{\alpha}u_x)_x + \kappa u^p \int_0^l u^q dx, & (x,t) \in (0,l) \times (0,+\infty), \\ u(0,t) = \int_0^l f(x)\, u(x,t) dx, & t \in (0,+\infty), \\ u(l,t) = \int_0^l g(x)\, u(x,t) dx, & t \in (0,+\infty), \\ u(x,0) = u_0(x) \geqslant 0, & x \in [0,l], \end{cases} \tag{1.1}$$

where $0 \leqslant \alpha < 1$, $p \geqslant 0$, q and κ are positive parameters, $f(x)$ and $g(x)$ are nonnegative continuous on $[0,l]$ and not identically zero, $u_0(x) \in C^{2+\chi}(0,l) \cap C[0,l]$ with $\chi \in (0,1)$, and satisfies the boundary compatibility conditions.

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The equation in (1.1) arose in the study of the conduction of heat related to the geometric shape of the body (see [2]). It is worthy to illustrate that the equation in problem (1.1) is degenerate and singular because the coefficients of u_{xx} and u_x will become zero and infinite as $x \to 0$, respectively.

There have been a substantial body of articles which considered the blow-up behavior of the solutions to different kinds of parabolic problems (see [1, 4, 5, 8–12, 15, 19–22] and the references therein). Especially, Chen et al. [3] dealt with problem (1.1) with $p = 0$ and $f(x) = g(x) \equiv 0$. Lin and Liu [13] studied problem (1.1) in higher dimensional space with $\alpha = p = 0$. Recently, Liu [14] investigated problem (1.1) with $p = 0$. Under some suitable assumptions, the authors of [3, 13, 14] obtained the conditions on the global existence and non-existence of the positive solutions, proved the global blow-up phenomenon and analysed the asymptotic behavior of the blow-up solutions.

Our main objectives of present article are to obtain the conditions for the occurrence of the blow-up in finite time or global existence, and to describe the blow-up rate and uniform blow-up profile of the blow-up solution near the blow-up time. In order to state our results better, we first let

$$\mathcal{N}_1 = \max\left\{\int_0^l f(x)dx, \int_0^l g(x)dx\right\}, \quad \mathcal{N}_2 = \min\left\{\int_0^l f(x)dx, \int_0^l g(x)dx\right\},$$

and

$$\mu(x) = \mathcal{N}_1 + \frac{\varepsilon_0}{2-\alpha}x^{1-\alpha}(l-x), \quad x \in [0,l], \tag{1.2}$$

where ε_0 is a suitable positive constant. One can observe that

$$\mathcal{C}_1 := \min_{x\in[0,l]} \mu(x) = \mathcal{N}_1 \text{ and } \mathcal{C}_2 := \max_{x\in[0,l]} \mu(x) = \mathcal{N}_1 + \frac{\varepsilon_0 l^{2-\alpha}(1-\alpha)^{1-\alpha}}{(2-\alpha)^{3-\alpha}}.$$

The main results of this article are stated as follows.

Theorem 1.1. *Assume that* $\mathcal{N}_2 \geqslant 1$. *If* $p + q \geqslant 1$, *then the solution of problem* (1.1) *blows up in finite time.*

Theorem 1.2. *Assume that* $\mathcal{N}_1 < 1$.

(i) *If* $p + q < 1$, *then the solution of problem* (1.1) *exists globally.*

(ii) *If* $p + q > 1$, *then the solution of problem* (1.1) *exists globally for small initial data and blows up in finite time for large initial data.*

(iii) *If* $p + q = 1$, *then the solution of problem* (1.1) *exists globally for sufficiently small* κ.

The following two results are about the blow-up rate and uniform blow-up profile. We first need to add some additional appropriate assumptions on $u_0(x)$:

(H_1) $(x^\alpha u_{0x})_x + \kappa u_0^p \int_0^l u_0^q dx > 0$ for $x \in (0, l)$, and

$$\lim_{x\to 0^+}\left[(x^\alpha u_{0x})_x + \kappa u_0^p \int_0^l u_0^q dx\right] = \lim_{x\to l^-}\left[(x^\alpha u_{0x})_x + \kappa u_0^p \int_0^l u_0^q dx\right] = 0.$$

(H_2) There exists a constant $\delta \geqslant \delta_0 > 0$ such that

$$(x^\alpha u_{0x})_x + \kappa u_0^p \int_0^l u_0^q dx - \delta u_0^{p+q} \geqslant 0, \text{ for } x \in (0, l).$$

(H_3) $(x^\alpha u_{0x})_x \leqslant 0$ in $(0, l)$.

Theorem 1.3. *Suppose that* $p+q>1$, $\mathcal{N}_1<1$ *and assumptions* (H_1)–(H_2) *hold. If the solution* $u(x,t)$ *of problem* (1.1) *blows up in finite time* T, *then there exist two positive constants* M_1, M_2 *such that*

$$M_1(T-t)^{-\frac{1}{p+q-1}} \leqslant \max_{x\in[0,l]} u(x,t) \leqslant M_2(T-t)^{-\frac{1}{p+q-1}}, \text{ for } 0<t<T.$$

Theorem 1.4. *Suppose that* $p+q>1$ *with* $0\leqslant p<1$, $\mathcal{N}_1<1$, *and assumptions* (H_1)–(H_3) *hold. If the solution* $u(x,t)$ *of problem* (1.1) *blows up in finite time* T, *then*

$$\lim_{t\to T}(T-t)^{\frac{1}{p+q-1}}u(x,t)=[\kappa l(p+q-1)]^{-\frac{1}{p+q-1}}.$$

This article is organized as follows. In Section 2, we state the maximum principle, comparison theorem, and existence and uniqueness results on the local solution of problem (1.1) as preliminaries. Section 3 is mainly about the conditions on the existence and non-existence of the global solution. The precise blow-up rate estimate of the blow-up solution is given in Section 4. Finally, we will establish the uniform blow-up profile in Section 5.

2. Preliminaries

We start with the definitions of the super-solution and sub-solution of problem (1.1). For convenience, we put $I_T=(0,l)\times(0,T)$ and $\bar{I}_T=[0,l]\times[0,T]$.

Definition 2.1. A nonnegative function $\bar{u}(x,t)$ is called a super-solution of problem (1.1) if $\bar{u}(x,t)\in C^{2,1}(I_T)\cap C(\bar{I}_T)$ satisfies

$$\begin{cases} \bar{u}_t \geqslant (x^\alpha \bar{u}_x)_x+\kappa\bar{u}^p\int_0^l \bar{u}^q dx, & (x,t)\in I_T,\\ \bar{u}(0,t)\geqslant \int_0^l f(x)\bar{u}(x,t)dx, & t\in(0,T),\\ \bar{u}(l,t)\geqslant \int_0^l g(x)\bar{u}(x,t)dx, & t\in(0,T),\\ \bar{u}(x,0)\geqslant \bar{u}_0(x), & x\in[0,l]. \end{cases} \tag{2.1}$$

Analogously, $\underline{u}(x,t)\in C^{2,1}(I_T)\cap C(\bar{I}_T)$ is called a sub-solution of problem (1.1) if it satisfies all the reversed inequalities in (2.1). We say that $u(x,t)$ is a solution of problem (1.1) if it is both a sub-solution and a super-solution of problem (1.1).

Now, by virtue of the analogous approaches as those in [6], we have the following maximum principle, which leads to the comparison theorem for problem (1.1).

Lemma 2.2 (Maximum principle)**.** *Let* $\omega(x,t)\in C^{2,1}(I_T)\cap C(\bar{I}_T)$ *satisfy*

$$\begin{cases} \omega_t-(x^\alpha\omega_x)_x\geqslant \theta_1(x,t)\omega+\theta_2(x,t)\int_0^l\theta_3(x,t)\omega dx, & (x,t)\in I_T,\\ \omega(0,t)\geqslant\int_0^l\theta_4(x)\omega(x,t)dx, & t\in(0,T),\\ \omega(l,t)\geqslant\int_0^l\theta_5(x)\omega(x,t)dx, & t\in(0,T), \end{cases}$$

where $\theta_i(x,t)$, $i=1,\cdots,5$, *are bounded functions, and* $\theta_2(x,t)$, $\theta_3(x,t)\geqslant 0$ *in* I_T, $\theta_4(x)$, $\theta_5(x)\geqslant 0$ *for* $x\in(0,l)$. *Then* $\omega(x,0)>0$ *in* $[0,l]$ *implies that* $\omega(x,t)>0$ *for* $(x,t)\in I_T$. *In additional, if one of the following conditions holds*

(a) $\theta_4(x)=\theta_5(x)\equiv 0$, *for* $x\in(0,l)$;

(b) $\theta_4(x)$, $\theta_5(x)\geqslant 0$, *for* $x\in(0,l)$ *and* $\mathcal{N}_1\leqslant 1$,

then $\omega(x,0)\geqslant 0$ *in* $[0,l]$ *implies that* $\omega(x,t)\geqslant 0$ *for* $(x,t)\in I_T$.

Theorem 2.3 (Comparison theorem)**.** *Let* $\overline{u}(x,t)$ *and* $\underline{u}(x,t)$ *be a nonnegative super-solution and sub-solution of problem* (1.1)*, respectively. Assume that* $\overline{u}(x,0) \geqslant \underline{u}(x,0)$*. Then* $\overline{u}(x,t) \geqslant \underline{u}(x,t)$ *holds in* $\overline{I}_T$.

In light of regularization method and Schauder's fixed point theorem, we can get the results on the existence and uniqueness of the local solution of problem (1.1).

Theorem 2.4 (Local existence and uniqueness)**.** *There exists a small positive real number* T *such that problem* (1.1) *admits a nonnegative solution* $u(x,t) \in C\left(\overline{I}_T\right) \cap C^{2,1}(I_T)$*. Furthermore, assume that the initial datum* $u_0(x)$ *is strictly positive for the case* $\min\{p,q\} < 1$*, then the local solution of problem* (1.1) *is unique.*

3. Global existence and blow-up in finite time

In this section, by using comparison principle and sub- and super-solution methods, we give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Setting $v(t) = \left[v_0^{1-p-q} - \kappa l(p+q-1)t\right]^{-\frac{1}{p+q-1}}$ with $v_0 \in \left(0, \min\limits_{x\in[0,l]} u_0(x)\right)$, then we have $v(t) \to \infty$ as $t \to \frac{v_0^{1-p-q}}{\kappa l(p+q-1)}$. Noticing that $\mathcal{N}_2 \geqslant 1$, we can easily show that $v(t)$ is a blow-up sub-solution of problem (1.1). The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2.
(i). Taking $\varepsilon_0 \in \left(0, \frac{(1-\mathcal{N})(2-\alpha)^{3-\alpha}}{l^{2-\alpha}(1-\alpha)^{1-\alpha}}\right)$ in (1.2), then by $\mathcal{N}_1 < 1$, we have $\mathcal{C}_2 < 1$. Thus, we can immediately claim that $\varphi_1(x) = \mathcal{A}_1\mu(x)$ with $\mathcal{A}_1 = \max\left\{\mathcal{C}_1^{-1}\max\limits_{x\in[0,l]} u_0(x), \left(\kappa l \varepsilon_0^{-1}\mathcal{C}_2^{p+q}\right)^{\frac{1}{1-p-q}}\right\}$ is a stationary super-solution of (1.1). Therefore, the solution of (1.1) is global.

(ii). Selecting $\mathcal{A}_2 = \left[\varepsilon_0(\kappa l)^{-1}\mathcal{C}_2^{-(p+q)}\right]^{\frac{1}{p+q-1}}$, then $\varphi_2(x) = \mathcal{A}_2\mu(x)$ is a global super-solution of problem (1.1) if $u_0(x) \leqslant \left[\varepsilon_0(\kappa l)^{-1}\mathcal{C}_2^{-(p+q)}\right]^{\frac{1}{p+q-1}}\mathcal{C}_1$. And hence, the solution of problem (1.1) exists globally.

Now, our attention focuses on the blow-up result. For $p > 1$, the blow-up conclusion holds obviously by the known results in [14, 15]. For $p \leqslant 1$, we let λ_1 be the first eigenvalue and $\zeta(x)$ be the corresponding eigenfunction of the problem[1]

$$-\left(x^{\alpha}\zeta_x\right)_x = \lambda\zeta, \quad 0 < x < l, \quad \zeta(0) = \zeta(l) = 0. \tag{3.1}$$

Let $\rho(t)$ be the unique solution of the following Cauchy problem

$$\begin{cases} \rho'(t) = -\lambda_1\rho(t) + \rho^{p+q}(t)\int_0^l \zeta^q(x)\,dx, & t > 0, \\ \rho(0) = \rho_0, \end{cases}$$

where $\rho_0 > \left[\lambda_1\left(\int_0^l \zeta^q dx\right)^{-1}\right]^{\frac{1}{p+q-1}}$. Then it is easy to prove that $\lim\limits_{t\to T^-}\rho(t) = \infty$, where

$$T = \frac{1}{\lambda_1(p+q-1)}\left\{\ln\int_0^l \zeta^q dx - \ln\left[\int_0^l \zeta^q dx - \lambda_1\rho_0^{-(p+q-1)}\right]\right\}.$$

Furthermore, if $u_0(x) \geqslant \rho_0\zeta(x)$, then we can show that $v(x,t) = \rho(t)\zeta(x)$ is a sub-solution of problem (1.1), and hence, $u(x,t)$ will become infinite in finite time by comparison principle.

(iii). It is not difficult to verify that, for any positive parameter $\mathcal{A}_3$, $\varphi_3(x) = \mathcal{A}_3\mu(x)$ is a super-solution of problem (1.1) provided that $\kappa < \varepsilon_0(\mathcal{C}_2 l)^{-1}$. Therefore, the solution of problem (1.1) exists globally. The proof of Theorem 1.2 is complete. □

[1] one can see [3, 16] for more details for λ_1 and $\zeta(x)$, such as the explicit expressions of them.

4. Blow-up rate estimate

In this section, following the methods in [7, 17], we give the blow-up rate estimate and the proof of Theorem 1.3.

Proof of Theorem 1.3. Putting $\mathcal{M}(t) = \max_{x\in[0,l]} u(x,t)$, then in light of Theorem 4.5 of [7], we can claim that

$$\mathcal{M}(t) \geqslant [\kappa l (p+q)]^{-\frac{1}{p+q-1}} (T-t)^{-\frac{1}{p+q-1}}, \tag{4.1}$$

holds for $t \in (0,T)$, which leads to the lower bound of the blow-up rate. In order to obtain the upper estimate, we introduce the auxiliary function $J(x,t) = u_t - \delta u^{p+q}$, where δ will be specialized later. Taking

$$\epsilon_1 = l^{\frac{q(p+q-1)}{(p+2q-1)^2}} \left[\frac{p+q}{\kappa(p+2q-1)}\right]^{\frac{q}{p+2q-1}}, \tag{4.2}$$

and calculating directly, we have

$$\begin{aligned} J_t - (x^\alpha J_x)_x - \epsilon_1 u^{-1}J^2 - \kappa q u^p \int_0^l u^{q-1} J dx - \left(2\epsilon_1\delta u^{p+q-1} + \kappa p u^{p-1}\int_0^l u^q dx\right) J \\ = \delta(p+q)(p+q-1)x^\alpha u^{p+q-2}|u_x|^2 + \kappa\delta q u^p \int_0^l u^{p+2q-1}dx \\ + \epsilon_1\delta^2 u^{2(p+q)-1} - \delta(p+q)u^{2p+q-1}\int_0^l u^q dx \\ \geqslant \kappa\delta q u^p \int_0^l u^{p+2q-1}dx + \epsilon_1\delta^2 u^{2(p+q)-1} - \delta(p+q)u^{2p+q-1}\int_0^l u^q dx. \end{aligned} \tag{4.3}$$

Applying Young's inequality, we find that

$$\begin{aligned} u^{p+q-1}\left(\int_0^l u^{p+2q-1}dx\right)^{\frac{q}{p+2q-1}} \\ \leqslant \frac{p+q-1}{p+2q-1}\left(\epsilon_1 u^{p+q-1}\right)^{\frac{p+2q-1}{p+q-1}} + \frac{q}{p+2q-1}\epsilon_1^{-\frac{p+2q-1}{q}}\int_0^l u^{p+2q-1}dx. \end{aligned} \tag{4.4}$$

Meanwhile, Hölder's inequality tells us that

$$\int_0^l u^q dx \leqslant l^{\frac{p+q-1}{p+2q-1}}\left(\int_0^l u^{p+2q-1}\right)^{\frac{q}{p+2q-1}}. \tag{4.5}$$

Putting (4.3), (4.4), (4.5) together, and keeping (4.2) in mind, we have

$$\begin{aligned} J_t - (x^\alpha J_x)_x - \epsilon_1 u^{-1}J^2 - \kappa q u^p \int_0^l u^{q-1} J dx - \left(2\epsilon_1\delta u^{p+q-1} + \kappa p u^{p-1}\int_0^l u^q dx\right) J \\ \geqslant \delta\left(\epsilon_1\delta - \kappa(p+q-1)\epsilon_1^{\frac{(p+2q-1)^2}{q(p+q-1)}}\right)u^{2(p+q)-1}. \end{aligned} \tag{4.6}$$

Choosing $\delta \geqslant \kappa(p+q-1)\epsilon_1^{\frac{(p-1)^2+3q(p+q-1)}{q(p+q-1)}} := \delta_0$, then from (4.6), it follows that

$$J_t - (x^\alpha J_x)_x - \epsilon_1 u^{-1}J^2 - \kappa q u^p \int_0^l u^{q-1} J dx - \left(2\epsilon_1\delta u^{p+q-1} + \kappa p u^{p-1}\int_0^l u^q dx\right) J \geqslant 0. \tag{4.7}$$

On the other hand, since $\mathcal{N}_1 < 1$, then in view of Jensen's inequality, we have, at the boundary point $x = 0$,

$$\begin{aligned} J(0,t) &= u_t(0,t) - \delta u^{p+q}(0,t) \\ &= \int_0^l f(x) u_t(x,t)\,dx - \delta \left(\int_0^l f(x) u(x,t)\,dx \right)^{p+q} \\ &= \int_0^l f(x) J(x,t)\,dx + \delta \int_0^l f(x) u^{p+q}(x,t)\,dx - \delta \left(\int_0^l f(x) u(x,t)\,dx \right)^{p+q} \\ &\geqslant \int_0^l f(x) J(x,t)\,dx - \delta \left(\int_0^l f(x) u(x,t)\,dx \right)^{p+q} \\ &\quad + \delta \left(\int_0^l f(x)\,dx \right)^{1-p-q} \left(\int_0^l f(x) u(x,t)\,dx \right)^{p+q} \\ &\geqslant \int_0^l f(x) J(x,t)\,dx. \end{aligned} \tag{4.8}$$

Analogously, at the boundary point $x = l$, we can also show that

$$J(l,t) \geqslant \int_0^l g(x) J(x,t)\,dx. \tag{4.9}$$

In addition, the assumptions (H_1) and (H_2) mean that

$$J(x,0) = (x^\alpha u_{0x})_x + \kappa u_0^p \int_0^l u^q dx - \delta u_0^{p+q} \geqslant 0, \quad \text{for } 0 < x < l. \tag{4.10}$$

Noticing that u is a positive bounded continuous function for $(x,t) \in I_T$, it follows from (4.7), (4.8), (4.9), (4.10) that $J(x,t) \geqslant 0$ in I_T, which leads to

$$u_t \geqslant \delta u^{p+q}. \tag{4.11}$$

Integrating (4.11) from t to T, we have

$$u(x,t) \leqslant [\delta(p+q-1)]^{-\frac{1}{p+q-1}} (T-t)^{-\frac{1}{p+q-1}}. \tag{4.12}$$

Combining (4.1) with (4.12), we have

$$M_1 (T-t)^{-\frac{1}{p+q-1}} \leqslant \max_{x\in[0,l]} u(x,t) \leqslant M_2 (T-t)^{-\frac{1}{p+q-1}},$$

where $M_1 = [\kappa l (p+q)]^{-\frac{1}{p+q-1}}$ and $M_2 = [\delta(p+q-1)]^{-\frac{1}{p+q-1}}$. The proof of Theorem 1.3 is complete. □

5. Uniform blow-up profile

Throughout this section, based on the general ideas of [18], by the similar arguments of [14, 15, 17, 20], we will discuss the uniform blow-up profile of problem (1.1) under the restrict condition $p + q > 1$ with $0 \leqslant p < 1$.

From the assumptions on $u_0(x)$, we can find a suitable small positive constant ε_1 and a nonnegative function $w_{0\varepsilon}(x)$ such that

(1) $w_{0\varepsilon} \in C^{2+\chi}(\varepsilon, l-\varepsilon) \cap C[\varepsilon, l-\varepsilon]$ with $\chi \in (0,1)$ and $\varepsilon \in (0, \varepsilon_1]$.

(2) $w_{0\varepsilon}(\varepsilon) = \int_{\varepsilon}^{l-\varepsilon} f(x) w_{0\varepsilon}(x)\,dx$ and $w_{0\varepsilon}(l-\varepsilon) = \int_{\varepsilon}^{l-\varepsilon} g(x) w_{0\varepsilon}(x)\,dx$.

(3) $w_{0\varepsilon}(x) < u_0(x)$ for $x \in (\varepsilon, 2\varepsilon) \cup (l-2\varepsilon, l-\varepsilon)$, and $w_{0\varepsilon}(x) = u_0(x)$ for $x \in [2\varepsilon, l-2\varepsilon]$.

(4) $(x^{\alpha} w_{0\varepsilon x})_x \leqslant 0$ for $x \in (\varepsilon, l-\varepsilon)$.

(5) $w_{0\varepsilon}$ is non-increasing with respect to ε in $(0, \varepsilon_1]$. Furthermore,

$$0 = \lim_{x \to \varepsilon^+} \left[(x^{\alpha} w_{0\varepsilon x})_x + \kappa w_{0\varepsilon}^p \int_{\varepsilon}^{l-\varepsilon} w_{0\varepsilon}^q\,dx \right] = \lim_{x \to (l-\varepsilon)^-} \left[(x^{\alpha} w_{0\varepsilon x})_x + \kappa w_{0\varepsilon}^p \int_{\varepsilon}^{l-\varepsilon} w_{0\varepsilon}^q\,dx \right].$$

(6) $(x^{\alpha} w_{0\varepsilon x})_x + \kappa w_{0\varepsilon}^p \int_{\varepsilon}^{l-\varepsilon} w_{0\varepsilon}^q\,dx \geqslant 0$ for $\varepsilon \in (0, \varepsilon_1]$ and $x \in (\varepsilon, l-\varepsilon)$.

Next, we focus on the following auxiliary problem

$$\begin{cases} w_{\varepsilon t} = (x^{\alpha} w_{\varepsilon x})_x + \kappa w_{\varepsilon}^p \int_{\varepsilon}^{l-\varepsilon} w_{\varepsilon}^q\,dx, & (x,t) \in (\varepsilon, l-\varepsilon) \times (0, +\infty), \\ w_{\varepsilon}(\varepsilon, t) = \int_{\varepsilon}^{l-\varepsilon} f(x) w_{\varepsilon}(x,t)\,dx, & t \in (0, +\infty), \\ w_{\varepsilon}(l-\varepsilon, t) = \int_{\varepsilon}^{l-\varepsilon} g(x) w_{\varepsilon}(x,t)\,dx, & t \in (0, +\infty), \\ w_{\varepsilon}(x, 0) = w_{0\varepsilon}(x), & x \in [0, l]. \end{cases} \tag{5.1}$$

By standard parabolic theory, we can prove that problem (5.1) admits a unique solution $w_{\varepsilon}(x,t)$. Moreover, the arguments of Section 2 in [20] tell us that $\lim_{\varepsilon \to 0^+} w_{\varepsilon}(x,t) = u(x,t)$, where $u(x,t)$ is the solution of problem (1.1).

Lemma 5.1. *Assume that* (H_1)–(H_3) *hold. Assume also that* $\mathcal{N}_1 < 1$ *and* $p + q > 1$ *with* $0 \leqslant p < 1$. *Then* $(x^{\alpha} u_x)_x \leqslant 0$ *holds for* $(x,t) \in I_T$.

Proof. Taking $\eta = (x^{\alpha} w_{\varepsilon x})_x$, then from (5.1), it follows that

$$\eta_t = (x^{\alpha} \eta_x)_x + \kappa p \int_{\varepsilon}^{l-\varepsilon} w_{\varepsilon}^q(x,t)\,dx \left[\eta w_{\varepsilon}^{p-1} + (p-1) x^{\alpha} w_{\varepsilon}^{p-2} |w_{\varepsilon x}|^2 \right]. \tag{5.2}$$

Recalling that $w_{\varepsilon} \geqslant 0$ and $0 \leqslant p < 1$, we can infer from (5.2) that

$$\eta_t - (x^{\alpha} \eta_x)_x - \kappa p \eta w_{\varepsilon}^{p-1} \int_{\varepsilon}^{l-\varepsilon} w_{\varepsilon}^q(x,t)\,dx \leqslant 0, \tag{5.3}$$

holds for any $(x,t) \in (\varepsilon, l-\varepsilon) \times (0,T)$. On the other hand, for any $t \in (0,T)$, we have

$$\begin{aligned} \eta(\varepsilon, t) &= \int_{\varepsilon}^{l-\varepsilon} f(x) w_{\varepsilon t}(x,t)\,dx - \kappa w_{\varepsilon}^p \int_{\varepsilon}^{l-\varepsilon} w_{\varepsilon}^q(x,t)\,dx \\ &= \int_{\varepsilon}^{l-\varepsilon} f(x) \left((x^{\alpha} w_{\varepsilon x})_x + \kappa w_{\varepsilon}^p \int_{\varepsilon}^{l-\varepsilon} w_{\varepsilon}^q(x,t)\,dx \right) dx \\ &\quad - \kappa \int_{\varepsilon}^{l-\varepsilon} w_{\varepsilon}^q(x,t)\,dx \left(\int_{\varepsilon}^{l-\varepsilon} f(x) w_{\varepsilon}(x,t)\,dx \right)^p \\ &\leqslant \int_{\varepsilon}^{l-\varepsilon} f(x) \eta(x,t)\,dx + \kappa \Lambda \int_{\varepsilon}^{l-\varepsilon} w_{\varepsilon}^q(x,t)\,dx, \end{aligned} \tag{5.4}$$

where

$$\Lambda = \int_{\varepsilon}^{l-\varepsilon} f(x) w_{\varepsilon}^p\,dx - \left(\int_{\varepsilon}^{l-\varepsilon} f(x) w_{\varepsilon}(x,t)\,dx \right)^p.$$

Under the assumption $\mathcal{N}_1 < 1$, one can see that $\Lambda < 0$ for $p \in [0,1)$. Combining (5.4) and $\Lambda < 0$, one can find that

$$\eta(\varepsilon,t) \leqslant \int_{\varepsilon}^{l-\varepsilon} f(x)\eta(x,t)\,dx, \quad t \in (0,T).$$

By the similar arguments, we also can verify that, for all $t \in (0,T)$,

$$\eta(l-\varepsilon,t) \leqslant \int_{\varepsilon}^{l-\varepsilon} g(x)\eta(x,t)\,dx.$$

Moreover, by virtue of $(x^{\alpha}w_{0\varepsilon x})_x \leqslant 0$, we know that $\eta(x,0) \leqslant 0$ for $x \in (0,l)$. Therefore, maximum principle implies that $\eta = (x^{\alpha}w_{\varepsilon x})_x \leqslant 0$ holds for all $(x,t) \in (\varepsilon, l-\varepsilon) \times (0,T)$. Furthermore, the arbitrariness of ε tells us that $(x^{\alpha}u_x)_x \leqslant 0$ in I_T. The proof of Lemma 5.1 is complete. □

Now, for convenience, we set

$$\psi(t) = \kappa \int_0^l u^q(x,t)\,dx, \quad \Psi(t) = \int_0^t \psi(\tau)\,d\tau,$$

and

$$\Theta(x,t) = \Psi(t) - \frac{1}{1-p}u^{1-p}(x,t).$$

Then, we have the following Lemmas.

Lemma 5.2. *Under the assumptions of Lemma 5.1, there exists a positive constant* M *such that*

$$\sup_{x\in K_d} \Theta(x,t) \leqslant \frac{M}{d^2}\left(1 + \int_0^t \Psi(\tau)\,d\tau\right), \tag{5.5}$$

holds in $[0,l] \times [\frac{T}{2}, T)$, *where*

$$K_d = \{x \in (0,l) : \text{dist}(x,0) \geqslant d,\ \text{dist}(x,l) \geqslant d\} \subset (0,l).$$

Proof. Let

$$\Phi(t) = \int_0^l \Theta(y,t)\zeta(y)\,dy,$$

where ζ is the first eigenfunction of problem (3.1) with $\|\zeta(x)\|_{L^1(0,l)} = 1$. Taking the derivative of $\Phi(t)$ with respect to t, we obtain

$$\begin{aligned}
\Phi'(t) &= \int_0^l \left(\psi(t) - u^{-p}u_t\right)\zeta(y)\,dy \\
&= -\int_0^l u^{-p}(y^{\alpha}u_y)_y\,\zeta(y)\,dy \\
&\leqslant \frac{\lambda_1}{1-p}\int_0^l u^{1-p}(y,t)\zeta(y)\,dy + \frac{l^{\alpha}\zeta_y|_{y=l}}{1-p}\left(\int_0^l f(y)u(y,t)\,dy\right)^{1-p} \\
&\leqslant \lambda_1 \int_0^l [\Psi(t) - \Theta(y,t)]\zeta(y)\,dy \\
&\leqslant \lambda_1\Psi(t) + \lambda_1\int_0^l \Theta^-(y,t)\zeta(y)\,dy,
\end{aligned} \tag{5.6}$$

where

$$\Theta^-(y,t) = \max\{-\Theta(y,t), 0\}.$$

From Lemma 5.1, we have that $u_t \leqslant \psi(t) u^p$. Setting $M_3 = \sup_{x\in(0,l)} u_0(x)$, and integrating the above inequality from 0 to t yields

$$u^{1-p}(x,t) \leqslant M_3^{1-p} + (1-p)\Psi(t), \tag{5.7}$$

which implies that

$$\Theta^-(y,t) \leqslant \frac{1}{1-p} M_3^{1-p}. \tag{5.8}$$

From (5.6) and (5.8), it follows that

$$\Phi'(t) \leqslant \lambda_1 \Psi(t) + \frac{\lambda_1}{1-p} M_3^{1-p}. \tag{5.9}$$

Integrating (5.9) over $(0,t)$, we arrive at

$$\Phi(t) \leqslant M_4 \left(1 + \int_0^t \Psi(\tau)\, d\tau\right),$$

where $M_4 = \max\left\{\lambda_1, \Phi(0) + \frac{\lambda_1 T}{1-p} M_3^{1-p}\right\}$. By [18, Lemma 4.5], we get the estimate (5.5). The proof of Lemma 5.3 is complete. □

Lemma 5.3. *Under the assumptions of Lemma 5.1, then*

$$\lim_{t\to T} \frac{u^{1-p}(x,t)}{(1-p)\Psi(t)} = \lim_{t\to T} \frac{|u(\cdot,t)|_\infty^{1-p}}{(1-p)\Psi(t)} = 1, \tag{5.10}$$

holds uniformly on any compact subset of $(0,l)$.

Proof. In light of (5.7), it is not difficult to show that

$$\lim_{t\to T} \psi(t) = \lim_{t\to T} \Psi(t) = \infty, \tag{5.11}$$

and hence, we have

$$\lim_{t\to T} \frac{u^{1-p}(x,t)}{(1-p)\Psi(t)} \leqslant 1. \tag{5.12}$$

On the other hand, from (5.5) and (5.7), it follows that

$$-\frac{M_3^{1-p}}{(1-p)\Psi(t)} \leqslant 1 - \frac{u^{1-p}(x,t)}{(1-p)\Psi(t)} \leqslant \frac{M}{d^2\Psi(t)} \left(1 + \int_0^t \Psi(\tau)\, d\tau\right). \tag{5.13}$$

From Theorem 1.3, we can deduce that

$$\int_0^t \Psi(\tau)\, d\tau \leqslant M_5 \int_0^t (T-\tau)^{-\frac{1-p}{p+q-1}}\, d\tau, \tag{5.14}$$

where $M_5 = \kappa l (p+q-1)(1-p)^{-1} M_2^q$. Meanwhile, combining (5.7) with Theorem 1.3, we see that there exists a suitable positive constant M_6 such that

$$\Psi(t) \geqslant M_6 (T-t)^{-\frac{1-p}{p+q-1}}. \tag{5.15}$$

In light of (5.14) and (5.15), we arrive at

$$\lim_{t\to T} \frac{\int_0^t \Psi(\tau)\, d\tau}{\Psi(t)} = 0. \tag{5.16}$$

Up to now, in view of (5.11), (5.13) and (5.16), we find that

$$\lim_{t\to T} \frac{u^{1-p}(x,t)}{(1-p)\Psi(t)} \geqslant 1. \tag{5.17}$$

Put (5.12) and (5.17) together, we obtain the desired result. The proof of Lemma 5.3 is complete. □

Proof of Theorem 1.4. From (5.10), it follows that $u(x,t) \sim [(1-p)\Psi(t)]^{\frac{1}{1-p}}$ as $t \to T$. Lebesgues dominated convergence theorem leads to

$$\Psi'(t) = \psi(t) = \kappa \int_0^l u^q(x,t)\,dx \sim \kappa l (1-p)^{\frac{q}{1-p}} \Psi^{\frac{q}{1-p}}(t), \quad t \to T.$$

And hence, by integrating, we obtain

$$\Psi(t) \sim \frac{1}{1-p}[\kappa l(p+q-1)]^{\frac{p-1}{p+q-1}} (T-t)^{\frac{p-1}{p+q-1}}, \quad t \to T. \tag{5.18}$$

Combining (5.10) with (5.18), we find that

$$u(x,t) \sim [\kappa l(p+q-1)]^{-\frac{1}{p+q-1}} (T-t)^{-\frac{1}{p+q-1}}, \quad t \to T,$$

which means immediately that

$$\begin{aligned}\lim_{t\to T}(T-t)^{\frac{1}{p+q-1}} u(x,t) &= \lim_{t\to T}(T-t)^{\frac{1}{p+q-1}} |u(\cdot,t)|_\infty \\ &= [\kappa l(p+q-1)]^{-\frac{1}{p+q-1}}.\end{aligned}$$

□

The proof of Theorem 1.4 is complete.

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