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Stability of a fractional difference equation of high order

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Abstract

In this paper we investigate the local stability, global stability, and boundedness of solutions of the recursive sequence

$$x_{n+1} = x_{n-p}\left(\frac{2\,x_{n-q} + a\,x_{n-r}}{x_{n-q} + a\,x_{n-r}}\right),$$

where $x_{-q+k} \neq -a\,x_{-r+k}$ for $k = 0, 1, \ldots, \min(q, r), a \in \mathbb{R}$, $p, q, r \geqslant 0$ with the initial condition $x_{-p}, x_{-p+1}, \ldots, x_{-q}, x_{-q+1}, \ldots, x_{-r}, x_{-r+1}, \ldots, x_{-1}$ and $x_0 \in (0, \infty)$. Some numerical examples will be given to illustrate our results.

Keywords: Difference equations, prime period two solution, boundedness character, locally asymptotically stable, global attractor, global stability, high orders.

2010 MSC: 39A10, 39A11, 39A99, 34C99.

1. Introduction

Theory of difference equations or discrete dynamical systems are diverse field which impacts almost every branch of pure and applied mathematics. Every dynamical system $a_{n+1} = f(a_n)$ determines a difference equation and vise versa. Recently, there has been great interest in studying difference equations. One of the reasons for this is a necessity for some techniques whose can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology, etc..

Recently there has been a lot of interest in studying the boundedness character and the periodic nature of nonlinear difference equations. Difference equations have been studied in various branches of mathematics for a long time. First results in qualitative theory of such systems were obtained by Poincaré and Perron at the end of the nineteenth and the beginning of twentieth centuries.

Many researchers have investigated the behavior of the solution of difference equations for example: in [3] Elabbasy et al. investigated the global stability, periodicity character and gave the solution of special

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case of the following recursive sequence

$$x_{n+1} = ax_n - \frac{b\, x_n}{cx_n - dx_{n-1}}.$$

Agarwal et al. [1] studied the solution of fourth-order rational recursive sequence

$$x_{n+1} = a\, x_n + \frac{b\, x_n\, x_{n-3}}{c\, x_{n-2} + d\, x_{n-3}}.$$

Elsayed [9]investigated the global stability and boundedness of solutions of the recursive sequence

$$x_{n+1} = a\, x_{n-1} + \frac{b\, x_{n-1}\, x_n}{c\, x_n + d\, x_{n-2}}.$$

For other important references, we refer the reader to [2, 4–8, 10, 12, 18].

Our goal in this paper is to investigate the local, global stability, and boundedness of solutions of the high order recursive sequence

$$x_{n+1} = x_{n-p} \left(\frac{2\, x_{n-q} \ + a\, x_{n-r}}{x_{n-q} \ + a\, x_{n-r}} \right), \tag{1.1}$$

where $x_{-q+k} \neq -a\; x_{-r+k}$ for $k = 0, 1, \ldots, \min(q, r)$, $a \in \mathbb{R}$, $p, q, r \geqslant 0$ with the initial condition $x_{-p}, x_{-p+1}, \ldots, x_{-q}, x_{-q+1}, \ldots, x_{-r}, x_{-r+1}, \ldots, x_{-1}$ and $x_0 \in (0, \infty)$.

Here, we recall some notations and results which will be useful in our investigation.

Definition 1.1. Let

$$F : I^{k+1} \to I,$$

where F is a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \ldots,$ x_{-1}, $x_0 \in I$ the difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$. An equilibrium point $\widetilde{x}$ of (1.2) is a point that satisties the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x}, \ldots, \widetilde{x})$. That is, the constant sequence $\{x_n\}$ with $x_n = \widetilde{x}$ for all $n \geqslant 0$ is a solution of (1.2) or equivalently, $\widetilde{x}$ is a fixed point of F.

Definition 1.2. Let $\widetilde{x} \in I$ be an equilibrium point of (1.2). Then we have:

(i) An equilibrium point $\widetilde{x}$ of (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-k}, x_{-k+1}, \ldots, x_{-1}$, $x_0 \in I$ with $|x_{-k} - \widetilde{x}| + |x_{-k+1} - \widetilde{x}| + \cdots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geqslant -k$.

(ii) An equilibrium point $\widetilde{x}$ of (1.2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-k}, x_{-k+1}, \ldots, x_{-1}$, $x_0 \in I$ with $|x_{-k} - \widetilde{x}| + |x_{-k+1} - \widetilde{x}| + \cdots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then
$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of (1.2) is called a global attractor if for every $x_{-k}, \ldots, x_{-1}$, $x_0 \in (0, \infty)$ we have
$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of (1.2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$ of (1.2) is called unstable if it is not locally stable.

Definition 1.3. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period r if $x_{n+r} = x_n$ for all $n \geqslant -p$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period r if r is the smallest positive integer having this property.

Definition 1.4. Equation (1.2) is called permanent and bounded if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ there exists a positive integer N which depends on these initial conditions such that $m \leqslant x_n \leqslant M$ for all $n \geqslant N$.

Definition 1.5. The linearized equation of (1.2) about the equilibrium point $\widetilde{x}$ is defined by the equation

$$z_{n+1} = \rho_0 z_n + \rho_1 z_{n-1} + \rho_2 z_{n-2} + \rho_3 z_{n-3} + \dots = 0,$$

where

$$\rho_0 = \frac{\partial F(\widetilde{x},\widetilde{x},\dots,\widetilde{x})}{\partial x_n},\ \rho_1 = \frac{\partial F(\widetilde{x},\widetilde{x},\dots,\widetilde{x})}{\partial x_{n-1}},\ \rho_2 = \frac{\partial F(\widetilde{x},\widetilde{x},\dots,\widetilde{x})}{\partial x_{n-2}},\ \rho_3 = \frac{\partial F(\widetilde{x},\widetilde{x},\dots,\widetilde{x})}{\partial x_{n-3}},\dots.$$

Definition 1.6 (Fibonacci sequence)**.** The sequence $\{F_m\}_{m=0}^{\infty} = \{1, 2, 3, 5, 8, 13, \dots\}$, i.e., $F_m = F_{m-1} + F_{m-2}$, $m \geqslant 0$, $F_{-2} = 0$, $F_{-1} = 1$ is called Fibonacci sequence.

Theorem 1.7 ([11])**.** *Assume that* $p_i \in R$, $i = 1, 2, \dots, k$. *Then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \dots + p_k x_n = 0, \quad n = 0, 1, 2, \dots.$$

Theorem 1.8 ([11])**.** *Let* $g : [a, b]^{k+1} \to [a, b]$ *be a continuous function, where* k *is a positive integer, and where* $[a, b]$ *is an interval of real numbers. Consider the difference equation* (1.2)*. Suppose that* F *satisfies the following conditions:*

1. *For each integer* i *with* $1 \leqslant i \leqslant k+1$, *the function* $F(z_1, z_2, \dots, z_{k+1})$ *is weakly monotonic in* z_i *for fixed* $z_1, z_2, \dots, z_{i-1}, z_{i+1}, \dots, z_{k+1}$.
2. *If* (m, M) *is a solution of the system*

$$m = F(m_1, m_2, \dots, m_{k+1}) \quad \textit{and} \quad M = F(M_1, M_2, \dots, M_{k+1}),$$

then $m = M$, *where for each* $i = 1, 2, \dots, k+1$, *we set*

$$m_i = \begin{cases} m & \textit{if } F \textit{ is non-decreasing in } z_i, \\ M & \textit{if } F \textit{ is non-increasing in } z_i, \end{cases} \quad \textit{and} \quad M_i = \begin{cases} M & \textit{if } F \textit{ is non-decreasing in } z_i, \\ m & \textit{if } F \textit{ is non-increasing in } z_i. \end{cases}$$

Then there exists exactly one equilibrium $\overline{x}$ *of* (1.2)*, and every solution of* (1.2) *converges to* $\widetilde{x}$.

2. The local stability of the solutions

In this section, we investigate the local stability character of the solutions of (1.1). Equation (1.1) has a unique equilibrium point and is given by

$$\widetilde{x} = \widetilde{x}\left(\frac{2\,\widetilde{x} + a\,\widetilde{x}}{\widetilde{x} + a\,\widetilde{x}}\right), \quad \text{or,} \quad \widetilde{x}\left(1 - \frac{a+2}{a+1}\right) = 0,$$

if $a \neq -1$, then the only positive equilibrium point of (1.1) is $\widetilde{x} = 0$.

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a function defined by

$$f(u, v, w) = u \left(\frac{2v + a w}{v + a w} \right).$$

Therefore it follows that

$$f_u(u, v, w) = \frac{2v + a w}{v + a w}, \quad f_v(u, v, w) = \frac{auw}{(v + a w)^2}, \quad f_w(u, v, w) = \frac{-auv}{(v + a w)^2},$$

we see that

$$\begin{cases} f_u(\widetilde{x}, \widetilde{x}, \widetilde{x}) = \frac{a+2}{a+1} = -a_2, \\ f_v(\widetilde{x}, \widetilde{x}, \widetilde{x}) = \frac{a}{(a+1)^2} = -a_1, \\ f_w(\widetilde{x}, \widetilde{x}, \widetilde{x}) = \frac{-a}{(a+1)^2} = -a_0. \end{cases}$$

The linearized equation of (1.1) about $\widetilde{x}$ is

$$y_{n+1} + a_0 y_{n-r} + a_1 y_{n-q} + a_2 y_{n-p} = 0. \tag{2.1}$$

The characteristic equation of the linearized equation (2.1) is

$$\lambda^{n+1} + a_0 \lambda^{n-r} + a_1 \lambda^{n-q} + a_2 \lambda^{n-p} = 0.$$

Theorem 2.1. *Assume that*

$$a < \frac{-1}{3}.$$

Then the equilibrium point of (1.1) *is locally asymptotically stable.*

Proof. It is follows by Theorem 1.7 that, (2.1) is asymptotically stable if

$$\left| \frac{a+2}{a+1} \right| + \left| \frac{a}{(a+1)^2} \right| + \left| \frac{-a}{(a+1)^2} \right| < 1,$$

or,

$$\frac{a+2}{a+1} + \frac{2a}{(a+1)^2} < 1,$$

and so,

$$a < \frac{-1}{3}.$$

Thus, the proof is now completed. □

3. Boundedness of the solutions

In this section, we investigate the boundedness of the positive solutions of (1.1).

Theorem 3.1. *Every solution of* (1.1) *is bounded from above by*

$$M = \max\{3x_{-(p+1)}, 3x_{-p}, \ldots, 3x_{-2}, 3x_{-1}, 3x_0\}.$$

Proof. Let $\{x_n\}_{n=-p}^{\infty}$ be a solution of (1.1). It follows from (1.1) that

$$\begin{aligned} x_{n+1} &= x_{n-p} \left\{ \frac{2x_{n-q} + a x_{n-r}}{x_{n-q} + a x_{n-r}} \right\} \\ &= x_{n-p} \left\{ \frac{2x_{n-q}}{x_{n-q} + a x_{n-r}} + \frac{a x_{n-r}}{x_{n-q} + a x_{n-r}} \right\} \\ &\leqslant x_{n-p} \left\{ \frac{2x_{n-q}}{x_{n-q}} + \frac{a x_{n-r}}{a x_{n-r}} \right\} \end{aligned}$$

$$\begin{aligned}&\leqslant x_{n-p}\left\{\frac{2\,x_{n-q}}{x_{n-q}}+\frac{a\,x_{n-r}}{a\,x_{n-r}}\right\}\\&\leqslant 3x_{n-p}.\end{aligned}$$

Then

$$x_{n+1}\leqslant 3x_{n-p}\quad \text{for all}\quad n\geqslant 0.$$

Then all the subsequences $\{3x_{(p+1)\ n\ +j}\}_{n=0}^{\infty}$ where $j=0,1,...,p$ are decreasing and so are bounded from above by $M=\max\{3x_{-(p+1)},3x_{-p},\dots,3x_{-2},3x_{-1},3x_0\}$. Thus, the proof is now completed. □

Theorem 3.2. *Every solution of* (1.1) *is bounded from down by*

$$M=\min\{x_{-(p+1)},x_{-p},\dots,x_{-2},x_{-1},x_0\}.$$

Proof. Let $\{x_n\}_{n=-p}^{\infty}$ be a solution of (1.1). Then from (1.1) we see that

$$x_{n+1}=x_{n-p}\left\{\frac{2\,x_{n-q}\ +a\,x_{n-r}}{x_{n-q}\ +a\,x_{n-r}}\right\}=x_{n-p}\left\{\frac{x_{n-q}}{x_{n-q}\ +a\,x_{n-r}}+1\right\}>x_{n-p}\ \text{for all}\ n\geqslant 1.$$

We can see that the right hand side can be written as

$$y_{n+1}=y_{n-p}.$$

So

$$\lim_{n\to\infty}y_n=\text{constant}.$$

Thus every solution is bounded from down by

$$M=\min\{x_{-(p+1)},x_{-p},\dots,x_{-2},x_{-1},x_0\}.$$ □

Remark 3.3. From Theorems 3.1 and 3.2 we can easily see that every solution of (1.1) is bounded.

4. Periodic solutions

In this section, we study the existence of periodic solutions of (1.1). The following theorem states the necessary and sufficient conditions that the equation has periodic solutions of prime period two for all cases depending on p, q, and r.

Theorem 4.1. *If one of the following conditions holds, then* (1.1) *has no positive solutions of prime period two:*

(1) *The positive integers p,q, and r are odd.*
(2) *The positive integers p,q, and r are even.*
(3) *The positive integers p,q are even and the positive integer r is odd.*
(4) *The positive integers p,r are even and the positive integer q is odd.*
(5) *The positive integers q,r are even and the positive integer p is odd.*
(6) *The positive integer p is even and the positive integers q,r are odd.*
(7) *The positive integer q is even and the positive integers p,r are odd.*
(8) *The positive integer r is even and the positive integers p,q are odd.*

Proof. We will prove case (1) and the other cases (2)-(8) come by the same way. Assume for the sake of contradiction that there exists distinctive positive real numbers $\Phi,\Psi\in(0,\infty)$ such that

$$\dots,\Phi,\Psi,\Phi,\Psi,\dots,$$

is a prime period two solution of (1.1). Consider p, q, and r are odd. In this case $x_{n+1} = x_{n-p} = x_{n-q} = x_{n-r}$. From (1.1) we have

$$\Phi = \Phi\left(\frac{2\Phi + a\,\Phi}{\Phi + a\,\Phi}\right), \quad \Psi = \Psi\left(\frac{2\Psi + a\,\Psi}{\Psi + a\,\Psi}\right).$$

Thus, we obtain

$$2\Phi^2 + a\,\Phi^2 = \Phi^2 + a\,\Phi^2, \tag{4.1}$$

and

$$2\Psi^2 + a\,\Psi^2 = \Psi^2 + a\,\Psi^2, \tag{4.2}$$

we try to solve these equations. From (4.1) and (4.2):

$$(\Psi^2 - \Phi^2) = 0.$$

This implies $\Phi = \Psi$. This is a contradiction. Thus, the proof of the theorem is now completed. □

5. Global stability

In this section we study the global asymptotic stability of the positive solutions of (1.1).

Theorem 5.1. *The equilibrium point $\tilde{x}$ of (1.1) is global attractor.*

Proof. Let p, q are real numbers and assume that $g : [p, q]^3 \longrightarrow [p, q]$ is function defined by $g(u, v, w) = u\,\left\{\frac{2\,v\,+a\,w}{v\,+a\,w}\right\}$, then we can easily see that the function $g(u, v, w)$ is increasing in u, v and decreasing in w. Suppose that (m, M) is a solution of the system

$$M = g(M, M, m) \quad \text{and} \quad m = g(m, m, M).$$

Then from (1.1), we see that

$$M = M\left\{\frac{2\,M\,+a\,m}{M\,+a\,m}\right\}, \quad m = m\left\{\frac{2\,m + a\,M}{m\,+a\,M}\right\},$$

or,

$$M = \left\{\frac{2\,M^{\,2} + a\,Mm}{M\,+a\,m}\right\}, \quad m = \left\{\frac{2\,m^2 + a\,Mm}{m + a\,M}\right\},$$

then

$$2\,M^{\,2} + a\,Mm = M^2 + a\,Mm, \qquad 2\,m^2 + a\,Mm = m^2 + a\,Mm.$$

Subtracting we obtain

$$M^2 - m^2 = 0.$$

Thus

$$M = m.$$

It follows by Theorem 1.8 that $\tilde{x}$ is a global attractor of (1.1) and then the proof is now completed. □

From Theorems 2.1 and 5.1, we arrive at the following result.

Theorem 5.2. *The positive equilibrium point $\tilde{x}$ of (1.1) is globally asymptotic stable.*

6. Applications

In this part we give the closed form solutions for some special cases of equation (1.1).

6.1. When $p = q = 0$ and $r = a = 1$

In this subsection, we formulate the solutions for (1.1) in special case when $p = q = 0$ and $r = a = 1$. This means that we deal with the following equation

$$x_{n+1} = x_n\left(\frac{2x_n + x_{n-1}}{x_n + x_{n-1}}\right), \quad n = 0, 1, \ldots. \tag{6.1}$$

Theorem 6.1. *Assume that $\{x_n\}_{n=-1}^{\infty}$ are solutions of equation (6.1). Then for $n = 0, 1, 2,$ we see that all solutions of equation (6.1) are given by the following formula*

$$x_n = \frac{\prod_{k=0}^{n-1} AF_k + BG_k}{\prod_{k=1}^{n-1} AG_k + BF_k}, \quad n = 0, 1, \ldots,$$

where $x_0 = x_1 = A$, $x_{-1} = B$, $n \geqslant 2$,

$$F_n = 3F_{n-1} - F_{n-2}, \quad F_0 = 1, F_1 = 2, \quad G_n = 3G_{n-1} - G_{n-2}, \quad G_0 = 0, G_1 = 1.$$

We note that $G_n = G_{n-1} + F_{n-1}$.

6.2. When $p = q = 0$, $r = 1$, and $a = -1$

In this subsection, we formulate the solutions for (1.1) in special case when $p = q = 0$, $r = 1$, and $a = -1$. This means that we deal with the following equation.

$$x_{n+1} = x_n\left(\frac{2x_n - x_{n-1}}{x_n - x_{n-1}}\right), \quad n = 0, 1, \ldots. \tag{6.2}$$

Theorem 6.2. *Assume that $\{x_n\}_{n=-1}^{\infty}$ are solutions of equation (6.2). Then for $n = 0, 1, 2,$ we see that all solutions of equation (6.2) are given by the following formula*

$$x_n = \frac{(AF_n - BF_{n-2})(AF_{n+1} - BF_n)}{A - B}, \quad n = 0, 1, \ldots,$$

where $x_0 = A$, $x_{-1} = B$, $n \geqslant 2$,

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1.$$

6.3. When $p = r = 0$ and $q = a = 1$

In this subsection, we formulate the solutions for equation (1.1) in special case when $p = r = 0$ and $q = a = 1$. This means that we deal with the following equation.

$$x_{n+1} = x_n\left(\frac{2x_{n-1} + x_n}{x_{n-1} + x_n}\right), \quad n = 0, 1, \ldots. \tag{6.3}$$

Theorem 6.3. *Assume that $\{x_n\}_{n=-1}^{\infty}$ are solutions of equation (6.3). Then for $n = 0, 1, 2,$ we see that all solutions of equation (6.3) are given by the following formula*

$$x_n = B\prod_{k=0}^{n-1} \frac{AQ_k + BR_k}{AP_k + BQ_k},$$

where $x_0 = A$, $x_{-1} = B$, $n \geqslant 2$,

$$\begin{aligned} P_n &= 2P_{n-1} + P_{n-2}, & P_0 &= 0, P_1 = 1, \\ Q_n &= 2Q_{n-1} + Q_{n-2}, & Q_0 &= 1, Q_1 = 1, \\ R_n &= 2R_{n-1} + R_{n-2}, & R_0 &= 0, R_1 = 2. \end{aligned}$$

Remark 6.4. The numbers given by the linear difference equation

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, P_1 = 1$$

are called Pell numbers. Also these numbers sometimes called 2-Fibonacci sequences.

Remark 6.5. The numbers given by the linear difference equation

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad Q_0 = 1, Q_1 = 1$$

are called Pell-Lucas numbers.

Remark 6.6. If we summarize $Q_3 = 7, Q_5 = 41, Q_7 = 239, \ldots,$ we give collection of prime numbers. This sequence of prime numbers are called Newman-Shanks-Williams primes (NSW primes). We can get these numbers by the following formula

$$S_{2n+1} = \frac{(1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1}}{2}, \quad n \geqslant 0.$$

7. Numerical examples

Here, we take some numerical examples.

Example 7.1. Figure 1 means that solution of (1.1) is bounded if $H_{-4} = 1, H_{-3} = 2, H_{-1} = 4, H_{-2} = 3, H_0 = 5$, $a = 0.5$.

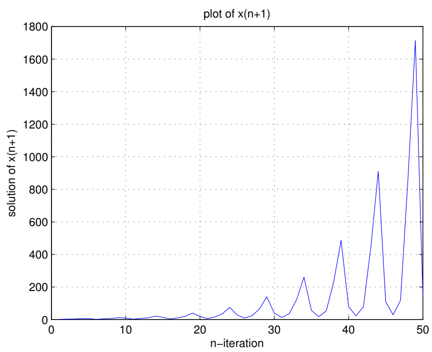

Figure 1: $\left(H_{n+1} = H_{n-4}\left(\frac{2H_{n-3}+0.5H_{n-2}}{H_{n-3}+0.5H_{n-2}}\right)\right)$.

Example 7.2. Figure 2 means that (1.1) has no positive prime period two solutions if $H_{-5} = 1, H_{-4} = 3, H_{-3} = 4, H_{-2} = 6, H_{-1} = 7, H_{-0} = 8$, $a = 100$.

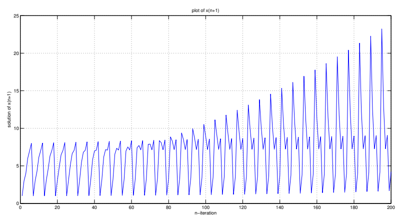

Figure 2: $\left(H_{n+1} = H_{n-5}\left(\frac{2H_{n-3}+100H_{n-1}}{H_{n-3}+100H_{n-1}}\right)\right)$.

Example 7.3. Figure 3 means that (1.1) is globally asymptotically stable if $H_{-6} = 1, H_{-5} = 2, H_{-3} = 4, H_{-4} = 3, H_{-1} = 6, H_{-2} = 5, H_0 = 7$, $a = -1$.

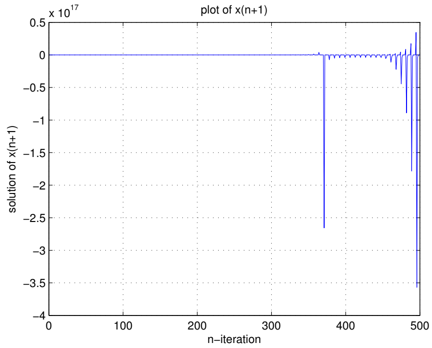

Figure 3: $\left(H_{n+1} = H_{n-6}\left(\frac{2H_{n-4}-H_{n-2}}{H_{n-4}-H_{n-2}}\right)\right)$.

Example 7.4. Figure 4 considers solution of (6.1) when $H_{-1} = 1, H_0 = 3$.

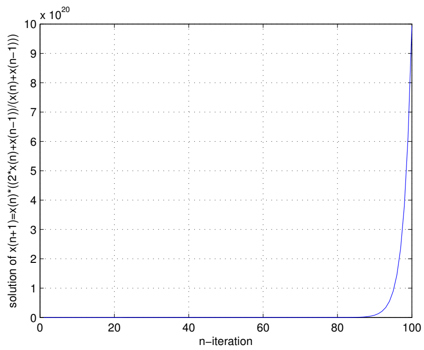

Figure 4: $\left(H_{n+1} = H_n\left(\frac{2H_n+H_{n-1}}{H_n+H_{n-1}}\right)\right)$.

Example 7.5. Figure 5 considers solution of (6.2) when $H_{-1} = 1,\ H_0 = 3$.

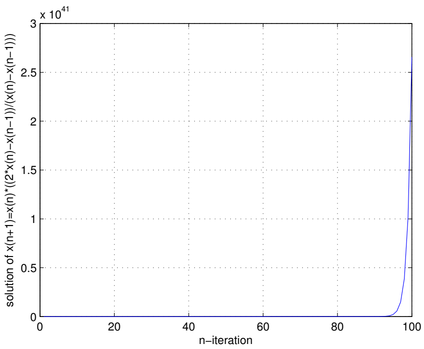

Figure 5: $\left(H_{n+1} = H_n\left(\frac{2H_n-H_{n-1}}{H_n-H_{n-1}}\right)\right)$.

Example 7.6. Figure 6 considers solution of (6.3) when $H_{-1} = 1,\ H_0 = 3$.

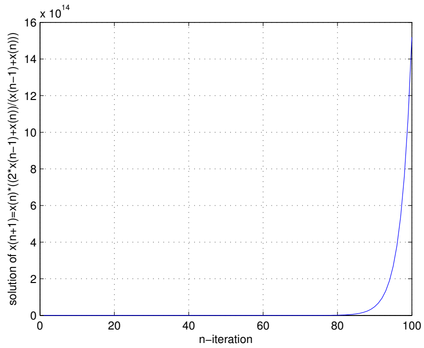

Figure 6: $\left(H_{n+1} = H_n\left(\frac{2H_{n-1}+H_n}{H_{n-1}+H_n}\right)\right)$.

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