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The implicit midpoint rule of nonexpansive mappings and applications in uniformly smooth Banach spaces

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Abstract

Let K be a nonempty closed convex subset of a Banach space E and $T : K \to K$ be a nonexpansive mapping. Using a viscosity approximation method, we study the implicit midpoint rule of a nonexpansive mapping T. We establish a strong convergence theorem for an iterative algorithm in the framework of uniformly smooth Banach spaces and apply our result to obtain the solutions of an accretive mapping and a variational inequality problem. The numerical example which compares the rates of convergence shows that the iterative algorithm is the most efficient. Our result is unique and the method of proof is of independent interest.

Keywords: Viscosity technique, implicit midpoint rule, nonexpansive, accretive, variational inequality problem.

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1. Introduction

Let K be a nonempty closed convex subset of a Hilbert space H and $T : K \to K$ be a nonlinear mapping. T is said to be nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$. A point $x \in K$ is said to be a fixed point of T if $Tx = x$. We shall denote the set of fixed points of T by $F(T)$. T is called a c-contraction if there exists $c \in (0, 1)$ such that

$$\|T(x) - T(y)\| \leqslant c\|x - y\| \text{ for all } x, y \in K.$$

We shall denote the collection of all contractions on K by Π_K. The nearest point projection $P_K : H \to K$ from H onto K is defined by

$$P_K x := \arg\min_{y \in K} \|x - y\|^2, \ x \in H.$$

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Let D be a subset of K and let S be a mapping from K to D. Then S is said to be sunny if $S(Sx+t(x-Sx)) = Sx$ whenever $Sx+t(x-Sx) \in K$ for $x \in K$ and $t \geqslant 0$. A mapping S from K into itself is said to be a retraction if $S^2 = S$. A set D is said to be a sunny nonexpansive retract of K if there exists a sunny nonexpansive retraction from K into D ([8, 24]). It is well known that if E is a smooth Banach space and K is a nonempty closed convex subset of E, then there exists at most one sunny nonexpansive retraction S from E onto K. In 1967, Halpern [9] considered the iterative sequence for a nonexpansive mapping T in a Hilbert space. He showed that the conditions (A_1) $\lim_{n\to\infty} \lambda_n = 0$ and (A_2) $\sum_{n=1}^{\infty} \lambda_n = \infty$ are essential for the convergence to a fixed point of T of the sequence $\{x_n\}$ defined by

$$x_1 \in K,\ x_{n+1} = \lambda_n u + (1-\lambda_n)Tx_n,\ n \in \mathbb{N}, \tag{1.1}$$

where $u \in K$ is a given point and $\lambda_n \in [0,1]$. Halpern [9] iteration attracted the attention of many researchers. In 1977, Lions [4] improved on the result of Halpern and showed that for $\{\lambda_n\}$ satisfying the conditions (A_1), (A_2), and (A_3): $\lim_{n\to\infty} |\lambda_n - \lambda_{n-1}|/\lambda_n^2 = 0$, $\{x_n\}$ converges strongly to a fixed point of T in a Hilbert space. In 1992, still in Hilbert space and for $\{\lambda_n\}$ satisfying the conditions (A_1), (A_2), and (A_4): $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < +\infty$, Wittmann [13] proved a strong convergence theorem for the sequence (1.1) to a fixed point of T. By considering various conditions either on $\{\lambda_n\}$ or on the space, there are also several theorems for the strong convergence of Halpern's iteration to a fixed point of T in Banach spaces (see, e.g., [4, 13, 19–21, 23]). Modification of Halpern [9] type iteration have also been studied by many authors. In 2000, Moudafi [16] introduced the concept of viscosity approximation method for selecting a particular fixed point of a given nonexpansive mapping. He considered an explicit viscosity method for nonexpansive mappings and defined the iterative sequence $\{x_n\}$ by

$$x_{n+1} = \lambda_n Q(x_n) + (1-\lambda_n)Tx_n, n \in \mathbb{N}, \tag{1.2}$$

where Q is a contraction on K and the nonexpansive mapping $T : K \to K$ is also defined on K, which is a nonempty closed convex subset of a real Hilbert space H. He showed that the sequence $\{x_n\}$ defined by (1.2) converges strongly to a fixed point of T with the conditions that (A_1), (A_2), and (A_5): $\lim_{n\to\infty} |\lambda_n - \lambda_{n-1}|/\lambda_n\lambda_{n-1} = 0$ are satisfied. One of the essential numerical methods for solving ordinary differential and differential algebraic equations is the implicit midpoint rule ([5, 6, 10, 22]). In 2014, Alghamdi et al. [3] presented a semi-implicit midpoint iteration for nonexpansive mappings in a Hilbert space. They proved a weak convergence theorem for the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N}, \tag{1.3}$$

where $\lambda_n \in (0,1)$ certifies certain conditions. Furthermore, in 2015, Xu et al. [29] used regularized semi-implicit midpoint by the contraction Q and defined the viscosity implicit midpoint sequence for a nonexpansive mapping T on K by

$$x_{n+1} = \lambda_n Q(x_n) + (1-\lambda_n)T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N}, \tag{1.4}$$

where $\lambda_n \in (0,1)$. Precisely, they proved the following strong convergence theorem.

Theorem 1.1 ([29])**.** *Let K be a nonempty closed convex subset of a Hilbert space H and $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose $Q : K \to K$ is a contraction with coefficient $\lambda \in [0,1)$ and assume that the sequence $\{\lambda_n\}$ satisfies the conditions (A_1), (A_2), and either (A_4) or $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n-1}} = 1$. Then the sequence $\{x_n\}$ generated by (1.4) converges in norm to a fixed point p of T, which is also the unique solution of the variational inequality*

$$\langle (I-Q)p, x-p \rangle \geqslant 0,\ \forall\, x \in F(T).$$

That is, p is the unique fixed point of the contraction $P_{F(T)}Q$, in other words, $P_{F(T)}Q(p) = p$.

Still in a Hilbert space, in 2015, Yao et al. [30] introduced the iterative sequence

$$x_{n+1} = \lambda_n Q(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N}, \tag{1.5}$$

where T and Q are as defined in Theorem 1.1 and $\lambda_n + \beta_n + \gamma_n = 1 \ \forall \ n \in \mathbb{N}$. They imposed suitable conditions on the parameters and obtained that the sequence $\{x_n\}$ generated by (1.5) converges strongly to $p = P_{F(T)}Q(p)$. In 2017, Luo et al. [15] extends the result of Xu et al. [29] to a uniformly smooth Banach space. Few among several other works on modified Halpern-type iteration include Qin et al. [18], Wang et al. [25] and the references contained in them. Also, some authors studied modified Halpern-type sequences for various classes of mappings (see e.g., Aibinu and Mewomo [1, 2], Chidume and Mutangandura [7], Hu and Wang [11] and Nandal and Chugh [17]). The following questions are of interest to us.

Problem 1.2. Do the main results of Yao et al. [30] which are in Hilbert spaces also hold in general Banach spaces?

Problem 1.3. Comparing the three implicit iterative schemes that we have mentioned, which one has the highest rate of convergence?

The purpose of this paper is to study the implicit midpoint procedure (1.5) in the framework of Banach spaces for approximating a fixed point of nonexpansive mappings. We prove a strong convergence theorem in a uniformly smooth Banach space for the sequence $\{x_n\}$ defined by (1.5) and illustrate with the numerical example that it is the most efficient among the three algorithms. Moreover, we obtain the results of Xu et al. [29], Luo et al. [15], and Yao et al. [30] as corollaries.

2. Preliminaries

A real Banach space E with norm $\|.\|$ is said to be strictly convex if for all $x, y \in E$, $\frac{\|x+y\|}{2} < 1$ with $\|x\| = \|y\| = 1$ and $x \neq y$. The modulus of convexity of E, $\delta_E : (0, 2] \to [0, 1]$ is defined by $\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| > \epsilon\right\}$, $0 \leqslant \epsilon \leqslant 1$. E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. Every uniformly convex space is reflexive and strictly convex. Let $U(x) := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then E is said to be smooth (or Gáteaux differentiable) if the limit

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U(x)$. It is said to have uniformly Gâteaux differentiable norm if for each $y \in U(x)$, the limit is attained uniformly for $x \in U(x)$. Furthermore, E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in U(x)$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined as

$$J(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\},$$

where E^* denotes the dual of E and $\langle ., . \rangle$ is the duality pairing between E and E^*. Recall that if E^* is strictly convex, then J is single-valued. Moreover, for a Banach space E with a uniformly Gâteaux differentiable norm, the normalized duality mapping J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^*.

We shall need the following lemmas in the sequel.

Lemma 2.1 ([23])**.** *Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$. Suppose that $u_{n+1} = (1 - \beta_n)u_n + \beta_n v_n$ for all $n \geqslant 0$ and $\limsup_{n\to\infty} (\|u_{n+1} - u_n\| - \|v_{n+1} - v_n\|) \leqslant 0$. Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.*

Lemma 2.2 ([27]). *Assume $\{a_n\}$ is a sequence of nonnegative real sequence such that*

$$a_{n+1} = (1-\sigma_n)a_n + \sigma_n\delta_n, \ \ n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that

(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([28]). *Let E be a uniformly smooth Banach space, K be a closed convex subset of E, $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and let $Q \in \Pi_K$. Then the sequence $\{x_t\}$ defined by $x_t = tQ(x_t) + (1-t)Tx_t$ converges strongly to a point in $F(T)$. If we define a mapping $S : \Pi_K \to F(T)$ by $S(Q) := \lim_{t\to 0} x_t$, $\forall\, Q \in \Pi_K$, then $S(Q)$ solves the following variational inequality:*

$$\langle (I-Q)S(Q), j(S(Q)-p)\rangle \leqslant 0, \ \ \forall\, Q \in \Pi_K.$$

Lemma 2.4 ([26, Lemma 2.12]). *Let E be a Banach space with a uniformly Gâteaux differentiable norm, K be a nonempty, closed, and convex subset of E, $Q : K \to K$ be a continuous operator, $T : K \to K$ be a nonexpansive operator, and $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Suppose that $\{z_t\}$ is a path in K defined by $z_t = tf(z_t) + (1-t)Tz_t, t \in (0,1)$ such that $z_t \to z$ as $t \to 0^+$. Then*

$$\limsup_{n\to\infty} \langle Q(z) - z, j(x_n - z)\rangle \leqslant 0.$$

3. Main results

Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $Q : K \to K$ a c-contraction. Suppose $\{\lambda_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$, and $\{\gamma_n\} \subset (0,1)$ are real sequences satisfying $\lambda_n + \beta_n + \gamma_n = 1$, $\forall\, n \in \mathbb{N}$. For arbitrary $x_1 \in K$, we consider the following iterative scheme for the sequence $\{x_n\}$ defined by (1.5).

Remark 3.1. It is known that the sequence $\{x_n\}$ is well defined [30].

We first give and prove a lemma which is useful in establishing our main result.

Lemma 3.2. *Let E be a uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and suppose $Q : K \to K$ is a c-contraction. For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}$ by (1.5). Then the sequence $\{x_n\}$ is bounded.*

Proof. We show that the sequence $\{x_n\}$ is bounded. For $p \in F(T)$,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\lambda_n\left(Q(x_n) - Q(p)\right) + \lambda_n\left(Q(p) - p\right) + \beta_n(x_n - p) + \gamma_n\left(T(\frac{x_n + x_{n+1}}{2}) - p\right)\| \\
&\leqslant \lambda_n\|Q(x_n) - Q(p)\| + \lambda_n\|Q(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|T(\frac{x_n + x_{n+1}}{2}) - p\| \\
&\leqslant \lambda_n\|Q(x_n) - Q(p)\| + \lambda_n\|Q(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|\frac{x_n + x_{n+1}}{2} - p\| \\
&\leqslant c\lambda_n\|x_n - p\| + \lambda_n\|Q(p) - p\| + \beta_n\|x_n - p\| + \frac{\gamma_n}{2}\|x_n - p\| + \frac{\gamma_n}{2}\|x_{n+1} - p\|.
\end{aligned}$$

We then have that

$$\begin{aligned}
\left(1 - \frac{\gamma_n}{2}\right)\|x_{n+1} - p\| &\leqslant \left(c\lambda_n + \beta_n + \frac{\gamma_n}{2}\right)\|x_n - p\| + \lambda_n\|Q(p) - p\|, \\
\frac{2-\gamma_n}{2}\|x_{n+1} - p\| &\leqslant \frac{2c\lambda_n + 2\beta_n + \gamma_n}{2}\|x_n - p\| + \lambda_n\|Q(p) - p\|,
\end{aligned}$$

$$\frac{1+\lambda_n+\beta_n}{2}\|x_{n+1}-p\| \leqslant \frac{2c\lambda_n+2\beta_n+1-(\lambda_n+\beta_n)}{2}\|x_n-p\|+\lambda_n\|Q(p)-p\|,$$
$$\frac{1+\lambda_n+\beta_n}{2}\|x_{n+1}-p\| \leqslant \frac{1+\beta_n+\lambda_n(2c-1)}{2}\|x_n-p\|+\lambda_n\|Q(p)-p\|.$$

Therefore,

$$\begin{aligned}
\|x_{n+1}-p\| &\leqslant \frac{1+\beta_n+\lambda_n(2c-1)}{1+\lambda_n+\beta_n}\|x_n-p\|+\frac{2\lambda_n}{1+\lambda_n+\beta_n}\|Q(p)-p\| \\
&= \left(1-\frac{2\lambda_n(1-c)}{1+\lambda_n+\beta_n}\right)\|x_n-p\|+\frac{2\lambda_n(1-c)}{1+\lambda_n+\beta_n}\frac{1}{1-c}\|Q(p)-p\| \\
&\leqslant \max\left\{\|x_n-p\|, \frac{1}{1-c}\|Q(p)-p\|\right\} \\
&\;\vdots \\
&\leqslant \max\left\{\|x_1-p\|, \frac{1}{1-c}\|Q(p)-p\|\right\}.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded and hence $\left\{T\left(\frac{x_n+x_{n+1}}{2}\right)\right\}$ is also bounded.

Obviously, for $p \in F(T)$,

$$\begin{aligned}
\|T(\frac{x_n+x_{n+1}}{2})\| &= \|T(\frac{x_n+x_{n+1}}{2})-p+p\| \\
&\leqslant \|T(\frac{x_n+x_{n+1}}{2})-Tp\|+\|p\| \\
&\leqslant \|\frac{x_n+x_{n+1}}{2}-p\|+\|p\| \\
&\leqslant \frac{1}{2}(\|x_n-p\|+\|x_{n+1}-p\|)+\|p\| \\
&\leqslant \max\left\{\|x_1-p\|, \frac{1}{1-c}\|Q(p)-p\|\right\}+\|p\|,
\end{aligned}$$

because $\|x_n-p\| \leqslant \max\left\{\|x_1-p\|, \frac{1}{1-c}\|Q(p)-p\|\right\}$. □

Theorem 3.3. *Let E be a uniformly smooth Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $Q : K \to K$ be a c-contraction. Suppose $\{\lambda_n\}$ satisfies (A_1) and (A_2), and $\{\beta_n\}$ satisfies*

(A_6) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$, *and*
(A_7) $\lim_{n\to\infty} |\beta_{n+1}-\beta_n| = 0$.

For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}$ by (1.5). Then as $n \to \infty$, the sequence $\{x_n\}$ converges in norm to a fixed point p of T, where p is the unique solution in $F(T)$ to the variational inequality:

$$\langle (I-Q)p, j(x-p)\rangle \geqslant 0, \ \forall\, x \in F(T).$$

Proof.

Step 1: Let the iterative process (1.5) be written as below:

$$\begin{aligned}
x_{n+1} &= \lambda_n Q(x_n)+\beta_n x_n+\gamma_n T(\frac{x_n+x_{n+1}}{2}) \\
&= \beta_n x_n+(1-\beta_n)\frac{\lambda_n Q(x_n)+\gamma_n T(\frac{x_n+x_{n+1}}{2})}{1-\beta_n} = \beta_n x_n+(1-\beta_n)y_n,
\end{aligned} \tag{3.1}$$

where $y_n = \frac{\lambda_n}{1-\beta_n}Q(x_n) + \frac{\gamma_n}{1-\beta_n}T(\frac{x_n+x_{n+1}}{2})$, $n \in \mathbb{N}$.

From condition (A_6), we have that

$$0 < \beta_n \leqslant \beta < 1, \text{ for some } \beta \in \mathbb{R}^+,$$

where $\mathbb{R}^+$ denotes the set of positive real numbers. Therefore,

$$1 - \beta_n \geqslant 1 - \beta. \tag{3.2}$$

Recall that Q is a c-contraction while $\{x_n\}$ and $\left\{T(\frac{x_n+x_{n+1}}{2})\right\}$ are bounded sequences. These guarantee that $\{y_n\}$ is bounded.

Step 2: We show that $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

We need to first show that $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. Observe that

$$\begin{aligned}
y_{n+1} - y_n &= \frac{\lambda_{n+1}}{1-\beta_{n+1}}Q(x_{n+1}) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}T(\frac{x_{n+1}+x_{n+2}}{2}) \\
&\quad - \left(\frac{\lambda_n}{1-\beta_n}Q(x_n) + \frac{\gamma_n}{1-\beta_n}T(\frac{x_n+x_{n+1}}{2})\right) \\
&= \frac{\lambda_{n+1}}{1-\beta_{n+1}}(Q(x_{n+1}) - Q(x_n)) + \left(\frac{\lambda_{n+1}}{1-\beta_{n+1}} - \frac{\lambda_n}{1-\beta_n}\right)Q(x_n) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(T(\frac{x_{n+1}+x_{n+2}}{2}) - T(\frac{x_n+x_{n+1}}{2})\right) \\
&\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)T(\frac{x_n+x_{n+1}}{2}) \\
&= \frac{\lambda_{n+1}}{1-\beta_{n+1}}(Q(x_{n+1}) - Q(x_n)) + \left(\frac{\lambda_{n+1}}{1-\beta_{n+1}} - \frac{\lambda_n}{1-\beta_n}\right)Q(x_n) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(T(\frac{x_{n+1}+x_{n+2}}{2}) - T(\frac{x_n+x_{n+1}}{2})\right) \\
&\quad + \left(\frac{1-\lambda_{n+1}-\beta_{n+1}}{1-\beta_{n+1}} - \frac{1-\lambda_n-\beta_n}{1-\beta_n}\right)T(\frac{x_n+x_{n+1}}{2}) \\
&= \frac{\lambda_{n+1}}{1-\beta_{n+1}}(Q(x_{n+1}) - Q(x_n)) + \left(\frac{\lambda_{n+1}}{1-\beta_{n+1}} - \frac{\lambda_n}{1-\beta_n}\right)Q(x_n) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(T(\frac{x_{n+1}+x_{n+2}}{2}) - T(\frac{x_n+x_{n+1}}{2})\right) \\
&\quad + \left(\frac{\lambda_n}{1-\beta_n} - \frac{\lambda_{n+1}}{1-\beta_{n+1}}\right)T(\frac{x_n+x_{n+1}}{2}) \\
&= \frac{\lambda_{n+1}}{1-\beta_{n+1}}(Q(x_{n+1}) - Q(x_n)) + \left(\frac{\lambda_n}{1-\beta_n} - \frac{\lambda_{n+1}}{1-\beta_{n+1}}\right)\left(T(\frac{x_n+x_{n+1}}{2}) - Q(x_n)\right) \\
&\quad + \frac{1-\lambda_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}\left(T(\frac{x_{n+1}+x_{n+2}}{2}) - T(\frac{x_n+x_{n+1}}{2})\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leqslant \frac{c\lambda_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + \left|\frac{\lambda_n}{1-\beta_n} - \frac{\lambda_{n+1}}{1-\beta_{n+1}}\right| \|T(\frac{x_n+x_{n+1}}{2}) - Q(x_n)\| \\
&\quad + \frac{1-\lambda_{n+1}-\beta_{n+1}}{2(1-\beta_{n+1})}\left(\|x_{n+2} - x_{n+1}\| + \|x_{n+1} - x_n\|\right).
\end{aligned} \tag{3.3}$$

We evaluate $\|x_{n+2} - x_{n+1}\|$.

$$\|x_{n+2} - x_{n+1}\| = \|\lambda_{n+1}Q(x_{n+1}) + \beta_{n+1}x_{n+1} + \gamma_{n+1}T(\frac{x_{n+1}+x_{n+2}}{2})$$

$$
\begin{aligned}
&\quad -\left(\lambda_n Q(x_n)+\beta_n x_n+\gamma_n T(\frac{x_n+x_{n+1}}{2})\right)\|\\
&=\|\lambda_{n+1}\left(Q(x_{n+1})-Q(x_n)\right)+(\lambda_{n+1}-\lambda_n)Q(x_n)+\beta_{n+1}(x_{n+1}-x_n)\\
&\quad +(\beta_{n+1}-\beta_n)x_n+\gamma_{n+1}(T(\frac{x_{n+1}+x_{n+2}}{2})-T(\frac{x_n+x_{n+1}}{2}))\\
&\quad +(\gamma_{n+1}-\gamma_n)T(\frac{x_n+x_{n+1}}{2})\|\\
&=\|\lambda_{n+1}\left(Q(x_{n+1})-Q(x_n)\right)+(\lambda_{n+1}-\lambda_n)Q(x_n)+\beta_{n+1}(x_{n+1}-x_n)\\
&\quad +(\beta_{n+1}-\beta_n)x_n+((\lambda_n-\lambda_{n+1})+(\beta_n-\beta_{n+1}))T(\frac{x_n+x_{n+1}}{2})\|\\
&\quad +(1-\lambda_{n+1}-\beta_{n+1})(T(\frac{x_{n+1}+x_{n+2}}{2})-T(\frac{x_n+x_{n+1}}{2}))\\
&=\|\lambda_{n+1}\left(Q(x_{n+1})-Q(x_n)\right)+(\lambda_n-\lambda_{n+1})\left(T(\frac{x_n+x_{n+1}}{2})-Q(x_n)\right)\\
&\quad +\beta_{n+1}(x_{n+1}-x_n)+(\beta_{n+1}-\beta_n)\left(x_n-T(\frac{x_n+x_{n+1}}{2})\right)\\
&\quad +(1-\lambda_{n+1}-\beta_{n+1})(T(\frac{x_{n+1}+x_{n+2}}{2})-T(\frac{x_n+x_{n+1}}{2}))\|\\
&\leqslant c\lambda_{n+1}\|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|\left(\|T(\frac{x_n+x_{n+1}}{2})\|+\|Q(x_n)\|\right)\\
&\quad +\beta_{n+1}\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\;\|x_n-T(\frac{x_n+x_{n+1}}{2})\|\\
&\quad +\frac{1-\lambda_{n+1}-\beta_{n+1}}{2}\left(\|x_{n+2}-x_{n+1}\|+\|x_{n+1}-x_n\|\right).
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
\left(1-\frac{1-\lambda_{n+1}-\beta_{n+1}}{2}\right)\|x_{n+2}-x_{n+1}\| &\leqslant \left(c\lambda_{n+1}+\beta_{n+1}+\frac{1-\lambda_{n+1}-\beta_{n+1}}{2}\right)\|x_{n+1}-x_n\|\\
&\quad +|\lambda_n-\lambda_{n+1}|\left(\|T(\frac{x_n+x_{n+1}}{2})\|+\|Q(x_n)\|\right)\\
&\quad +|\beta_{n+1}-\beta_n|\;\|x_n-T(\frac{x_n+x_{n+1}}{2})\|.
\end{aligned}
$$

$$
\begin{aligned}
\frac{1+\lambda_{n+1}+\beta_{n+1}}{2}\|x_{n+2}-x_{n+1}\| &\leqslant \frac{1+\beta_{n+1}+2c\lambda_{n+1}-\lambda_{n+1}}{2}\|x_{n+1}-x_n\|\\
&\quad +|\lambda_n-\lambda_{n+1}|\left(\|T(\frac{x_n+x_{n+1}}{2})\|+\|Q(x_n)\|\right)\\
&\quad +|\beta_{n+1}-\beta_n|\;\|x_n-T(\frac{x_n+x_{n+1}}{2})\|.
\end{aligned}
$$

Let where $M=\sup\left\{\frac{1}{1-\beta_{n+1}}\max\left\{\sup\left\{\|T(\frac{x_n+x_{n+1}}{2})\|+\|Q(x_n)\|\right\},\ \sup\left\{\|x_n-T(\frac{x_n+x_{n+1}}{2})\|\right\}\right\}\right\}$.
It follows that

$$
\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leqslant \frac{1+\beta_{n+1}+2c\lambda_{n+1}-\lambda_{n+1}}{1+\lambda_{n+1}+\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{2|\beta_{n+1}-\beta_n|}{1+\lambda_{n+1}+\beta_{n+1}}\|x_n-T(\frac{x_n+x_{n+1}}{2})\|\\
&\quad +\frac{2|\lambda_n-\lambda_{n+1}|}{1+\lambda_{n+1}+\beta_{n+1}}\left(\|T(\frac{x_n+x_{n+1}}{2})\|+\|Q(x_n)\|\right)\\
&=\left(1-\frac{2\lambda_{n+1}(1-c)}{1+\lambda_{n+1}+\beta_{n+1}}\right)\|x_{n+1}-x_n\|+\frac{2|\beta_{n+1}-\beta_n|}{1+\lambda_{n+1}+\beta_{n+1}}\|x_n-T(\frac{x_n+x_{n+1}}{2})\| \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2|\lambda_n-\lambda_{n+1}|}{1+\lambda_{n+1}+\beta_{n+1}}\left(\|T(\frac{x_n+x_{n+1}}{2})\|+\|Q(x_n)\|\right)\\
&\leqslant \|x_{n+1}-x_n\|+\frac{2M}{1+\lambda_{n+1}+\beta_{n+1}}\left(|\lambda_n-\lambda_{n+1}|+|\beta_{n+1}-\beta_n|\right)(1-\beta_{n+1}).
\end{aligned}
$$

By substituting (3.4) into (3.3), we get

$$
\begin{aligned}
\|y_{n+1}-y_n\| &\leqslant \frac{c\lambda_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\left|\frac{\lambda_n}{1-\beta_n}-\frac{\lambda_{n+1}}{1-\beta_{n+1}}\right|M+\frac{1-\lambda_{n+1}-\beta_{n+1}}{2(1-\beta_{n+1})}(2\|x_{n+1}-x_n\|)\\
&\quad+\frac{1-\lambda_{n+1}-\beta_{n+1}}{2(1-\beta_{n+1})}\left(\frac{2M}{1+\lambda_{n+1}+\beta_{n+1}}\left(|\lambda_n-\lambda_{n+1}|+|\beta_{n+1}-\beta_n|\right)(1-\beta_{n+1})\right)\\
&=\frac{1-\beta_{n+1}+\lambda_{n+1}(c-1)}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\left|\frac{\lambda_n}{1-\beta_n}-\frac{\lambda_{n+1}}{1-\beta_{n+1}}\right|M\\
&\quad+M\left(\lambda_n-\lambda_{n+1}+|\beta_{n+1}-\beta_n|\right)\\
&=\left(1-\frac{\lambda_{n+1}(1-c)}{1-\beta_{n+1}}\right)\|x_{n+1}-x_n\|+M\left(|\lambda_n-\lambda_{n+1}|+|\beta_{n+1}-\beta_n|+\left|\frac{\lambda_n}{1-\beta_n}-\frac{\lambda_{n+1}}{1-\beta_{n+1}}\right|\right).
\end{aligned}
$$

Thus,

$$
\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\right)\leqslant 0.
$$

Hence, by Lemma 2.1, we have

$$
\lim_{n\to\infty}\|y_n-x_n\|=0.
$$

Step 3: We show that $\|x_n-Tx_n\|\to 0$ as $n\to\infty$.

We observe from (3.1) that

$$
x_{n+1}-x_n=\beta_n x_n+(1-\beta_n)y_n-x_n=(1-\beta_n)y_n-(1-\beta_n)x_n=(1-\beta_n)(y_n-x_n).
$$

Therefore

$$
\|x_{n+1}-x_n\|\leqslant(1-\beta_n)\|y_n-x_n\|\to 0 \text{ as } n\to\infty. \tag{3.5}
$$

Also, from (1.5), we obtain that

$$
\begin{aligned}
\|x_n-Tx_n\| &\leqslant \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\|\\
&=\|x_n-x_{n+1}\|+\lambda_n\|Q(x_n)-Tx_n\|+\beta_n\|x_n-Tx_n\|\\
&\quad+(1-\lambda_n-\beta_n)\|T(\frac{x_n+x_{n+1}}{2})-Tx_n\|\\
&=\|x_n-x_{n+1}\|+\lambda_n\|Q(x_n)-Tx_n\|+\beta_n\|x_n-Tx_n\|+(1-\lambda_n-\beta_n)\|\frac{x_n+x_{n+1}}{2}-x_n\|\\
&=\|x_n-x_{n+1}\|+\lambda_n\|Q(x_n)-Tx_n\|+\beta_n\|x_n-Tx_n\|+\frac{(1-\lambda_n-\beta_n)}{2}\|x_n-x_{n+1}\|.
\end{aligned}
$$

By (3.2), we obtain that

$$
\begin{aligned}
\|x_n-Tx_n\| &\leqslant \frac{3-\lambda_n-\beta_n}{2(1-\beta_n)}\|x_n-x_{n+1}\|+\frac{\lambda_n}{1-\beta_n}\|Q(x_n)-Tx_n\|\\
&\leqslant \frac{3-\lambda_n-\beta_n}{2(1-\beta)}\|x_n-x_{n+1}\|+\frac{\lambda_n}{1-\beta}\|Q(x_n)-Tx_n\|\to 0 \text{ as } n\to\infty.
\end{aligned} \tag{3.6}
$$

Step 4: For $t\in(0,1)$ and $Q\in\Pi_K$, define the sequence $\{x_t\}$ by $x_t=tQ(x_t)+(1-t)Tx_t$. By Lemma 2.3, x_t strongly converges to a fixed point q of T, which is also a solution to the variational inequality

$$
\langle (I-Q)q, j(x-q)\rangle\geqslant 0,\ x\in F(T).
$$

We show that $\limsup_{n\to\infty} \langle Q(q) - q, j(x_{n+1} - q)\rangle \leqslant 0$. It is generally known that a contraction map is continuous. Therefore, using Lemma 2.4 and since $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - Tx_n\| \to 0$ as $n \to 0$ (by (3.5) and (3.6), respectively), we get

$$\limsup_{n\to\infty} \langle Q(q) - q, j(x_{n+1} - q)\rangle \leqslant 0. \tag{3.7}$$

Step 5: Lastly, we prove that $x_n \to q$.

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \lambda_n \langle Q(x_n) - Q(q), j(x_{n+1} - q)\rangle + \lambda_n \langle Q(q) - q, j(x_{n+1} - q)\rangle \\
&\quad + \beta_n \langle x_n - q, j(x_{n+1} - q)\rangle + (1 - \lambda_n - \beta_n) \left\langle T(\frac{x_n + x_{n+1}}{2}) - q, j(x_{n+1} - q \right\rangle \\
&\leqslant c\lambda_n\|x_n - q\| \, \|x_{n+1} - q\| + \lambda_n \langle Q(q) - q, j(x_{n+1} - q)\rangle \\
&\quad + \beta_n\|x_n - q\| \, \|x_{n+1} - q\| + \frac{1 - \lambda_n - \beta_n}{2}(\|x_n - q\| + \|x_{n+1} - q\|)\|x_{n+1} - q\| \\
&= c\lambda_n\|x_n - q\| \, \|x_{n+1} - q\| + \lambda_n \langle Q(q) - q, j(x_{n+1} - q)\rangle \\
&\quad + \beta_n\|x_n - q\| \, \|x_{n+1} - q\| + \frac{1 - \lambda_n - \beta_n}{2}(\|x_n - q\| \, \|x_{n+1} - p\| + \|x_{n+1} - q\|^2) \\
&= \left(c\lambda_n + \beta_n + \frac{1 - \lambda_n - \beta_n}{2}\right) \|x_n - q\| \, \|x_{n+1} - q\| + \frac{1 - \lambda_n - \beta_n}{2}\|x_{n+1} - q\|^2 \\
&\quad + \lambda_n \langle Q(q) - q, j(x_{n+1} - q)\rangle \\
&\leqslant \frac{1 + \beta_n - (1 - 2c)\lambda_n}{4}(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + \frac{1 - \lambda_n - \beta_n}{2}\|x_{n+1} - q\|^2 \\
&\quad + \lambda_n \langle Q(p) - q, j(x_{n+1} - q)\rangle \\
&\leqslant \frac{1 + \beta_n - (1 - 2c)\lambda_n}{4}\|x_n - q\|^2 \\
&\quad + \frac{3 - \beta_n - (3 - 2c)\lambda_n}{4}\|x_{n+1} - q\|^2 + \lambda_n \langle Q(q) - q, j(x_{n+1} - q)\rangle .
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leqslant \frac{1 + \beta_n - (1 - 2c)\lambda_n}{1 + \beta_n + \lambda_n(3 - 2c)}\|x_n - q\|^2 + \frac{4\lambda_n}{1 + \beta_n + \lambda_n(3 - 2c)} \langle Q(q) - q, j(x_{n+1} - q)\rangle \\
&= \left(1 - \frac{4(1 - c)\lambda_n}{1 + \beta_n + \lambda_n(3 - 2c)}\right) \|x_n - q\|^2 \\
&\quad + \frac{4\lambda_n}{1 + \beta_n + \lambda_n(3 - 2c)} \langle Q(q) - q, j(x_{n+1} - q)\rangle .
\end{aligned} \tag{3.8}$$

By applying Lemma 2.2 to (3.7) and (3.8), we deduce that $x_n \to q$ as $n \to \infty$. □

Remark 3.4. Our result extends and improves the results of Luo et al. [15] and Yao et al. [30] which are stated below.

Corollary 3.5 ([15])**.** *Let K be a closed convex subset of a uniformly smooth Banach space E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and $Q : K \to K$ a contraction with coefficient $c \in [0, 1)$. Let $\{x_n\}$ be a sequence generated by the following viscosity implicit midpoint rule:*

$$x_{n+1} = \lambda_n Q(x_n) + (1 - \lambda_n)T\left(\frac{x_n + x_{n+1}}{2}\right), n \geqslant 0,$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ such that it satisfies the satisfies the conditions (A_1), (A_2), and either (A_4) or $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n-1}} = 1$. Then $\{x_n\}$ converges strongly to a fixed point p of T, which also solves the variational inequality

$$\langle (I - Q)p, j(x - p)\rangle \geqslant 0, \ \forall\, x \in F(T).$$

Proof. Take $\beta_n = 0$ in equation (1.5). Then since $\lambda_n + \beta_n + \gamma_n = 1$, $\forall\, n \in \mathbb{N}$, we have that $\gamma_n = 1 - \lambda_n$. Consequently, equation (1.5) becomes

$$x_{n+1} = \lambda_n Q(x_n) + (1 - \lambda_n) T\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N}.$$

Hence, the result follows from Theorem 3.3 by taking $\beta_n = 0$. □

Corollary 3.6. *The result of Xu et al.* [29] *is also obtained as a corollary by considering a Hilbert space in Corollary 3.5.*

Corollary 3.7 ([30])**.** *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $Q : K \to K$ be a c-contraction. For given $x_0 \in K$ arbitrarily, let the sequence $\{x_n\}$ be generated by the manner*

$$x_{n+1} = \lambda_n Q(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \geqslant 0, \tag{3.9}$$

where $\{\lambda_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$, and $\{\gamma_n\} \subset (0, 1)$ are three sequences satisfying $\lambda_n + \beta_n + \gamma_n = 1$ for all $n \geqslant 0$. Assume that $\{\lambda_n\}$ satisfies (A_1) and (A_2) and $\{\beta_n\}$ satisfies

(A_6) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$, *and*
(A_8) $\lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0$.

Then the sequence $\{x_n\}$ generated by (3.9) converges strongly to $p = P_{F(T)}Q(p)$.

4. Applications

4.1. Accretive mappings

Recall that a nonlinear mapping $A : K \to E$ is called accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geqslant 0 \;\forall\, x, y \in K.$$

A is m-accretive if $R(I + rA) = E$ for all $r > 0$, where I is the identity operator. The set of zeros of A is denoted by $A^{-1}(0)$, that is $A^{-1}(0) = \{z \in D(A) : 0 \in A(z)\}$. We denote the resolvent of A by $J_r^A = (I + rA)^{-1}$ for each $r > 0$. It is known that if A is m-accretive then $J_r^A : E \to E$ is nonexpansive and $F(J_r^A) = A^{-1}(0)$ for each $r > 0$. Consequently, we can deduce the result below from Theorem 3.3.

Theorem 4.1. *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E and $Q : K \to K$ be a c-contraction. Let $A : K \to E$ be an accretive mapping such that $R(I + rA) = E$ for all $r > 0$ with $A^{-1}(0) \neq \emptyset$. Suppose $\{\lambda_n\}$ satisfies (A_1) and (A_2) and $\{\beta_n\}$ satisfies*

(A_6) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$, *and*
(A_7) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$.

For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}$ by

$$x_{n+1} = \lambda_n Q(x_n) + \beta_n x_n + \gamma_n J_r^A\left(\frac{x_n + x_{n+1}}{2}\right), n \in \mathbb{N},$$

where $\{\lambda_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$, and $\{\gamma_n\} \subset (0, 1)$ are real sequences satisfying $\lambda_n + \beta_n + \gamma_n = 1$, $\forall\, n \in \mathbb{N}$. Then as $n \to \infty$, the sequence $\{x_n\}$ converges in norm to $p \in A^{-1}(0)$, where p is the unique solution to the variational inequality:

$$\langle (I - Q)p, j(x - p)\rangle \geqslant 0,\; \forall\, x \in A^{-1}(0).$$

4.2. Variational inequality problems

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$ and let K be a nonempty closed convex subset of H. The variational inequality problem is finding $x^* \in K$ such that

$$\langle Ax^*, x - x^* \rangle \geqslant 0 \text{ for all } x \in K. \tag{4.1}$$

We denote the set of all solutions of the variational inequality (4.1) by $VI(C, A)$. We shall consider the system of general variational inequalities in Banach spaces recently introduced by Katchang and Kumam [14]. Given two operators $A_1, A_2 : K \to E$, where E is a real Banach space, where K is a nonempty closed convex subset E. The authors considered the problem of finding $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} \langle \alpha_1 A_1 y^* + x^* - y^*, j(x - x^*) \rangle \geqslant 0, & \forall\, x \in K, \\ \langle \alpha_2 A_2 y^* + x^* - y^*, j(x - x^*) \rangle \geqslant 0, & \forall\, x \in K, \end{cases} \tag{4.2}$$

where α_1 and α_2 are two positive real numbers. Recall that a nonlinear mapping $A : K \to E$ is called μ-inverse strongly accretive if there exist $j(x - y) \in J(x - y)$ and $\mu > 0$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geqslant \mu \|Ax - Ay\|^2, \ \forall\, x, y \in K.$$

We need the two Lemmas below to establish our next result.

Lemma 4.2 ([12])**.** *Let K be a nonempty closed convex subset of a real Banach space E and let $\alpha_1, \alpha_2 > 0$ and $A_1, A_2 : K \to E$ be two mappings. Let $G : K \to K$ be defined by*

$$G(x) = S_K[S_K(x - \alpha_2 A_2 x) - \alpha_1 A_1 S_K(x - \alpha_2 A_2 x)], \ \forall\, x \in K,$$

where S_K is a sunny nonexpansive retraction from E onto K. If $I - \alpha_1 A_1$ and $I - \alpha_2 A_2$ are nonexpansive mappings, then G is nonexpansive.

Lemma 4.3 ([14])**.** *Let K be a nonempty closed convex subset of a real smooth Banach space E. Let S_K be the sunny nonexpansive retraction from E onto K. Let $A_1, A_2 : K \to E$ be two possibly nonlinear mappings. For given $x^*, y^* \in K$, (x^*, y^*) is a solution of problem (4.2) if and only if $x^* = S_K(y^* - \alpha_1 A_1 y^*)$, where $y^* = S_K(x^* - \alpha_2 A_2 x^*)$.*

Remark 4.4. Observe that Lemma 4.3 implies that

$$x^* = S_K[S_K(x^* - \alpha_2 A_2 x^*) - \alpha_1 A_1 S_K(x^* - \alpha_2 A_2 x^*)].$$

That is, x^* is a fixed point of the mapping G, defined in Lemma 4.2. Thus, we can conclude the result below from Theorem 3.3.

Theorem 4.5. *Let K be a nonempty closed convex subset of a 2-uniformly smooth Banach space E and $Q : K \to K$ be a c-contraction. Let $A_1, A_2 : K \to E$ be two possibly nonlinear mappings and G be a mapping defined in Lemma 4.2 with $F(G) \neq \emptyset$. Let S_K be a sunny nonexpansive retraction from E onto K. Suppose $\{\lambda_n\}$ satisfies (A_1) and (A_2) and $\{\beta_n\}$ satisfies*

(A_6) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$, *and*
(A_7) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$.

For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}$ by

$$\begin{cases} x_{n+1} = \lambda_n Q(x_n) + \beta_n x_n + \gamma_n y_n, \\ y_n = S_K(u_n - \alpha_1 A_1 u_n), \\ u_n = S_K(v_n - \alpha_2 A_1 v_n), \\ v_n = \frac{x_n + x_{n+1}}{2}, \end{cases}$$

where $\{\lambda_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$, and $\{\gamma_n\} \subset (0, 1)$ are real sequences satisfying $\lambda_n + \beta_n + \gamma_n = 1$, $\forall\, n \in \mathbb{N}$. Then as $n \to \infty$, the sequence $\{x_n\}$ converges in norm to a fixed point p of G, where p is the unique solution to the variational inequality:

$$\langle (I - Q)p, j(x - p) \rangle \geqslant 0, \ \forall\, x \in F(G).$$

Remark 4.6. Nonlinear mappings that satisfy Theorem 4.5 are readily available. Let L be the 2-uniformly smooth constant of a 2-uniformly smooth Banach space and $A_1, A_2 : K \to E$ be μ_1-inverse strongly accretive and μ_2-inverse strongly accretive, respectively. If $0 < \alpha_1 < \frac{\mu_1}{L^2}$ and $0 < \alpha_2 < \frac{\mu_2}{L^2}$, then $I - \alpha_1 A_1$ and $I - \alpha_2 A_2$ are nonexpansive [12].

5. Numerical examples

Example 5.1. Let $\mathbb{R}$ be the real line with the Euclidean norm. Let $Q, T : \mathbb{R} \to \mathbb{R}$ be maps defined by $Q(x) = \frac{1}{4}x$ and $T(x) = 2 - x$ for all $x \in \mathbb{R}$, respectively. It is obvious that T is a nonexpansive mapping and $F(T) = \{1\}$. Let $\{z_n\}, \{y_n\}$, and $\{x_n\}$ be the sequences generated by (1.3), (1.4), and (1.5) respectively. We find that $\{z_n\}, \{y_n\}$, and $\{x_n\}$ strongly converge to 1 (by [16], Theorem 1.1 of [29], and Theorem 3.3, respectively). Take $\lambda_n = \frac{2}{4n+5}$, $n \in \mathbb{N}$ in (1.3) and (1.4). Notice that the parameters in (1.5) are arbitrary sequences satisfying the conditions stated in Theorem 3.3. Therefore, the sequence $\{\lambda_n\}$ in (1.5) is not necessarily the same as the one in (1.3) and (1.4). Thus, for the iterative scheme defined by (1.5), we choose $\lambda_n = \frac{1}{4n+5}, \beta_n = \frac{n+4}{4n+5}$ and $\gamma_n = \frac{3n}{4n+5}$ for all $n \in \mathbb{N}$. One can rewrite (1.3), (1.4), and (1.5) respectively as follows

$$z_{n+1} = \frac{2n+1}{2n+3} z_n + \frac{2}{2n+3}, \tag{5.1}$$

$$y_{n+1} = -\frac{4n+2}{12n+13} y_n + \frac{4(4n+3)}{12n+13}, \tag{5.2}$$

$$x_{n+1} = \frac{17-2n}{2(11n+10)} x_n + \frac{12n}{11n+10}. \tag{5.3}$$

Using Matlab 2015a and by taking $z_1 = y_1 = x_1 = 0$, the results for (5.1), (5.2), and (5.3) are displayed in Table 1 and Figure 1. The graphs show that the three algorithms converge to 1 with the iterative algorithm (1.5) having the highest rate of convergence for the viscosity implicit midpoint rule. Therefore, it is the most efficient among the three algorithms.

Remark 5.2. It is worth of mentioning that the efficiency of (1.5) depends on the choice of suitable control parameters.

The next example displays the result where λ_n is the same for all the three iterative schemes.

Example 5.3. Let Q and T be as defined in Example 5.1. Then for the iterative scheme defined by (1.5), choose $\lambda_n = \frac{2}{4n+5}, \beta_n = \frac{n+1}{4n+5}$ and $\gamma_n = \frac{3n+2}{4n+5}$ for all $n \in \mathbb{N}$. The equation (5.3) then becomes

$$x_{n+1} = \frac{1-n}{2(11n+12)} x_n + \frac{4(3n+2)}{11n+12}.$$

The results are presented in Figure 2 and Table 2 with the algorithm (1.4) having the highest rate of convergence.

The next example compares the convergence rate where where λ_n is greater for (1.5).

Example 5.4. Let Q and T be as defined in Example 5.1. Then for the iterative scheme defined by (1.5), choose $\lambda_n = \frac{4}{4n+5}, \beta_n = \frac{n+1}{4n+5}$, and $\gamma_n = \frac{3n}{4n+5}$ for all $n \in \mathbb{N}$. The equation (5.3) then becomes

$$x_{n+1} = \frac{4-n}{2(11n+10)} x_n + \frac{12n}{11n+10}.$$

The results are presented in Figure 3 and Table 3 with the algorithm (1.4) having the highest rate of convergence.

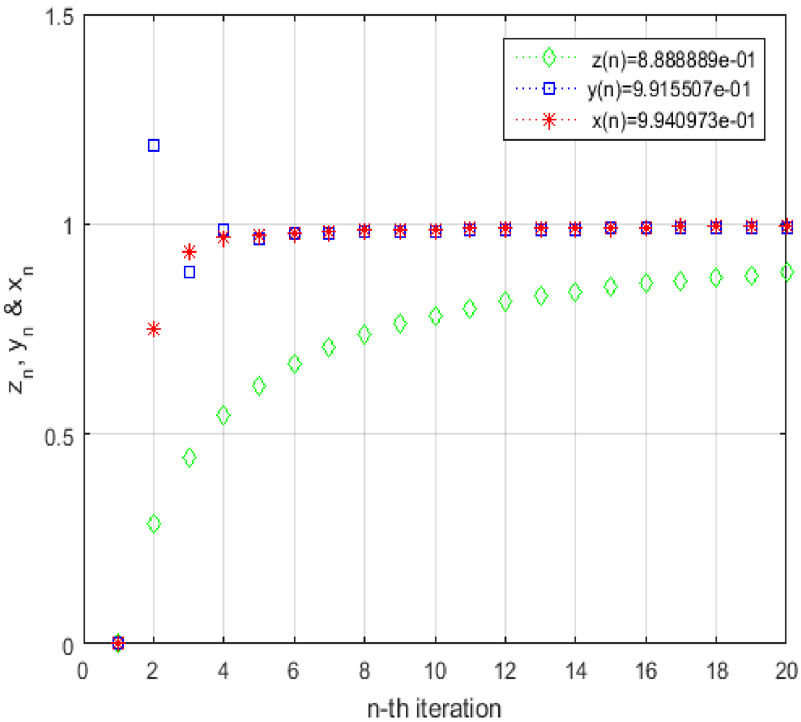

Figure 1: Comparison of the rates of convergence for the iterative schemes (1.3), (1.4), and (1.5) with different values for λ_n.

Table 1: Comparison of the rates of convergence for the iterative schemes (1.3), (1.4), and (1.5) with different values for λ_n.

iteration (n)	z_n e-01	y_n e-01	x_n e-01
1	0	0	0
2	4.444444	8.847215	9.331395
3	5.454545	9.848363	9.666505
4	6.153846	9.63474	9.751273
5	6.666667	9.758785	9.794450
6	7.058824	9.765324	9.824042
7	7.368421	9.797972	9.846041
8	7.619048	9.815513	9.863092
9	7.826087	9.832695	9.876711
10	8.000000	9.846179	9.887849
11	8.148148	9.857905	9.897130
12	8.275862	9.867888	9.904986
13	8.387097	9.876588	9.911723
14	8.484848	9.884205	9.917565
15	8.571429	9.890939	9.922679
16	8.648649	9.896932	9.927194
17	8.717949	9.902301	9.931210
18	8.780488	9.907139	9.934805
19	8.837209	9.911520	9.938042
20	8.888889	9.915507	9.940973

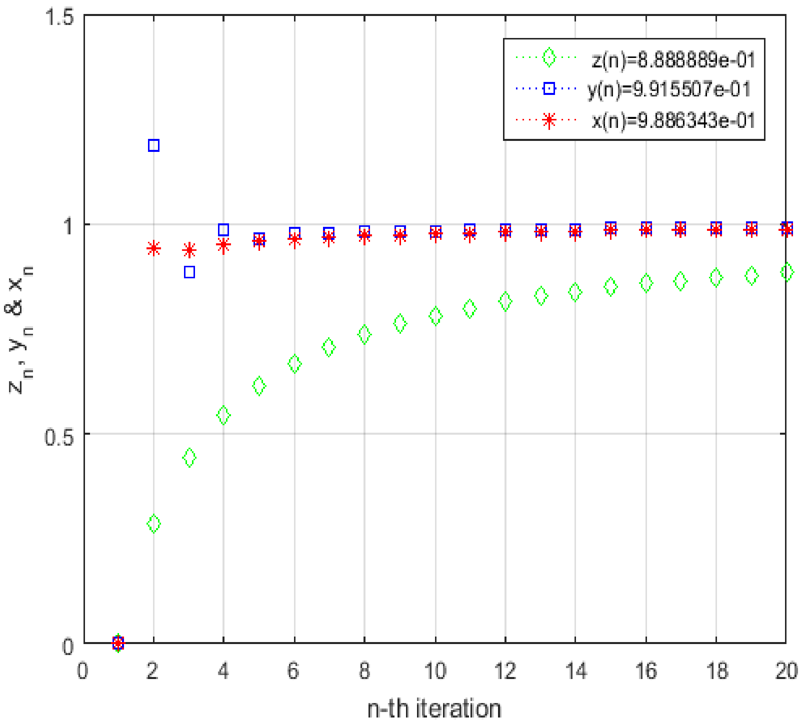

Figure 2: Comparison of the rates of convergence for the iterative schemes (1.3), (1.4), and (1.5) with same value for λ_n.

Table 2: Comparison of the rates of convergence for the iterative schemes (1.3), (1.4), and (1.5) with same value for λ_n.

iteration (n)	z_n e-01	y_n e-01	x_n e-01
1	0	0	0
2	4.444444	8.847215	9.359477
3	5.454545	9.848363	9.498599
4	6.153846	9.63474	9.582173
5	6.666667	9.758785	9.642168
6	7.058824	9.765324	9.687045
7	7.368421	9.797972	9.721907
8	7.619048	9.815513	9.749772
9	7.826087	9.832695	9.772558
10	8.000000	9.846179	9.791537
11	8.148148	9.857905	9.807591
12	8.275862	9.867888	9.821348
13	8.387097	9.876588	9.833268
14	8.484848	9.884205	9.843696
15	8.571429	9.890939	9.852897
16	8.648649	9.896932	9.861074
17	8.717949	9.902301	9.868389
18	8.780488	9.907139	9.874973
19	8.837209	9.911520	9.880929
20	8.888889	9.915507	9.886343

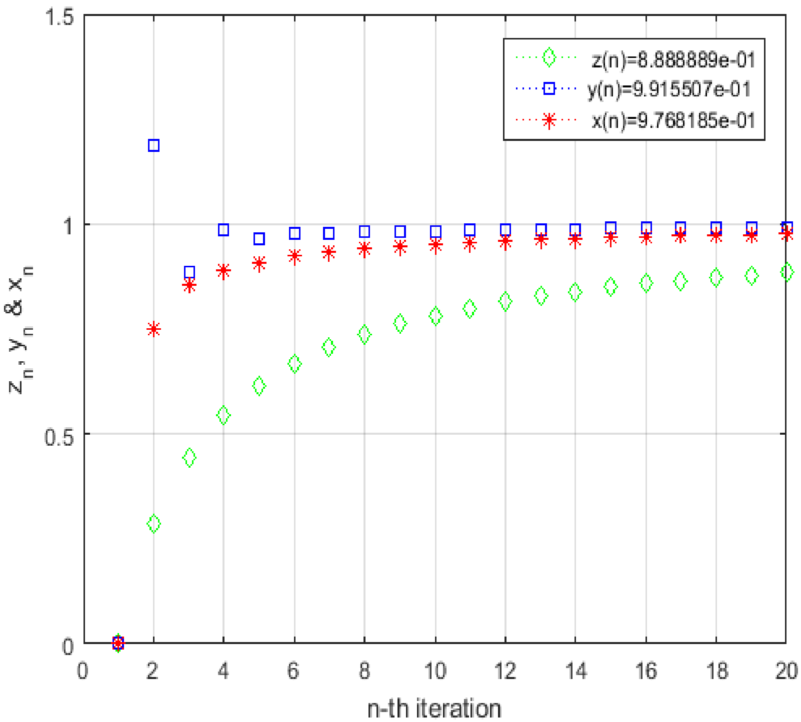

Figure 3: Comparison of the rates of convergence for the iterative schemes (1.3), (1.4), and (1.5) where λ_n is greater for (1.5).

Table 3: Comparison of the rates of convergence for the iterative schemes (1.3), (1.4), and (1.5) where λ_n is greater for (1.5).

iteration (n)	z_n e-01	y_n e-01	x_n e-01
1	0	0	0
2	4.444444	8.847215	8.546512
3	5.454545	9.848363	8.888889
4	6.153846	9.63474	9.094017
5	6.666667	9.758785	9.234368
6	7.058824	9.765324	9.336746
7	7.368421	9.797972	9.414827
8	7.619048	9.815513	9.476384
9	7.826087	9.832695	9.526181
10	8.000000	9.846179	9.567303
11	8.148148	9.857905	9.601842
12	8.275862	9.867888	9.631264
13	8.387097	9.876588	9.656630
14	8.484848	9.884205	9.678726
15	8.571429	9.890939	9.698147
16	8.648649	9.896932	9.715351
17	8.717949	9.902301	9.730698
18	8.780488	9.907139	9.744473
19	8.837209	9.911520	9.756906
20	8.888889	9.915507	9.768185

Example 5.5. Let $E = \mathbb{R}^2$ with the usual norm and $Q, T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Q(x) = \frac{1}{2}x$ and $T(x) = 0$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, respectively. Take $\lambda_n = \frac{4}{4n+5}$, $\beta_n = \frac{1}{4} - \frac{1}{4n+5}$, and $\gamma_n = \frac{12n+3}{4(4n+5)}$ for all $n \in \mathbb{N}$. Observe that λ_n, β_n, and γ_n satisfy the conditions of Theorem 3.3 and T is nonexpansive. Indeed, for $x, y \in \mathbb{R}^2$

$$\|Tx - Ty\| = 0 \leqslant \|x - y\|.$$

Also, it is obvious that $F(T) = \{0\}$. Therefore, $\{x_n\}$ strongly converges to 0. A simple computation shows that (1.5) is equivalent to:

$$x_{n+1} = \frac{4n+9}{4(4n+5)} x_n. \tag{5.4}$$

Choosing the initial point for (5.4) to be $(1.0, 1.2)$, Table 4 and Figure 4 show the results from the Matlab 2015a.

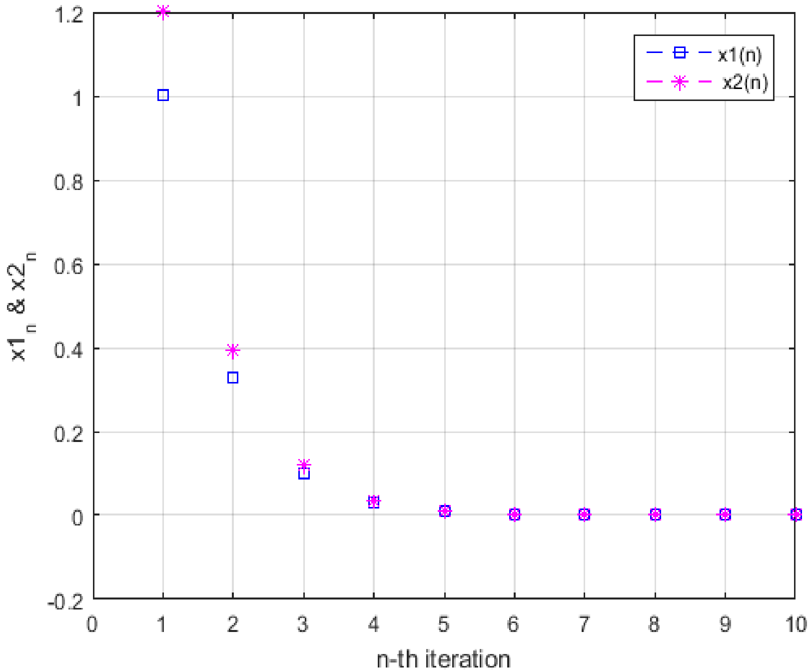

Figure 4: Two dimensional figure for (5.4).

Table 4: Values of iteration for (5.4).

iteration (n)	$x_1(n)$	$x_2(n)$
1	1.0	1.2
2	1.009615e-01	1.211538e-01
3	3.004808e-02	3.605769e-02
4	8.713942e-03	1.045673e-02
5	2.478966e-03	2.974760e-03
6	6.948618e-04	8.338341e-04
7	1.924955e-04	2.309946e-04
8	5.281889e-05	6.338266e-05
9	1.437847e-05	1.725417e-05
10	3.888057e-06	4.665668e-06

Conclusion 5.6. We have considered the implicit midpoint rule of nonexpansive mappings, using the viscosity approximation method in the framework of Banach spaces. Our method of proof is of independent interest and our result is an improvement on the existing results in the literature. The numerical examples show the application of our work and the efficiency of the algorithm over the existing ones. Moreover, we obtained the results of Xu et al. [29], Yao et al. [30], and Luo et al. [15] as corollaries. It is observed that the iterative scheme (1.5) converges faster than (1.4) with the following two conditions:

(i) the value of λ_n in (1.5) is less than the value of λ_n in (1.4);
(ii) the sum of values of λ_n and γ_n in (1.5) is greater than the value of λ_n in (1.4).

Open question

Can the implicit midpoint rule be applied to approximate a fixed point of non-affine nonexpansive mappings such as sin x? For instance, taking $T(x) = \sin x$ in the implicit midpoint rule, a simplest form of the equation in $\mathbb{R}$ which one would obtain is

$$y = x + \sin(x + y),$$

where y is to be made the subject of the formula in order to get an explicit equation like (5.1), (5.2), or (5.3).

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