Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 12 (2019), 75–85
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

A motion of complex curves in $\mathbb{C}^3$ and the nonlocal nonlinear Schrödinger equation

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Abstract

This paper shows that soliton solutions to the nonlocal nonlinear Schrödinger equation (NNLS) proposed recently by Ablowitz and Musslimani [M. J. Ablowitz, Z. H. Musslimani, Phys. Rev. Lett., **110** (2013), 5 pages] describe a motion of three distinct complex curves in $\mathbb{C}^3$ with initial data being suitably restricted. This gives a geometric interpretation of NNLS.

Keywords: Complex moving curve, geometric interpretation, uniqueness.

2010 MSC: 53A04, 35Q55, 35Q60.

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1. Introduction

The study of moving curves in a Riemannian or pseudo-Riemannian manifold, especially in the Euclidean or pseudo-Euclidean spaces and their relation to integrable equations, is an attractive topic in differential geometry (see [6, 9, 11, 12, 17, 19–21, 23]). Pioneering work by Hasimoto [19] showed that the equation of motion of a vortex filament regarded as a space curve, i.e., the famous Da Rios equation [5]: $\mathbf{X}_t = \kappa\mathbf{B}$, where $\mathbf{X}(x,t)$ is the position of the curve, κ stands for the curvature and $\mathbf{B}$ denotes the binormal vector at arclength x and time t at the point $\mathbf{X}(x,t)$, was equivalent to the well-known, integrable nonlinear Schrödinger equation (NLS$^+$): $i\varphi_t + \varphi_{xx} + 2|\varphi|^2\varphi = 0$. Meanwhile, it is also proved in [6, 7] that the defocusing nonlinear Schrödinger equation (NLS$^-$): $i\varphi_t + \varphi_{xx} - 2|\varphi|^2\varphi = 0$ and the nonlinear heat system:

$$\begin{cases} q_t = q_{xx} + 2qrq, \\ r_t = -r_{xx} - 2rqr, \end{cases}$$

describe respectively the binormal motion of timelike and spacelike curves in $\mathbb{R}^{2,1}$: $\mathbf{X}_t = \pm\kappa\mathbf{B}$. Using the Hasimoto transformation that relates space curves and complex curvature functions, Lamb [21] generalized the above result by demonstrating the link between the motion of certain space curves and

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doi: 10.22436/jnsa.012.02.02

Received: 2018-04-15 Revised: 2018-08-05 Accepted: 2018-08-14

soliton-bearing equations. Murugesh and Balakrishnana [23] had devised a method which derived explicitly the three distinct evolving curves that correspond to NLS$^+$, and the tangent vector of the first of these curves, the binormal vector of the second and the normal vector of the third, are shown to satisfy to the integrable Landau-Lifshitz equation (i.e., the Heisenberg ferromagnetic model):

$$\mathbf{s}_t = \mathbf{s} \times \mathbf{s}_{xx}, \ \mathbf{s}^2 = 1. \tag{1.1}$$

Fukumoto and Miyazaki showed in [12] that the complex mKdV equation: $\varphi_t = \varphi_{xxx} + 3|\varphi|^2\varphi_x$ is equivalent to the motion of space curves in $\mathbb{R}^3$ satisfying $\mathbf{X}_t = \frac{1}{2}\kappa^2\mathbf{T} + \kappa_x\mathbf{N} + \kappa\tau\mathbf{B}$, where τ is the torsion curvature of curves, $\mathbf{T}$ and $\mathbf{N}$ are respectively the tangent and principal normal vectors at arclength x and time t at the point $\mathbf{X}(x,t)$. Motion of curves in $\mathbb{S}^2$ and $\mathbb{S}^3$ were considered by Doliwa and Santini in [11]. In general, Gürses [17] established a connection between the curves moving in a 3-space with arbitrary signature (-1 or 3) and soliton equations. Furthermore, it is shown that many integrable nonlinear PDEs including the sine-Gordon equation, the NLS equation, the mKdV equation, and the KdV equation, may arise from two-dimensions surfaces with vanishing Gaussian curvature, flat surfaces.

Recently Ablowitz and Musslimani (see [2]) proposed a nonlocal nonlinear Schrödinger equation (NNLS)

$$i\varphi_t(x,t) + \varphi_{xx}(x,t) + 2\epsilon\varphi(x,t)\varphi^*(-x,t)\varphi(x,t) = 0, \tag{1.2}$$

where * stands for the complex conjugation and $\epsilon = \pm$ signals the focusing (+) and defocusing (-) nonlinearity. The key point is NNLS (1.2) has qualitative properties other than the standard NLS and its classical generalizations (refer to [22, 28]). For example, in the focusing case, NNLS$^+$ ($\epsilon = +1$) admits both bright (sech-type) and dark (tanh-type) soliton states [26], while NLS$^+$ supports only bright soliton solutions. By using the Hirota bilinear method and the reduction formulas, Gürses and Pekcan [18] have found one-, two-, and three-soliton solutions of the NLS and NNLS equations. From geometrical point of view of moving complex curves, a quite relevant question arises: what is the link between the motion of space curves in the (real or complex) Euclidean or Minkowski 3-space and NNLS?

The purpose of this paper is to give positive answers to the above question. The main idea applying here is inspired from the work of Ding et al. in [9], which determined a motion of space curves in $\mathbb{R}^{2,1}$ inducing the KdV equation. In this paper, we show that if an initial curve $\mathbf{X}_0(x) = \mathbf{X}(x,t)|_{t=0}$ at $t = 0$ is generated by the complex curvature $\kappa(x,0) = \kappa_0(x)$ and the complex torsion curvature $\tau(x,0) = \tau_0(x)$ satisfying

$$\kappa_0(x) = \epsilon\kappa_0^*(-x)\exp[i\int_0^x(\tau_0(y) + \tau_0^*(-y))dy], \tag{1.3}$$

then the quantity:

$$\kappa(x,t) = \epsilon\kappa^*(-x,t)\exp[i\int_0^x(\tau(y,t) + \tau^*(-y,t))dy]$$

is preserved by complex curves $\mathbf{X}(x,t)$ evolving by the evolution equation:

$$\mathbf{X}_t = \mathbf{X}_x \times \mathbf{X}_{xx} = \kappa\mathbf{B}, \text{ with } \mathbf{X}_x \cdot \mathbf{X}_x = 1 \tag{1.4}$$

in $\mathbb{C}^3$, where $\mathbf{B}$ is binormal vector at the point $\mathbf{X}(x,t)$ in $\mathbb{C}^3$, $\times$ denotes the cross product between vectors in $\mathbb{C}^3$, i.e., the cross product of $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{C}^3$ by

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1),$$

and $\cdot$ denotes the standard holomorphic inner product in $\mathbb{C}^3$ (see [25]). This fact comes mainly from the uniqueness (in a suitable space of functions) of solutions to the initial-value problem of the coupled NLS system of equations (Ablowitz-Kaup-Newell-Segur (AKNS)) (see [1, 3, 24])

$$\begin{cases} i\varphi_{1t} - \varphi_{1xx} + 2\varphi_1^2\varphi_2 = 0, \\ i\varphi_{2t} + \varphi_{2xx} - 2\varphi_1\varphi_2^2 = 0, \end{cases} \tag{1.5}$$

where $\varphi_1(x,t)$ and $\varphi_2(x,t)$ are complex dynamical variables, which is equivalent to the above evolution equation (1.4). We call the above system of coupled equations a nonlinear Schrödinger system (NLS system). Now, by setting $\mathbf{s}(x,t) = \frac{d}{dx}\mathbf{X}(x,t) \in \mathbb{C}^3$, we see that Eq. (1.4) is equivalent to the coupled Landau-Lifshitz equations (CLL)

$$\mathbf{s}_t = \mathbf{s} \times \mathbf{s}_{xx}, \ \mathbf{s} = (s_1, s_2, s_3) \in \mathbb{C}\mathbb{S}^2 \hookrightarrow \mathbb{C}^3. \tag{1.6}$$

When $\mathbf{s} \in \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, Eq. (1.6) returns to Eq. (1.1). In [10, 13], the NLS system (1.5) is gauge equivalent to CLL (1.6) and vice versa. Here, by using the viewpoint of complex moving curves in $\mathbb{C}^3$, Eq. (1.4) is equivalent to the NLS system (1.5), in which the associated complex function $\varphi_1(x,t) = -\kappa(x,t)\exp\left(-i\int_0^x \tau(y,t)dy\right)$ and $\varphi_2(x,t) = \kappa(x,t)\exp\left(i\int_0^x \tau(y,t)dy\right)$ evolves to the NLS system (1.5), where $\kappa(x,t)$ and $\tau(x,t)$ stand for the complex curvature and the complex torsion curvature at $\mathbf{X}(x,t)$ respectively. This implies that soliton solutions $\varphi_2(x,t)$ to NNLS describe the motion (1.4) of complex curves $\mathbf{X}(x,t)$ in $\mathbb{C}^3$ with initial data at $t = 0$ being restricted by the relation (1.3). Hence NNLS arises from the motion (1.4) of complex curves just with the initial-data being suitably restricted. By using gauge equivalent way, Ding et al. in [10] have given an accurate characterization of the gauge-equivalent magnetic structure of NNLS, but here we give the direct interrelations between NNLS and the motion of complex curves in $\mathbb{C}^3$. Finally, by using Murugesh and Balakrishnana's unified formalism [23], any given solution of the NLS system (1.5) gets associated with three distinct complex curve evolutions, that is the tangent vector of the first of these curves, the binormal vector of the second and the normal vector of the third satisfy to CLL (1.6). These connections enable us to find the three surfaces swept out by the moving curves associated with the NNLS$^+$.

2. Motion of complex curves in $\mathbb{C}^3$

Consider the bilinear form (the standard holomorphic inner product) $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^3$ defined by $\langle X, Y\rangle = X\cdot Y = X_1Y_1 + X_2Y_2 + X_3Y_3, \forall X, Y \in \mathbb{C}^3$. If a vector $\alpha \in \mathbb{C}^3$ satisfies $\alpha\cdot\alpha = 1$, then α is called the complex unit vector. The set of all complex unit vectors is the complex 2-sphere $\mathbb{C}\mathbb{S}^2(1)$. One may know that there are only three complex linearly independent vectors in $\mathbb{C}^3$. An 3×3 complex matrix $A = (a_{ij})$ is said to be orthogonal if the column vectors that make up A are orthonormal, that is, if $\sum_{k=1}^3 a_{ki}a_{kj} = \delta_{ij}$, $1 \leqslant i,j \leqslant 3$. Here δ_{ij} is the Kronecker delta. Equivalently, A is orthogonal if it preserves the standard holomorphic inner product, namely if $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{C}^3$. Still another equivalent definition is that A is orthogonal if $A^TA = I$, i.e., if $A^T = A^{-1}$. Here, A^T is the transpose of A. The set of all 3×3 complex matrices A which preserve this form (i.e., such that $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{C}^3$) is the complex orthogonal group $O(3;\mathbb{C})$, and it is a subgroup of the general linear groups $GL(3;\mathbb{C})$.

The study of the special complex curves in $\mathbb{C}^3$ including minimal curves [14, 16], null curves [15] is an attractive topic in differential geometry. Due to $\mathbb{C}^3$ is equipped with the standard holomorphic inner product, the complex curves in $\mathbb{C}^3$ have new significance. Next, we consider that a complex curve $\mathbf{X}(x) = (z_1(x), z_2(x), z_3(x))$ in $\mathbb{C}^3$ satisfies reality condition: $\mathbf{X}'(x)\cdot\mathbf{X}'(x) = 1, \forall x$.

The Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of a curve $\alpha(x)$ in $\mathbb{R}^3$ is an orthonormal frame of vector fields $\mathbf{T}$ (tangent vector), $\mathbf{N}$ (normal vector) and $\mathbf{B}$ (binormal vector) moving along the curve $\alpha(x)$. The linear hull of $\mathbf{T}$ is the tangent line, the linear hull of $\mathbf{T}, \mathbf{N}$ is the osculating plane spanned by the vectors $\alpha'(x)$ and $\alpha''(x)$. Similar to $\mathbb{R}^3$, there are the Frenet frame and formulae of complex curves with reality condition in $\mathbb{C}^3$.

To find a Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in a similar way, we define the vector fields $\mathbf{T}(x)$, $\mathbf{N}(x)$, and $\mathbf{B}(x)$ on $\mathbf{X}(x)$ by

$$\mathbf{T}(x) = \mathbf{X}'(x), \ \ \mathbf{N}(x) = \frac{\mathbf{X}''(x)}{\kappa(x)}, \ \ \mathbf{B}(x) = \mathbf{T}(x)\times\mathbf{N}(x), \tag{2.1}$$

where $\kappa(x) = \sqrt{\mathbf{X}''(x)\cdot\mathbf{X}''(x)} \neq 0$.

Since $\mathbf{T}\cdot\mathbf{T}$=1 from the first of Eqs. (2.1) which indicates that the tangent vector $\mathbf{T}$ completely locates in the complex 2-sphere $\mathbb{C}\mathbb{S}^2(1)$. We note that $\mathbf{X}\cdot\mathbf{Y} = 0, \forall\ \mathbf{X}, \mathbf{Y} \in \mathbb{C}^3$ expresses a notion of orthogonality. An important difference with real inner products, is that $\mathbf{X} \cdot \mathbf{Y} = 0$ implies that the whole complex lines $\mathbb{C}\mathbf{X}$ and $\mathbb{C}\mathbf{Y}$ (which are planed in the real dimensional sense) are orthogonal to each other, i.e., $\mathbf{X}_0 \cdot \mathbf{Y}_0 = 0$ for any $\mathbf{X}_0 \in \mathbb{C}\mathbf{X}$ and $\mathbf{Y}_0 \in \mathbb{C}\mathbf{Y}$. We have

Lemma 2.1. $\mathbf{T}, \mathbf{N}, \mathbf{B}$ *are the unit orthogonal and complex linearly independent vectors.*

Proof. Since $\mathbf{T}, \mathbf{N}, \mathbf{B}$ given by (2.1) and reality condition: $\mathbf{T}\cdot\mathbf{T}$=1, we have

$$\begin{aligned}
&\mathbf{T} \cdot \mathbf{N} = \mathbf{X}' \cdot \frac{\mathbf{X}''}{\kappa} = \frac{1}{2\kappa}(\mathbf{X}' \cdot \mathbf{X}')' = 0,\\
&\mathbf{N} \cdot \mathbf{N} = \frac{1}{\kappa^2}\mathbf{X}'' \cdot \mathbf{X}'' = \frac{1}{||\mathbf{X}''||^2}\mathbf{X}'' \cdot \mathbf{X}'' = 1,\\
&\mathbf{T} \cdot \mathbf{B} = \mathbf{T} \cdot (\mathbf{T} \times \mathbf{N}) = 0, \ \mathbf{N} \cdot \mathbf{B} = \mathbf{N} \cdot (\mathbf{T} \times \mathbf{N}) = 0,\\
&\mathbf{B} \cdot \mathbf{B} = (\mathbf{T} \times \mathbf{N}) \cdot (\mathbf{T} \times \mathbf{N}) = 1,
\end{aligned}$$

i.e., for the standard holomorphic inner product on $\mathbb{C}^3$, the frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is orthogonal. If there are three complex-valued functions $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1\mathbf{T} + \lambda_2\mathbf{N} + \lambda_3\mathbf{B} = 0, \tag{2.2}$$

for the standard holomorphic inner product with $\mathbf{T}$ in the both sides of equations (2.2), we have $\lambda_1 = 0$. Similarly, $\lambda_2 = \lambda_3 = 0$. Hence $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are the unit orthogonal and complex linearly independent vectors. □

Similarly, the frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of the curve $\mathbf{X}(x)$ in $\mathbb{C}^3$ is also called the Frenet frame of a complex curve in $\mathbb{C}^3$. An important difference with the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of a curve in $\mathbb{R}^3$, is that the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of a complex curve in $\mathbb{C}^3$ is a family of complex-valued vector functions.

Lemma 2.2. *For the derivatives of the Frenet frame* $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ *we have the following analogue of the Frenet formulae:*

$$\begin{aligned}
\mathbf{T}'(x) &= \qquad\qquad \kappa(x)\mathbf{N}(x),\\
\mathbf{N}'(x) &= -\kappa(x)\mathbf{T}(x) + \qquad\qquad +\tau(x)\mathbf{B}(x),\\
\mathbf{B}'(x) &= \qquad\qquad -\tau(x)\mathbf{N}(x),
\end{aligned} \tag{2.3}$$

where $\tau(x) = \mathbf{N}'(x) \cdot \mathbf{B}(x)$. *The complex valued-functions* $\kappa(x), \tau(x)$ *are called the complex curvature and the complex torsion curvature of* $\mathbf{X}(x)$, *respectively.*

Proof. By Lemma 2.1, it is shown that from the construction that $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are an orthonormal frame. Consequently, expressing these derivatives $\mathbf{T}', \mathbf{N}', \mathbf{B}'$ as linear combinations of $\mathbf{T}, \mathbf{N}, \mathbf{B}$, the matrix of coeffcients so obtained must be skew symmetric. Therefore, since the first of Eqs. (2.3) follows immediately from Eqs. (2.1), we have only to prove that the last equation of (2.3) is true. But $\mathbf{B}' \cdot \mathbf{B} = 0$ and $\mathbf{B}' \cdot \mathbf{T} = -\mathbf{B} \cdot \mathbf{T}' = -\kappa(\mathbf{B} \cdot \mathbf{N}) = 0$ imply that $\mathbf{B}' = (\mathbf{B}' \cdot \mathbf{N})\mathbf{N}$, therefore, it is sufficient to show that $\mathbf{B}' \cdot \mathbf{N} = -\mathbf{B} \cdot \mathbf{N}' = -\tau$. □

Now, we consider the motion of the complex curves $\mathbf{X}(x, t)$ in $\mathbb{C}^3$ evolving by the evolution equation (1.4). We assert that the relation $\mathbf{X}_x \cdot \mathbf{X}_x = 1$ is preserved invariant under the evolution equation (1.4), that is, if $\mathbf{X} = \mathbf{X}(x, t)$ evolves according to Eq. (1.4) with $\mathbf{X}_x \cdot \mathbf{X}_x = 1$ at $t = 0$, then $\mathbf{X}_x \cdot \mathbf{X}_x = 1$ for any $t > 0$. In fact, it suffices to prove that $\frac{d}{dt}(\mathbf{X}_x^2) = 0$ holds for any t:

$$\frac{d}{dt}(\mathbf{X}_x^2) = 2\mathbf{X}_{xt} \cdot \mathbf{X}_x = 2\mathbf{X}_{tx} \cdot \mathbf{X}_x = 2(\mathbf{X}_x \times \mathbf{X}_{xxx}) \cdot \mathbf{X}_x = 0.$$

Writing matrix S by

$$S = \begin{pmatrix} s_3 & s_1 - is_2 \\ s_1 + is_2 & -s_3 \end{pmatrix},$$

where $s_1^2+s_2^2+s_3^2=1$, then $S=S(x,t)$ is a 2×2 matrix with $S^2=I$ (I stands for the unit matrix) and $\mathrm{tr}S=0$. CLL (1.6) reads as the following complex matrix equation,

$$S_t=-\frac{i}{2}[S,S_{xx}]. \tag{2.4}$$

When S is a Hermitian matrix (i.e., $S^\dagger=S$), Eq. (2.4) returns to LL (1.1). This motivates the introduce in geometry the concept of Schrödinger flows (or maps) (see [8, 27] or [4]). It is proved in [10] that CLL (1.6) is exactly the equation of Schrödinger flows from $\mathbb{R}^1$ to the complex 2-sphere $\mathbb{C}S^2(1)\hookrightarrow\mathbb{C}^3$.

3. Geometric interpretation of NNLS

This section shows that soliton solutions to NNLS describe a motion of complex curves in $\mathbb{C}^3$ with initial data being suitably restricted.

Proposition 3.1. *The complex curves $\mathbf{X}(x,t)$ in $\mathbb{C}^3$ evolving by the evolution equation (1.4) (or the coupled Landau-Lishitz equations (2.4)) is equivalent to the NLS system (1.5).*

Proof. It is a direct computation by using the Frenet-Serret formula (2.3) that Eq. (1.4) can be rewritten as

$$\mathbf{T}_t=\kappa_s\mathbf{B}-\kappa\tau\mathbf{N}.$$

Now we introduce complexifing the Hasimoto frame:

$$\mathbf{e}_1=\mathbf{T},\qquad \mathbf{e}_2=\frac{1}{\sqrt{2}}(\mathbf{N}+i\mathbf{B})\exp(i\int_0^x\tau(y)dy),\qquad \mathbf{e}_3=\frac{1}{\sqrt{2}}(\mathbf{N}-i\mathbf{B})\exp(-i\int_0^x\tau(y)dy).$$

Then the Frenet-Serret formula (2.3) becomes

$$\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}_x=\begin{pmatrix}0&\varphi_1&\varphi_2\\ -\varphi_1&0&0\\ -\varphi_2&0&0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}, \tag{3.1}$$

where

$$\varphi_1(x,t)=\frac{\kappa(x,t)}{\sqrt{2}}\exp\left(-i\int_0^x\tau(y,t)dy\right),\qquad \varphi_2(x,t)=\frac{\kappa(x,t)}{\sqrt{2}}\exp\left(i\int_0^x\tau(y,t)dy\right).$$

On the other hand, it is also a direct verification that Eq. (1.4) now reads

$$\mathbf{e}_{1t}=\varphi_1\mathbf{e}_2+\varphi_2\mathbf{e}_3.$$

By using the relations: $\mathbf{e}_1\cdot\mathbf{e}_2=\mathbf{e}_1\cdot\mathbf{e}_3=\mathbf{e}_2\cdot\mathbf{e}_2=\mathbf{e}_3\cdot\mathbf{e}_3=0$ and $\mathbf{e}_2\cdot\mathbf{e}_3=1$, we arrive at

$$\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}_t=\begin{pmatrix}0&-i\varphi_{1x}&i\varphi_{2x}\\ -i\varphi_{2x}&a_2&0\\ i\varphi_{1x}&0&-a_2\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}, \tag{3.2}$$

where a_2 will be determined. The integrability condition of (3.1) and (3.2) implies

$$a_{2x}=i(\varphi_{1x}\varphi_2+\varphi_{2x}\varphi_1),\quad \varphi_{1t}+i\varphi_{1xx}+a_2\varphi_1=0,\quad \varphi_{2t}-i\varphi_{2xx}-a_2\varphi_2=0. \tag{3.3}$$

One may solve the first of Eqs. (3.3) to obtain

$$a_2=i\varphi_1\varphi_2+c(t)$$

for some real function $c(t)$ depending only on t. Substituting it into the second and the third of Eqs. (3.3) we have

$$\begin{cases} \varphi_{1t} + i(\varphi_{1xx} + \varphi_1^2\varphi_2) + c(t)\varphi_1 = 0, \\ \varphi_{2t} - i(\varphi_{2xx} + \varphi_1\varphi_2^2) - c(t)\varphi_2 = 0. \end{cases}$$

This differential system is equivalent to the NLS system (1.5) by the transform:

$$\varphi_1 \to -\sqrt{2}\varphi_1 \exp\left(-\int_0^t c(\tilde{t})d\tilde{t}\right), \ \varphi_2 \to \sqrt{2}\varphi_2 \exp\left(\int_0^t c(\tilde{t})d\tilde{t}\right). \qquad \square$$

Now, we restrict ourselves to a solution $(\varphi_1(x,t), \varphi_2(x,t))$ to the initial value problem of the NLS system (1.5) with $\varphi_1 - c_1, \varphi_2 - c_2 \in C^1([0,T); H^2(\mathbb{R}))$ for some constants c_1 and c_2 and $0 < T \leqslant +\infty$. We claim that, under this circumstance, solutions to the initial-value problem (i.e., $\varphi_1|_{t=0} = \varphi_1^0(x), \varphi_2|_{t=0} = \varphi_2^0(x)$ with $\varphi_1^0(x) - c_1, \varphi_2^0(x) - c_2 \in H^2(\mathbb{R})$) of the NLS system (1.5) are unique (e.g. refer to [10]).

Returning to the complex moving curves $\mathbf{X}(x,t)$ in $\mathbb{C}^3$, we have the following conclusion.

Theorem 3.2. *Suppose an initial curve $\mathbf{X}_0(x) = \mathbf{X}(x,t)|_{t=0}$ at $t = 0$ is generated by the complex curvature $\kappa_0(x)$ and the complex torsion curvature $\tau_0(x)$ satisfying*

$$\kappa_0(x) = \epsilon\kappa_0^*(-x)\exp[i\int_0^x (\tau_0(y) + \tau_0^*(-y))dy], \tag{3.4}$$

where $\varphi_2(x,t)$ is a solution to the nonlocal nonlinear Schrödinger equation:

$$i\varphi_2(x,t) + \varphi_{2xx}(x,t) + 2\epsilon\varphi_2(x,t)\varphi_2^*(-x,t)\varphi_2(x,t) = 0$$

with $\varphi_2 - c \in C^1([0,T); H^2(\mathbb{R}))$ for some constant c, then the complex moving curves $\mathbf{X}(x,t)$ by Eq. (1.4) preserve the relation: $\varphi_1(x,t) = -\epsilon\varphi_2^*(-x,t)$ invariant and take $\varphi_2(x,t)$ to satisfy the NNLS (1.2).

Proof. This is because of the above claim of the uniqueness of solutions, since $(\varphi_1(x,t) = -\epsilon\varphi_2^*(-x,t), \varphi_2(x,t))$ is a solution to Eq. (1.5) with $\varphi_1 - c, \varphi_2 - c \in C^1([0,T); H^2(\mathbb{R}))$ and satisfies the initial data (3.4) $\Leftrightarrow \varphi_1(x,0) = -\epsilon\varphi_2^*(-x,0)$. This fact indicates that soliton solutions $\varphi_2(x,t)$ to NNLS describe complex curves $\mathbf{X}(x,t)$ in $\mathbb{C}^3$ evolving by Eq. (1.4) with initial data $\mathbf{X}_0(x) = \mathbf{X}(x,t)|_{t=0}$ at $t = 0$ generated by the complex curvature κ_0 and the complex torsion curvature τ_0 satisfying the relation (3.4). Hence NNLS arises from the above motion of complex curves just with the initial-data being suitably restricted. $\square$

Remark 3.3. By the way, in a similar way by the restriction $\varphi_1(x,t) = -\epsilon\varphi_2^*(x,t)$, the same conclusion is also valid for the nonlinear Schrödinger equation: $i\varphi_2(x,t) + \varphi_{2xx}(x,t) + 2\epsilon|\varphi_2(x,t)|^2\varphi_2(x,t) = 0$. Since $\varphi_1(x,t) = -\kappa\exp(-i\int_0^x \tau(y)dy) = -\epsilon\kappa^*\exp(-i\int_0^x \tau^*(y)dy) = -\epsilon\varphi_2^*(x,t)$ implies $\kappa = \epsilon\kappa^*$ and $\tau = \tau^*$. If $\epsilon = 1$, then $\kappa(x,t), \tau(x,t) \in \mathbb{R}$, i.e., the complex moving curves $\mathbf{X}(x,t)$ completely locate in $\mathbb{R}^3$. If $\epsilon = -1$, then $i\kappa(x,t), \tau(x,t) \in \mathbb{R}$, i.e., the complex moving curves $\mathbf{X}(x,t)$ completely locate in $\mathbb{R}^{2,1}$. This indicates again that NLS arises from a motion of space curves, which coincides with the geometric interpretation of NLS displayed in [6, 7, 19].

4. Construction of the three complex curves in $\mathbb{C}^3$ associated with the NLS system

In this section, we show that the equivalence of the NLS system (1.5) to CLL (1.6) in three ways.

By using Murugesh and Balakrishnana's unified formalism [23], we have the following Proposition.

Proposition 4.1. *The tangent vector of the first of complex curves, the binormal vector of the second, and the normal vector of the third satisfy to CLL (1.6), are proved to be equivalent to the NLS system (1.5).*

Proof.

(I) From Proposition 3.1, we show that the tangent vector $\mathbf{T}_1 = \mathbf{T}_1(x,t)$ of complex curves $\mathbf{X}_1 = \mathbf{X}_1(x,t)$ in $\mathbb{C}^3$ satisfy CLL (1.6) is equivalent to the NLS system (1.5).

(II) Let the binormal vector $\mathbf{B}_2 = \mathbf{B}_2(x,t)$ of complex curves $\mathbf{X}_2 = \mathbf{X}_2(x,t)$ in $\mathbb{C}^3$ satisfy to CLL (1.6), i.e.,

$$\mathbf{B}_{2t} = \mathbf{B}_2 \times \mathbf{B}_{2xx}, \ \mathbf{B}_2 \in \mathbb{CS}^2(1). \tag{4.1}$$

By using the Frenet-Serret formula (2.3), Eq. (4.1) can be rewritten as

$$\mathbf{B}_{2t} = \kappa_2\tau_2\mathbf{N}_2 + \tau_{2s}\mathbf{B}_2.$$

Now, let

$$\mathbf{m}_1 = \mathbf{B}_2, \qquad \mathbf{m}_2 = \frac{1}{\sqrt{2}}(\mathbf{N}_2 + i\mathbf{T}_2)\exp(i\int_0^x \kappa_2(y,t)dy), \qquad \mathbf{m}_3 = \frac{1}{\sqrt{2}}(\mathbf{N}_2 - i\mathbf{T}_2)\exp(-i\int_0^x \kappa_2(y,t)dy).$$

Then the Frenet-Serret formula (2.3) now becomes

$$\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{pmatrix}_x = \begin{pmatrix} 0 & \phi_1 & \phi_2 \\ -\phi_1 & 0 & 0 \\ -\phi_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{pmatrix},$$

where

$$\phi_1(x,t) = -\frac{\tau_2(x,t)}{\sqrt{2}}\exp\left(-i\int_0^x \kappa_2(y,t)dy\right), \quad \phi_2(x,t) = -\frac{\tau_2(x,t)}{\sqrt{2}}\exp\left(i\int_0^x \kappa_2(y,t)dy\right).$$

Similarly, we show that the binormal vector $\mathbf{B}_2$ of the second of complex curves $\mathbf{X}_2$ in $\mathbb{C}^3$ satisfy to CLL (1.6) is equivalent to the NLS system (1.5).

(III) Let the normal vector $\mathbf{N}_3 = \mathbf{N}_3(x,t)$ of complex curves $\mathbf{X}_3 = \mathbf{X}_3(x,t)$ in $\mathbb{C}^3$ satisfy to CLL (1.6), i.e.,

$$\mathbf{N}_{3t} = \mathbf{N}_3 \times \mathbf{N}_{3xx}, \ \mathbf{N}_3 \in \mathbb{CS}^2(1). \tag{4.2}$$

By using the Frenet-Serret formula (2.3), Eq. (4.2) can be rewritten as

$$\mathbf{N}_{3t} = \tau_{3s}\mathbf{T}_3 + \kappa_{3s}\mathbf{B}_3.$$

Now, let

$$\mathbf{n}_1 = \mathbf{N}_3, \ \mathbf{n}_2 = \frac{1}{\sqrt{2}}(\mathbf{T}_3 + i\mathbf{B}_3), \ \mathbf{n}_3 = \frac{1}{\sqrt{2}}(\mathbf{T}_3 - i\mathbf{B}_3).$$

Then the Frenet-Serret formula (2.3) reads

$$\begin{pmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{pmatrix}_x = \begin{pmatrix} 0 & \chi_1 & \chi_2 \\ -\chi_1 & 0 & 0 \\ -\chi_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{pmatrix},$$

where

$$\chi_1(x,t) = -\frac{1}{\sqrt{2}}(\kappa_3 - \sqrt{-1}\tau_3), \quad \chi_2(x,t) = -\frac{1}{\sqrt{2}}(\kappa_3 + \sqrt{-1}\tau_3).$$

Similarly, it is shown that the normal vector $\mathbf{N}_3$ of the third of complex curves $\mathbf{X}_3$ in $\mathbb{C}^3$ satisfy to CLL (1.6) is equivalent to the NLS system (1.5). □

Now, we turn to the construction of the three moving complex curves associated with the NLS system (1.5) (see [23]). The corresponding expressions for the position vectors $\mathbf{X}_1(x,t)$, $\mathbf{X}_2(x,t)$ and $\mathbf{X}_3(x,t)$, creating the three moving complex curves, are obtained as follows.

(I) The position vector $\mathbf{X}_1$ for the first moving is obtained by integrating $\mathbf{T}_1 = \mathbf{s}$. Namely,

$$\mathbf{X}_1(x,t) = \int^x \mathbf{T}_1(y,t)dy = \int^x \mathbf{s}(y,t)dy. \tag{4.3}$$

(II) Here, the binormal $\mathbf{B}_2$ satisfies CLL (1.6). Hence $\mathbf{B}_2 = \mathbf{s}$. Using the Frenet-Serret formula (2.3), then the unit tangent vector $\mathbf{T}_2 = \frac{\mathbf{s}\times\mathbf{s}_x}{|\mathbf{s}_x|}$, yielding the expression for the position vector creating the second moving complex curve as

$$\mathbf{X}_2(x,t) = \int^x \mathbf{T}_2(y,t)dy = \int^x \frac{\mathbf{s}(y,t)\times\mathbf{s}_y(y,t)}{|\mathbf{s}_y(y,t)|}dy. \tag{4.4}$$

(III) Here, since the normal vector $\mathbf{N}_3$ satisfies CLL (1.6), $\mathbf{N}_3 = \mathbf{s}$. Hence the unit tangent vector $\mathbf{T}_3$ can be expressed in term of $\mathbf{s}$ as follows:

$$\mathbf{T}_3 = \frac{\mathbf{s}\times\mathbf{s}_x \sin\alpha - \mathbf{s}_x\cos\alpha}{\kappa_1}, \tag{4.5}$$

where $\alpha = \int^x \tau_1(y,t)dy + C(t)$, and $C(t)$ is an arbitrary function of time t. Thus Eq. (4.5) leads to the following expression for the position vector of the third moving complex curve in $\mathbb{C}^3$:

$$\mathbf{X}_3(x,t) = \int^x \mathbf{T}_3(y,t)dy = \int^x \frac{\mathbf{s}(y,t)\times\mathbf{s}_y(y,t)\sin\alpha - \mathbf{s}_y(y,t)\cos\alpha}{\kappa_1}dy. \tag{4.6}$$

Combining Theorem 3.2, Proposition 4.1, and Eqs. (4.3), (4.4), (4.6), we have the following theorem.

Theorem 4.2.

(a) *Suppose an initial curves* $\mathbf{X}_1^0(x) = \mathbf{X}_1(x,t)|_{t=0}$ *at* $t=0$ *is generated by* $\kappa_1^0(x)$ *and* $\tau_1^0(x)$ *satisfying* $\kappa_1^0(x) = \epsilon\kappa_1^{0*}(-x)\exp[i\int_0^x(\tau_1^0(y)+\tau_1^{0*}(-y))dy]$, *where* $\varphi_2(x,t)$ *is a solution to NNLS with* $\varphi_2 - c \in C^1([0,T);H^2(\mathbb{R}))$ *for some constant* c, *then the tangent vector* $\mathbf{T}_1$ *of complex curves* $\mathbf{X}_1(x,t)$ *in* $\mathbb{C}^3$ *satisfy CLL (1.6)) preserve the relation:* $\varphi_1(x,t) = -\epsilon\varphi_2^*(-x,t)$ *invariant and take* $\varphi_2(x,t)$ *to satisfy the NNLS (1.2).*

(b) *Suppose an initial curves* $\mathbf{X}_2^0(x) = \mathbf{X}_2(x,t)|_{t=0}$ *at* $t=0$ *is generated by* $\kappa_2^0(x)$ *and* $\tau_2^0(x)$ *satisfying* $\tau_2^0(x) = \epsilon\tau_2^{0*}(-x)\exp[i\int_0^x(\kappa_2^0(y)+\kappa_2^{0*}(-y))dy]$, *where* $\phi_2(x,t)$ *is a solution to NNLS with* $\phi_2 - c \in C^1([0,T);H^2(\mathbb{R}))$ *for some constant* c, *then the binormal vector* $\mathbf{B}_2$ *of complex curves* $\mathbf{X}_2(x,t)$ *in* $\mathbb{C}^3$ *satisfy to CLL (1.6) preserve the relation:* $\phi_1(x,t) = -\epsilon\phi_2^*(-x,t)$ *invariant and take* $\phi_2(x,t)$ *to satisfy the NNLS (1.2).*

(c) *Suppose an initial curves* $\mathbf{X}_3^0(x) = \mathbf{X}_3(x,t)|_{t=0}$ *at* $t=0$ *is generated by* $\kappa_3^0(x)$ *and* $\tau_3^0(x)$ *satisfying* $\kappa_3^0(x) - \epsilon\kappa_3^{0*}(-x) = i(\tau_3^0(x) - \epsilon\tau_3^{0*}(-x))$, *where* $\chi_2(x,t)$ *is a solution to NNLS with* $\chi_2 - c \in C^1([0,T);H^2(\mathbb{R}))$ *for some constant* c, *then the normal vector* $\mathbf{N}_3$ *of complex curves* $\mathbf{X}_3(x,t)$ *in* $\mathbb{C}^3$ *satisfy to CLL (1.6) preserve the relation:* $\chi_1(x,t) = -\epsilon\chi_2^*(-x,t)$ *invariant and take* $\chi_2(x,t)$ *to satisfy the NNLS (1.2).*

This theorem shows that soliton solutions to NNLS describe a motion of three distinct complex curves in $\mathbb{C}^3$ with initial data being suitably restricted. This gives a geometric interpretation of NNLS.

5. Example: soliton solutions

A solution of the NLS system (1.5) is given by

$$\varphi_1(x,t) = -\kappa(x,t)\exp(-i\sigma), \ \ \varphi_2(x,t) = \kappa(x,t)\exp(i\sigma), \tag{5.1}$$

where $\sigma = \int_0^x \tau(y,t)dy$. It is a direct verification that $\kappa(x,t)$ and $\tau(x,t)$ satisfy

$$\begin{cases} \kappa_t = -2\kappa_x\tau - \kappa\tau_x, \\ \tau_t = (\frac{\kappa_{xx}}{\kappa} - \tau^2 + 2\kappa^2)_x. \end{cases} \tag{5.2}$$

Now we introduce $\eta = x + 2i\nu t + d_{11}$, where parameters ν and d_{11} are complex. Then $\kappa(x,t) = \kappa(\eta)$, $\tau(x,t) = \tau(\eta)$ and

$$\kappa_t = 2i\nu\kappa',\ \tau_t = 2i\nu\tau',\ \kappa_x = \kappa',\ \tau_x = \tau'. \tag{5.3}$$

Substituting Eqs. (5.3) in Eqs. (5.2) gives

$$\begin{cases} 2i\nu\kappa' = -2\kappa'\tau - \kappa\tau', \\ 2i\nu\tau' = (\frac{\kappa''}{\kappa} - \tau^2 + 2\kappa^2)'. \end{cases} \tag{5.4}$$

One may solve the first of Eqs. (5.4) to obtain

$$((i\nu + \tau)\kappa^2)' = 0. \tag{5.5}$$

Here, we assume that κ and τ satisfy $\lim_{x\to+\infty} \kappa = 0$ and τ is bounded. Then one may solve Eq. (5.5) to give $\tau = -i\nu$ and substituting in the second of Eqs. (5.4) gives

$$\frac{\kappa''}{\kappa} + 2\kappa^2 = \pm a^2, \tag{5.6}$$

where a is the constant of integration. Its solution of Eq. (5.6) can be written

$$\kappa = a\sec(-ia\eta) \text{ or} - a\csc(-ia\eta). \tag{5.7}$$

It is a direct verification that

$$\sigma = -i\nu x + (a^2 + \nu^2)t + d_2, \tag{5.8}$$

where d_2 is the constant of integration. Here, using the first of Eq. (5.7) (i.e., $\kappa = a\sec(-ia\eta)$) and Eq. (5.8) in Eq. (5.1) yields

$$\varphi_1(x,t) = -a\sec(P)\exp(iQ), \quad \varphi_2(x,t) = a\sec(P)\exp(-iQ),$$

where $P = -ia\eta = -iax + 2a\nu t + d_1,\ d_1 = -iad_{11}$, and $Q = i\nu x - (a^2 + \nu^2)t - d_2$. Note that

$$\varphi_1(x,t) = -\varphi_2^*(-x,t) (\Leftrightarrow \varphi_2(x,t) \text{ is solution of NNLS}^+) \Leftrightarrow \text{all parameters } a, \nu, d_1, d_2 \text{ are real.}$$

Next, we return to CLL (1.6). A soliton solution of CLL (1.6) is given by (see [13])

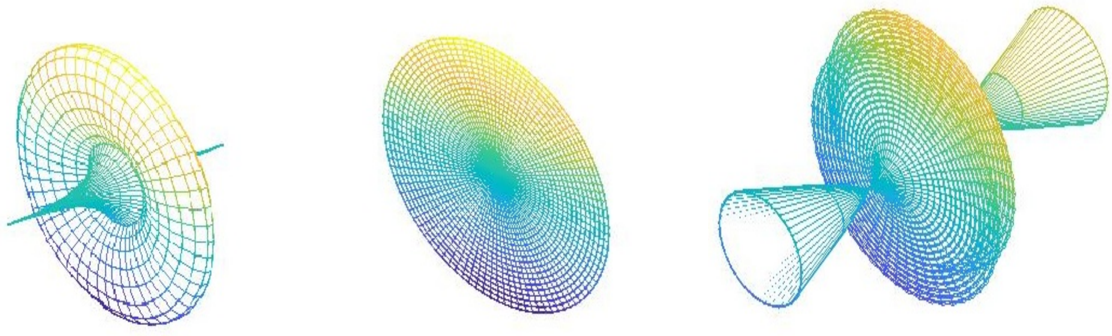
$$\mathbf{s} = \left(\frac{2ia\sec P}{a^2 - \nu^2}(\nu\sin Q + a\cos Q\tan P), \frac{2ia\sec P}{a^2 - \nu^2}(\nu\cos Q - a\sin Q\tan P), 1 - \frac{2a^2\sec^2 P}{a^2 - \nu^2}\right) \hookrightarrow \mathbb{C}^3, \tag{5.9}$$

where all parameters a, ν, d_1, and d_2 are real, which corresponding to the soliton solution $\varphi_2(x,t) = a\sec(P)\exp(-iQ)$ of NNLS$^+$. The three moving curves that correspond to the soliton solution $\varphi_2(x,t)$ of NNLS$^+$ are found by substituting Eq. (5.9) in Eqs. (4.3), (4.4), (4.6), respectively. Note that $\varphi_2^*(-x,t) = a\sec P\exp(iQ) \neq \varphi_2(x,t)$, then the solution φ_2 is not a solution of NLS$^+$. For the sake of illustration, let us consider the special case $\nu = 0$. We have the following three swept-out surfaces:

(I) The position vector $\mathbf{X}_1(x,t)$ for the first moving is

$$\mathbf{X}_1(x,t) = \frac{1}{a}(-2\cos Q\sec P, 2\sin Q\sec P, i(P - 2\tan P)) \in \mathbb{C}^3. \tag{5.10}$$

It is a direct verification that $\kappa_1 = 2a\sec P$ and $\tau_1 = 0$. Note that $\kappa_1^*(-x,0) = \kappa_1(x,0)$, i.e., $\kappa_1(x,0)$ and $\tau_1(x,0)$ satisfying relation (3.4). This surface is given in Fig. 1 (a).



(a) (b) (c)

Figure 1: Surface swept-out by the moving curve $\mathbf{X}_1(x,t)$ (Eq. (5.10)) for $a = 1, d_1 = \pi$, and $d_2 = 2$; Surface swept-out by the moving curve $\mathbf{X}_2(x,t)$ (Eq. (5.11)) for $a = 1, d_1 = \frac{\pi}{2}$, and $d_2 = -2$; Surface swept-out by the moving curve $\mathbf{X}_3(x,t)$ (Eq. (5.12)) for $a = 1, d_1 = \frac{\pi}{2}$, and $d_2 = -2$.

(II) The position vector $\mathbf{X}_2(x,t)$ for the second moving is

$$\mathbf{X}_2(x,t) = \frac{i}{a}\left(P\sin Q, -P\cos Q, 0\right) \in \mathbb{C}^3. \tag{5.11}$$

Here, $\kappa_2 = 0$ and $\tau_2 = 2a\sec P$. This planar surface is given in Fig. 1 (b).

(III) The position vector $\mathbf{X}_3(x,t)$ for the third moving is

$$\begin{aligned}\mathbf{X}_3(x,t) = \frac{i}{a}(&-P\sin Q\sin\alpha + \cos Q(2\tan P - P)\cos\alpha,\\ &-P\cos Q\sin\alpha - \sin Q(2\tan P - P)\cos\alpha, 2i\sec P) \in \mathbb{C}^3,\end{aligned} \tag{5.12}$$

where $\alpha = a^2 t$. Here, $\kappa_3 = \kappa_1\cos\alpha, \tau_3 = \kappa_1\sin\alpha$. This surface is given in Fig. 1 (c).

6. Conclusion

In this paper, by using the idea of uniqueness of solutions, we show that, if an initial curve $\mathbf{X}_1^0(x)$ at $t = 0$ is generated by the complex curvature $\kappa_0(x)$ and the complex torsion curvature $\tau_0(x)$ satisfying $\kappa_1^0(x) = \epsilon\kappa_1^{0*}(-x)\exp[i\int_0^x(\tau_1^0(y) + \tau_1^{0*}(-y))dy]$, then the relation: $\kappa_1(x,t) = \epsilon\kappa_1^*(-x,t)\exp[i\int_0^x(\tau_1(y,t) + \tau_1^*(-y,t))dy]$ is preserved by the tangent vector $\mathbf{T_1}$ of complex curves $\mathbf{X}_1(x,t)$ in $\mathbb{C}^3$ satisfy CLL (1.6). This implies that soliton solutions $\varphi_2(x,t) = \kappa_1(x,t)\exp\left(i\int_0^x \tau_1(y,t)dy\right)$ to NNLS describe the above motion of complex curves $\mathbf{X}_1(x,t)$ in $\mathbb{C}^3$ with initial data at $t = 0$ being restricted by $\kappa_1^0(x) = \epsilon\kappa_1^{0*}(-x)\exp[i\int_0^x(\tau_1^0(y) + \tau_1^{0*}(-y))dy]$. Similarly, soliton solutions $\phi_2(x,t) = -\tau_2(x,t)\exp\left(i\int_0^x \kappa_2(y,t)dy\right)$ to NNLS describe the binormal vector $\mathbf{B}_2(x,t)$ of complex curves $\mathbf{X}_2(x,t)$ in $\mathbb{C}^3$ satisfy to CLL (1.6) with initial data at $t = 0$ being restricted by $\tau_2^0(x) = \epsilon\tau_2^{0*}(-x)\exp[i\int_0^x(\kappa_2^0(y) + \kappa_2^{0*}(-y))dy]$, and soliton solutions $\chi_2(x,t) = -(\kappa_3(x,t) + \sqrt{-1}\tau_3(x,t))$ to NNLS describe the normal vector $\mathbf{N}_3(x,t)$ of complex curves $\mathbf{X}_3(x,t)$ in $\mathbb{C}^3$ satisfy to CLL (1.6) with initial data at $t = 0$ being restricted by $\kappa_3^0(x) - \epsilon\kappa_3^{0*}(-x) = i(\tau_3^0(x) - \epsilon\tau_3^{0*}(-x))$. Hence NNLS arises from the motion of three distinct complex curves just with the initial data being suitably restricted. This gives a geometric interpretation for NNLS.

Acknowledgment

I would like to express my sincere gratitude to my advisor Q. Ding, for his instructive advice and useful suggestions on this paper. I am deeply grateful of his help in the completion of this paper. The author would like to thank the anonymous referees for their valuable comments which helped improve the paper significantly.

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