Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 12 (2019), 124–134
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Implicit hybrid methods for solving fractional Riccati equation

Muhammed I. Syam*, Azza Alsuwaidi, Asia Alneyadi, Safa Al Refai, Sondos Al Khaldi

Department of Mathematical Sciences, United Arab Emirates University, UAE.

Abstract

In this paper, we modify the implicit hybrid methods for solving fractional Riccati equation. Similar methods are implemented for the ordinary derivative and we are the first who implement it for fractional derivative case. This approach is of higher order comparing with the existing methods in the literature. We study the convergence, zero stability, consistency, and region of absolute stability. Numerical results are presented to show the efficiency of the proposed method.

Keywords: Fractional Riccati equation, implicit hybrid methods, convergence.

2010 MSC: 35C08, 74J35.

1. Introduction

Riccati equation is used for constructing the exact solutions of nonlinear fractional partial differential equations. Riccati equation appears in a wide range of contexts such that physics, biology, engineering, signal processing, systems identification, control theory, finance, and fractional dynamics. Fractional Riccati equation plays an important role in several physical and engineering applications. Since it is not easy task to find the exact solution of the fractional Riccati equation, several researchers investigated its solution numerically such as the Legendre wavelet operational matrix method [3], Adomian decomposition method [18], homotopy perturbation method [13], the Laplace transform and homotopy perturbation method [2], fractional Chebyshev finite difference method [10], the polynomial least squares method [4], and the Bezier curves [7]. In addition, artificial neural networks [20], the optimal homotopy asymptotic method [8] and the Laplace-Adomian-Pade method [12], Bäcklund transformation [17], and He's variational iteration method [9], are used to solved this problem. More methods can be found in [1, 5, 6, 22–26].

Fractional derivatives have several definitions. In this paper, we use the conformable fractional derivative which was presented in [11]. The definition is given as follows.

Definition 1.1. Let $y : [0, \infty) \to \mathbb{R}$ be a given function. The conformable fractional derivative of y of order α is defined by

$$D^{\alpha} y(t) = \lim_{h \to 0} \frac{y\left(t + h\, t^{1-\alpha}\right) - y(t)}{h}, \; t > 0, \; 0 < \alpha < 1.$$

*Corresponding author
Email address: m.syam@uaeu.ac.ae (Muhammed I. Syam)

doi: 10.22436/jnsa.012.02.06

Received: 2018-07-20 Revised: 2018-08-05 Accepted: 2018-08-16

Some properties of the conformable fractional derivative are given in the following theorem.

Theorem 1.2. *Let* $0 < \alpha < 1$ *and* u, v *are* α*-differentiable at* $t > 0$. *Then,*

1. $D^{\alpha}c = 0$ *for all constants* c;
2. $D^{\alpha}t^{\beta} = \beta t^{\beta-\alpha}$ *for all* $\beta \in \mathbb{R}$;
3. $D^{\alpha}(u \circ v)(t) = t^{1-\alpha}v'(t)\,u'(v(t))$;
4. $D^{\alpha}(uv)(t) = u(t)\,D^{\alpha}v(t) + v(t)D^{\alpha}u(t)$.

In this paper, we consider the following class of fractional order Riccati differential equations of the form

$$D^{\alpha}y(t) = a(t)\,y(t) + b(t)\,y^2(t) + r(t),\ 0 < t < 1,\ 0 < \alpha < 1 \tag{1.1}$$

subject to

$$y(0) = \vartheta, \tag{1.2}$$

where $\lim_{t\to 0}\frac{y(t)}{t}$ and $\lim_{t\to 0}\frac{y(t)}{t^{1-\alpha}}$ exist. Let $y(t) = t^{\alpha-1}u(t)$. Then, Eqs. (1.1)-(1.2) become

$$u'(t) = \left(\frac{a(t)}{t^{1-\alpha}} - \frac{(\alpha-1)}{t}\right)u(t) + b(t)\left(\frac{u(t)}{t^{1-\alpha}}\right)^2 + r(t) = h(t,u) \tag{1.3}$$

subject to

$$u(0) = 0.$$

We organize our paper as follows. In section 2, we present some definitions and results which we use in this paper. The proposed methods for solving Eqs. (1.1)-(1.2) are presented in Section 3. Some theoretical results will be presented in Section 4. Numerical results will be presented in Section 5. We draw some conclusions in Section 6.

2. Preliminaries

In this section, we review the implicit hybrid methods as well as some definitions related to these methods. Let $\{t_0, t_1, \ldots, t_M\}$ be a uniform partition of $[0,1]$ with $t_i = i\,\epsilon$, $i = 0, 1, \ldots, M$, and $h = \frac{1}{M}$.

Definition 2.1. A k-step hybrid formula is defined by

$$\sum_{i=0}^{k} a_i u_{n+i} + \sum_{i=0}^{l} a_{n+v_i} u_{n+v_i} = \epsilon \sum_{i=0}^{k} b_i h_{n+i} + \epsilon \sum_{i=0}^{l} b_{n+v_i} h_{n+v_i},$$

where $a_k = 1$, a_0 and b_0 are nonzeros, $v \notin \{0, 1, \ldots, k\}$, $u_{n+i} = u(t_n + i\,\epsilon)$ and $h_{n+v_i} = h(t_{n+v_i}, t_{n+v_i})$. For more details, see [14].

Definition 2.2. Let

$$\mathcal{L}[u[t_n];\epsilon] = \sum_{i=0}^{k} a_i u_{n+i} + \sum_{i=0}^{l} a_{n+v_i} u_{n+v_i} = \epsilon \sum_{i=0}^{k} b_i h_{n+i} + \epsilon \sum_{i=0}^{l} b_{n+v_i} h_{n+v_i} = c_0 u_n + c_1 u_n' + \cdots .$$

If $c_0 = 0, c_1 = 0, \ldots, c_{p+1} = 0, c_{p+1} \neq 0$, then the order of the method is p and the error constant is c_{p+1}, see [14].

Definition 2.3 ([14]). A linear multi-step method is said to be consistent if it has order at least one.

Definition 2.4 ([14]). If no zeros of the first characteristic polynomial have modulus greater than one and every root of modulus one has multiplicity not greater than one, then it is called zero stable.

Definition 2.5 ([14]). If the method is consistent and zero stable, it is convergent.

3. Methods of solution

In this section, we derive the proposed methods. We approximate the solution of Eq. 1.3 by

$$u(t) = \sum_{k=0}^{1+\gamma} a_k\, t^k \tag{3.1}$$

and its first derivative by

$$u'(t) = \sum_{k=1}^{1+\gamma} k a_k\, t^{k-1}, \tag{3.2}$$

where γ is the number of points. Let $\{t_0, \ t_1, t_2, \ldots, t_M\}$ be a uniform partition of $[0,1]$ with $\epsilon = \frac{1}{M}$ and $t_i = i\,\epsilon$. Collocate Eq. (3.1) at $t_{n+\frac{1}{\gamma}}$ and interpolate Eq. (3.2) at $t_{n+\frac{i}{\gamma}}$, $i = 0, 1, \ldots, r$ to get

$$Aa = R, \tag{3.3}$$

where $(a)_i = a_i$,

$$A_{i,j} = \begin{cases} t_{n+\frac{1}{\gamma}}^{j-1}, & i = 1, \\ (j-1)t_{n+\frac{i-2}{\gamma}}^{j-2}, & i > 1, \end{cases} \qquad R_i = \begin{cases} u_{n+\frac{1}{\gamma}} = u(t_{n+\frac{1}{\gamma}}), & i = 1, \\ h_{n+\frac{i-1}{\gamma}} = h(t_{n+\frac{i-1}{\gamma}}, u_{n+\frac{i-1}{\gamma}}), & i > 1, \end{cases}$$

for $i, j = 1, 2, \ldots, 2+\gamma$. Let $t_{n+\frac{1}{\gamma}} = t - \epsilon\, s$. Then, $t_{n+\frac{i}{\gamma}} = t - \epsilon\, s + \frac{(i-1)\,\epsilon}{r}$ and $t_n = t - \epsilon\, s - \frac{\epsilon}{r}$ for $i = 1, 2, \ldots, r$. Solving System (3.3), we get

$$u(t) = \alpha_0(s)\, u_{n+\frac{1}{\gamma}} + \sum_{i=0}^{r} \beta_i(s)\, h_{n+\frac{i}{\gamma}}, \tag{3.4}$$

where α_0 and $\beta_i(s)$ are given in Table 1.

Table 1: The values of α_0 and $\beta_i(s)$ for $\gamma = 3, 4, 5$.

	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$
$\alpha_0(s)$	1	1	1
$\beta_0(s)$	$\frac{\epsilon}{72}(13 - 108t^2 + 108t^3 + 243t^4$	$\frac{\epsilon}{180}(17 + 180s^2 + 160s^3 - 1440s^4 - 1536s^5)$	$\frac{\epsilon}{7200}(621 - 6000s^2 - 15000s^3 + 65625s^4 + 225000s^5 + 156250s^6)$
$\beta_1(s)$	$\frac{\epsilon}{72}(13 + 72t + 54t^2 - 216t^3 - 243t^4)$	$\frac{\epsilon}{360}(114 - 1080s^2 + 480s^3 + 5760s^4 + 4608s^5)$	$\frac{\epsilon}{7200}(1566 + 18000s^2 + 35000s^3 - 206250s^4 - 525000s^5 - 312500s^6)$
$\beta_2(s)$	$\frac{\epsilon}{72}(-1 + 36t^2 + 108t^3 + 81t^4)$	$\frac{\epsilon}{360}(34 + 360s + 600s^2 - 1600s^3 - 4800s^4 - 3072s^5)$	$\frac{\epsilon}{7200}(1566 - 36000s^2 - 10000s^3 + 318750s^4 + 600000s^5 + 312500s^6)$
$\beta_3(s)$	—	0	0
$\beta_4(s)$	—	—	0

For $t = t_{n+\frac{i}{\gamma}}$, $s = \frac{i-1}{\gamma}$ for $i = 0, 1, 2, 3, \ldots, \gamma$ and Eq. (3.4) becomes for the three points, four points, and five points, respectively.

1. Three points:

$$y_{n+1} = y_{n+\frac{1}{3}} + \frac{\epsilon}{72}\left(8f_{n+\frac{1}{3}} + 32f_{n+\frac{2}{3}} + 8f_{n+1}\right), \tag{3.5}$$

$$y_{n+\frac{2}{3}} = y_{n+\frac{1}{3}} + \frac{\epsilon}{72}\left(-f_n + 13f_{n+\frac{1}{3}} + 13f_{n+\frac{2}{3}} - f_{n+1}\right), \tag{3.6}$$

$$y_n = y_{n+\frac{1}{3}} + \frac{\epsilon}{72}\left(-9f_n - 19f_{n+\frac{1}{3}} + 5f_{n+\frac{2}{3}} - f_{n+1}\right). \tag{3.7}$$

2. Four points:

$$y_{n+1} = y_{n+\frac{1}{4}} + \frac{\epsilon}{2880}\left(-27f_n + 378f_{n+\frac{1}{4}} + 648f_{n+\frac{1}{2}} + 918f_{n+\frac{3}{4}} + 243f_{n+1}\right), \tag{3.8}$$

$$y_{n+\frac{3}{4}} = y_{n+\frac{1}{4}} + \frac{\epsilon}{360}\left(-f_n + 34f_{n+\frac{1}{4}} + 114f_{n+\frac{1}{2}} + 34f_{n+\frac{3}{4}} - f_{n+1}\right), \tag{3.9}$$

$$y_{n+\frac{2}{4}} = y_{n+\frac{1}{4}} + \frac{\epsilon}{2880}\left(-19f_n + 346f_{n+\frac{1}{4}} + 456f_{n+\frac{1}{2}} - 72f_{n+\frac{3}{4}} + 11f_{n+1}\right), \tag{3.10}$$

$$y_n = y_{n+\frac{1}{4}} + \frac{\epsilon}{2880}\left(-251f_n - 646f_{n+\frac{1}{4}} + 264f_{n+\frac{1}{2}} - 106f_{n+\frac{3}{4}} + 19f_{n+1}\right). \tag{3.11}$$

3. Five points:

$$y_{n+1} = y_{n+\frac{1}{5}} + \frac{\epsilon}{7200}\left(448f_{n+\frac{1}{5}} + 2048f_{n+\frac{2}{5}} + 768f_{n+\frac{3}{5}} + 2048f_{n+\frac{4}{5}} + 448f_{n+1}\right), \tag{3.12}$$

$$y_{n+\frac{4}{5}} = y_{n+\frac{1}{5}} + \frac{\epsilon}{7200}\left(-27f_n + 621f_{n+\frac{1}{5}} + 1566f_{n+\frac{2}{5}} + 1566f_{n+\frac{3}{5}} + 621f_{n+\frac{4}{5}} - 27f_{n+1}\right), \tag{3.13}$$

$$y_{n+\frac{3}{5}} = y_{n+\frac{1}{5}} + \frac{\epsilon}{7200}\left(-16f_n + 544f_{n+\frac{1}{5}} + 1824f_{n+\frac{2}{5}} + 544f_{n+\frac{3}{5}} - 16f_{n+\frac{4}{5}}\right), \tag{3.14}$$

$$y_{n+\frac{2}{5}} = y_{n+\frac{1}{5}} + \frac{\epsilon}{7200}\left(-27f_n + 637f_{n+\frac{1}{5}} + 1022f_{n+\frac{2}{5}} - 258f_{n+\frac{3}{5}} + 77f_{n+\frac{4}{5}} - 11f_{n+1}\right), \tag{3.15}$$

$$y_n = y_{n+\frac{1}{5}} + \frac{\epsilon}{7200}\left(-475f_n - 1427f_{n+\frac{1}{5}} + 798f_{n+\frac{2}{5}} - 482f_{n+\frac{3}{5}} + 173f_{n+\frac{4}{5}} - 27f_{n+1}\right). \tag{3.16}$$

Then, we solve the above systems iteratively.

4. Analysis of the methods

In this section, we analyze Eqs. (3.5)-(3.7), (3.8)-(3.11), and (3.12)-(3.16). Using the Taylor series to expand Eqs. (3.5)-(3.16) we have the following.

a) Three points:

$$y_{n+1} - y_{n+\frac{1}{3}} - \frac{\epsilon}{72}\left(8f_{n+\frac{1}{3}} + 32f_{n+\frac{2}{3}} + 8f_{n+1}\right) = -\frac{3\epsilon^6 y_n^{(6)}}{655360} + \cdots,$$

$$y_{n+\frac{2}{3}} - y_{n+\frac{1}{3}} - \frac{\epsilon}{72}\left(-f_n + 13f_{n+\frac{1}{3}} + 13f_{n+\frac{2}{3}} - f_{n+1}\right) = \frac{11\epsilon^5 y_n^{(5)}}{6174960} + \cdots,$$

$$y_n - y_{n+\frac{1}{3}} - \frac{\epsilon}{72}\left(-9f_n - 19f_{n+\frac{1}{3}} + 5f_{n+\frac{2}{3}} - f_{n+1}\right) = \frac{19\epsilon^5 y_n^{(5)}}{6174960} + \cdots.$$

Thus, the order of Eq. (3.5) is 5 and the error constant is $= -\frac{3\epsilon^6}{655360}$. Also, the order of the block system (3.5)-(3.7) is $(4, 4, 4)^T$ and the error constant is $\left(0, \frac{11\epsilon^5}{6174960}, \frac{19\epsilon^5}{6174960}\right)^T$. The first and section characteristic functions are $\pi_1(r) = r - r^{\frac{1}{3}}$ and $\pi_2(r) = \frac{1}{72}\left(8r^{\frac{1}{3}} + 32r^{\frac{2}{3}} + 8r\right)$. Thus,

1. the roots of $\pi_1(r)$ are 0 and 1, thus, all roots are simple;
2. $\pi_1(1) = 0$;
3. $\pi'_1(1) = \pi_2(1)$.

Hence, Eq. (3.5) is consistent, zero stable, and convergent. The interval of absolute stability is (0.367855, 2.71847) and the region of absolute stability is given in Figure 1.

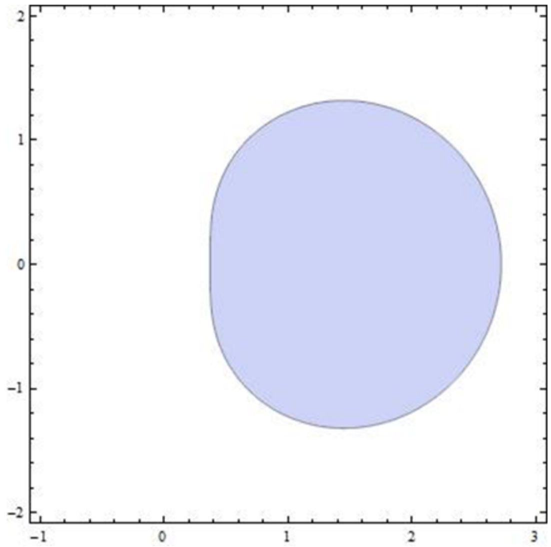

Figure 1: Region of absolute stability of three points method.

Let

$$A_1 = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, Y_{1,n} = \begin{pmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \end{pmatrix} A_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, Y_{2,n} = (y_n), F_{1,n} = (f_n), \ F_{2,n} = \begin{pmatrix} f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 \\ \frac{-\epsilon}{72} \\ \frac{-9\epsilon}{72} \end{pmatrix}, \ A_4 = \begin{pmatrix} \frac{8\epsilon}{72} & \frac{32\epsilon}{72} & \frac{8\epsilon}{72} \\ \frac{13\epsilon}{72} & \frac{13\epsilon}{72} & \frac{-\epsilon}{72} \\ \frac{-19\epsilon}{72} & \frac{5\epsilon}{72} & \frac{-\epsilon}{72} \end{pmatrix}.$$

Then, System (3.5)-(3.7) can be written in the matrix form as

$$A_1 Y_{1,n} = A_2 Y_{2,n} + A_3 F_{1,n} + A_4 F_{2,n}.$$

Multiply both sides of last equation by A_1^{-1} to get

$$B_1 Y_{1,n} = B_2 Y_{2,n} + B_3 F_{1,n} + B_4 F_{2,n},$$

where $B_1 = I_3$,

$$B_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Normalize last system to get

$$\hat{B}_2 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\det\left(sB_1 - \hat{B}_2\right) = (s+1)s^2.$$

Since the roots of the above equation which has modulus 1 is simple, the block method is zero stable as $\epsilon \to 0$.

b) Four points:

$$y_{n+1} - y_{n+\frac{1}{4}} - \frac{\epsilon}{2880}\left(-27f_n + 378f_{n+\frac{1}{4}} + 648f_{n+\frac{1}{2}} + 918f_{n+\frac{3}{4}} + 243f_{n+1}\right) = -\frac{3\epsilon^6}{655360} y_n^{(6)} + \cdots,$$

$$y_{n+\frac{3}{4}} - y_{n+\frac{1}{4}} - \frac{\epsilon}{360}\left(-f_n + 34f_{n+\frac{1}{4}} + 114f_{n+\frac{1}{2}} + 34f_{n+\frac{3}{4}} - f_{n+1}\right) = \frac{\epsilon^7}{12386304}y_n^{(7)} + \cdots,$$
$$y_{n+\frac{2}{4}} - y_{n+\frac{1}{4}} - \frac{\epsilon}{2880}\left(-19f_n + 346f_{n+\frac{1}{4}} + 456f_{n+\frac{1}{2}} - 72f_{n+\frac{3}{4}} + 11f_{n+1}\right) = -\frac{11\epsilon^6}{5898240}y_n^{(6)} + \cdots,$$
$$y_n - y_{n+\frac{1}{4}} - \frac{\epsilon}{2880}\left(-251f_n - 646f_{n+\frac{1}{4}} + 264f_{n+\frac{1}{2}} - 106f_{n+\frac{3}{4}} + 19f_{n+1}\right) = -\frac{3\epsilon^6}{655360}y_n^{(6)} + \cdots.$$

Thus, the order of Eq. (3.8) is 5 and the error constant is $= -\frac{3\epsilon^6}{655360}$. Also, the order of the block system (3.8)-(3.11) is $(5,5,5,5)^T$ and the error constant is $\left(-\frac{3\epsilon^6}{655360}, 0, -\frac{11\epsilon^6}{5898240}, -\frac{3\epsilon^6}{655360}\right)^T$. The first and section characteristic functions are $\pi_1(r) = r - r^{\frac{1}{4}}$ and $\pi_2(r) = -\frac{1}{2880}\left(-27 + 378r^{\frac{1}{4}} + 648r^{\frac{1}{2}} + 918r^{\frac{3}{4}} + 243r\right)$. Thus,

1. the roots of $\pi_1(r)$ are 0 and 1, thus, all roots are simple;
2. $\pi_1(1) = 0$;
3. $\pi'_1(1) = \pi_2(1)$.

Hence, Eq. (3.8) is consistent, zero stable, and convergent. The interval of absolute stability is (0.381829, 2.68863) and the region of absolute stability is given in Figure 2.

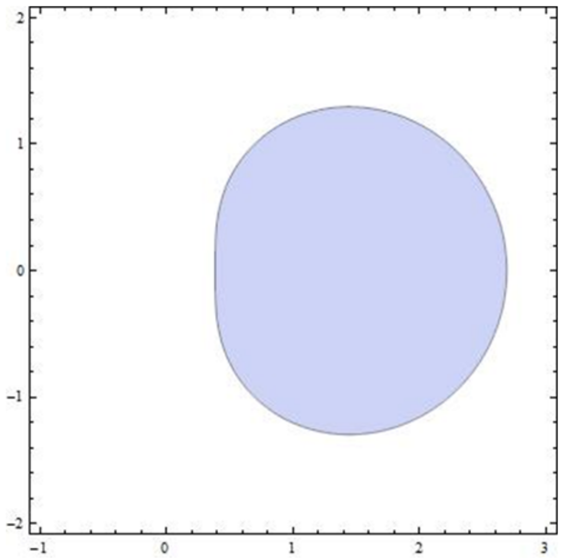

Figure 2: Region of absolute stability of the four points method.

Let

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad Y_{1,n} = \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad Y_{2,n} = (y_n), F_{1,n} = (f_n), \ F_{2,n} = \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \frac{-27\epsilon}{2880} \\ \frac{-\epsilon}{360} \\ \frac{-19\epsilon}{2880} \\ \frac{-251\epsilon}{2880} \end{pmatrix}, \quad A_4 = \begin{pmatrix} \frac{378\epsilon}{2880} & \frac{648\epsilon}{2880} & \frac{918\epsilon}{2880} & \frac{243\epsilon}{2880} \\ \frac{34\epsilon}{360} & \frac{114\epsilon}{360} & \frac{34\epsilon}{360} & \frac{-\epsilon}{360} \\ \frac{346\epsilon}{2880} & \frac{456\epsilon}{2880} & \frac{-72\epsilon}{2880} & \frac{-\epsilon}{2880} \\ \frac{-646\epsilon}{2880} & \frac{264\epsilon}{2880} & \frac{-106\epsilon}{2880} & \frac{19\epsilon}{2880} \end{pmatrix}.$$

Then, system (3.8)-(3.11) can be written in the matrix form as

$$A_1 Y_{1,n} = A_2 Y_{2,n} + A_3 F_{1,n} + A_4 F_{2,n}.$$

Multiply both sides of last equation by A_1^{-1} to get

$$B_1 Y_{1,n} = B_2 Y_{2,n} + B_3 F_{1,n} + B_4 F_{2,n},$$

where $B_1 = I_4$,

$$B_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

Normalize last system to get

$$\hat{B}_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\det\left(sB_1 - \hat{B}_2\right) = (s+1)s^3.$$

Since the roots of the above equation which has modulus 1 is simple, the block method is zero stable as $\epsilon \to 0$.

c) Five points:

$$\begin{aligned}
&y_{n+1} - y_{n+\frac{1}{5}} - \frac{\epsilon}{7200}\left(448 f_{n+\frac{1}{5}} + 2048 f_{n+\frac{2}{5}} + 768 f_{n+\frac{3}{5}} + 2048 f_{n+\frac{4}{5}} + 448 f_{n+1}\right) \\
&\quad = \frac{-8\epsilon^7}{73828125} y_n^{(7)} + \cdots, \\
&y_{n+\frac{4}{5}} - y_{n+\frac{1}{5}} - \frac{\epsilon}{7200}\left(-27 f_n + 621 f_{n+\frac{1}{5}} + 1566 f_{n+\frac{2}{5}} + 1566 f_{n+\frac{3}{5}} + 621 f_{n+\frac{4}{5}} - 27 f_{n+1}\right) \\
&\quad = \frac{13\epsilon^7}{175000000} y_n^{(7)} + \cdots, \\
&y_{n+\frac{3}{5}} - y_{n+\frac{1}{5}} - \frac{\epsilon}{7200}\left(-16 f_n + 544 f_{n+\frac{1}{5}} + 1824 f_{n+\frac{2}{5}} + 544 f_{n+\frac{3}{5}} - 16 f_{n+\frac{4}{5}}\right) \\
&\quad = \frac{\epsilon^7}{59062500} y_n^{(7)} + \cdots, \\
&y_{n+\frac{2}{5}} - y_{n+\frac{1}{5}} - \frac{\epsilon}{7200}\left(-27 f_n + 637 f_{n+\frac{1}{5}} + 1022 f_{n+\frac{2}{5}} - 258 f_{n+\frac{3}{5}} + 77 f_{n+\frac{4}{5}} - 11 f_{n+1}\right) \\
&\quad = \frac{271\epsilon^7}{4725000000} y_n^{(7)} + \cdots, \\
&y_n - y_{n+\frac{1}{5}} - \frac{\epsilon}{7200}\left(-475 f_n - 1427 f_{n+\frac{1}{5}} + 798 f_{n+\frac{2}{5}} - 482 f_{n+\frac{3}{5}} + 173 f_{n+\frac{4}{5}} - 27 f_{n+1}\right) \\
&\quad = \frac{863\epsilon^7}{4725000000} y_n^{(7)} + \cdots.
\end{aligned}$$

Thus, the order of Eq. (3.12) is 6 and the error constant is $= \frac{-8\epsilon^7}{73828125}$. Also, the order of the block System (3.12)-(3.16) is $(6,6,6,6,6)^T$ and the error constant is $\left(\frac{-8\epsilon^7}{73828125}, \frac{13\epsilon^7}{175000000}, \frac{\epsilon^7}{59062500}, \frac{271\epsilon^7}{4725000000}, \frac{863\epsilon^7}{4725000000}\right)^T$.

The first and section characteristic functions are $\pi_1(r) = r - r^{\frac{1}{5}}$ and $\pi_2(r) = \frac{\epsilon}{7200}\left(448 r^{\frac{1}{5}} + 2048 r^{\frac{2}{5}} + 768 r^{\frac{3}{5}} + 2048 r^{\frac{4}{5}} + 448 r\right)$. Thus,

1. the roots of $\pi_1(r)$ are 0 and 1, thus, all roots are simple;
2. $\pi_1(1) = 0$;
3. $\pi'_1(1) = \pi_2(1)$.

Hence, Eq. (3.12) is consistent, zero stable, and convergent. The interval of absolute stability is (0.367879, 2.71828) and the region of absolute stability is given in Figure 3.

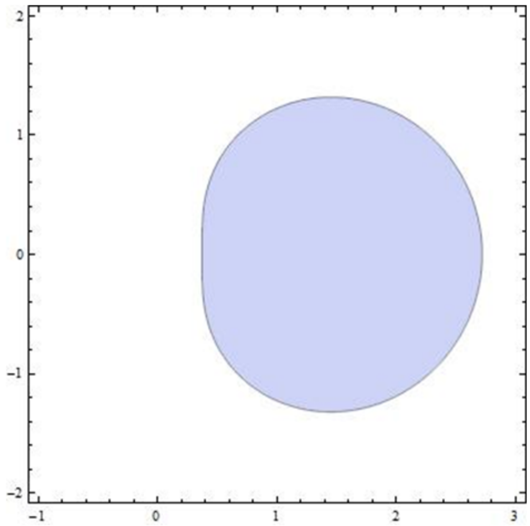

Figure 3: Region of Absolute stability for the five points method.

Let

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, Y_{1,n} = \begin{pmatrix} y_{n+\frac{1}{5}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{4}{5}} \\ y_{n+1} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, Y_{2,n} = (y_n), F_{1,n} = (f_n), \ F_{2,n} = \begin{pmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 \\ \frac{-27\epsilon}{7200} \\ \frac{-16\epsilon}{7200} \\ \frac{-27\epsilon}{7200} \\ \frac{-475\epsilon}{7200} \end{pmatrix}, \ A_4 = \begin{pmatrix} \frac{448\epsilon}{7200} & \frac{2048\epsilon}{7200} & \frac{768\epsilon}{7200} & \frac{2048\epsilon}{7200} & \frac{448\epsilon}{7200} \\ \frac{621\epsilon}{7200} & \frac{1566\epsilon}{7200} & \frac{1566\epsilon}{7200} & \frac{621\epsilon}{7200} & \frac{-27\epsilon}{7200} \\ \frac{544\epsilon}{7200} & \frac{1824\epsilon}{7200} & \frac{544\epsilon}{7200} & \frac{-16\epsilon}{7200} & 0 \\ \frac{637\epsilon}{7200} & \frac{1022\epsilon}{7200} & \frac{-258\epsilon}{7200} & \frac{77\epsilon}{7200} & \frac{-11\epsilon}{7200} \\ \frac{-1427\epsilon}{7200} & \frac{798\epsilon}{7200} & \frac{-482\epsilon}{7200} & \frac{173\epsilon}{7200} & \frac{-27\epsilon}{7200} \end{pmatrix}.$$

Then, System (3.12)-(3.16) can be written in the matrix form as

$$A_1 Y_{1,n} = A_2 Y_{2,n} + A_3 F_{1,n} + A_4 F_{2,n}.$$

Multiply both sides of last equation by A_1^{-1} to get

$$B_1 Y_{1,n} = B_2 Y_{2,n} + B_3 F_{1,n} + B_4 F_{2,n},$$

where $B_1 = I_4$,

$$B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Normalize last system to get

$$\hat{B}_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\det\left(sB_1 - \hat{B}_2\right) = (s+1)s^4.$$

Since the roots of the above equation which has modulus 1 is simple, the block method is zero stable as $\epsilon \to 0$.

5. Numerical results

In this section, we present two examples of our results. Comparison with [15, 16, 19, 21] will be presented.

Example 5.1. Consider the following problem

$$D^{\alpha} y\,(t) = 1 + 2y\,(t) - y^2\,(t)\,,\ 0 \leqslant t \leqslant 1$$

subject to

$$y\,(0) = 0.$$

The exact solution is

$$y\,(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{\sqrt{2}-1}{2+2\sqrt{2}}\right).$$

In Table 2, we compare of the absolute errors of our results when $\epsilon = 0.1$ and the results in [15, 16, 19, 21] for $\alpha = 0.75$.

Table 2: The absolute errors of our results when $\epsilon = 0.1$ and the results in [15, 16, 19, 21] for $\alpha = 0.75$.

t	Method [21]	Method [16]	Method [19]	Method [15]	Three points	Four points	Five points
0.2	0.23	0.23	0.19	0.34	2.1×10^{-4}	3.1×10^{-5}	2.6×10^{-6}
0.4	0.37	0.37	0.32	0.46	2.4×10^{-4}	3.2×10^{-5}	2.7×10^{-6}
0.5	0.39	0.39	0.38	0.44	2.8×10^{-4}	3.6×10^{-5}	2.9×10^{-6}
0.6	0.38	0.38	0.42	0.40	3.1×10^{-4}	3.9×10^{-5}	3.1×10^{-6}
0.8	0.28	0.28	0.45	0.25	3.3×10^{-4}	4.1×10^{-5}	3.2×10^{-6}
1	0.13	0.13	0.40	0.11	3.4×10^{-4}	4.2×10^{-5}	3.5×10^{-6}

Example 5.2. Consider the following problem

$$D^{\alpha} y\,(t) = -y\,(t) - y^2\,(t) + r(t),\ 0 \leqslant t \leqslant 1$$

subject to

$$y\,(0) = 0,$$

where

$$r(t) = \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} + t^2 + t^4.$$

The exact solution is

$$y(t) = t^2.$$

In Table 3, we present our results for $\alpha = 0.65$.

Table 3: Our results when $\epsilon = 0.1$ for $\alpha = 0.65$.

t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Our method	0.01	0.04	0.09	0.16	0.25	0.36	0.49	0.64	0.81	1

6. Conclusion

In this paper, we modify the implicit hybrid methods for solving fractional Riccati equation. Similar methods are implemented for the ordinary derivative and we are the first who implement it for the fractional derivative case. This approach is of higher order compared with the existing methods in the literature. We study the convergence, zero stability, consistency, and region of absolute stability. Numerical results are presented to show the efficiency of the proposed method. We notice the following.

1. The order of the one step-three hybrid point method is $(4, 4, 4)$.
2. The order for the one step-three hybrid point method is $(5, 5, 5, 5)$.
3. The order of the one step-three hybrid point method is $(6, 6, 6, 6, 6)$.
4. All methods are consistent, zero stable, and convergent.
5. We draw the region of absolute stability for the three, four, and five points in Figures 1-3, respectively.
6. From Example 5.1, we see that our results are more accurate than the results in [15, 16, 19, 21].
7. In Example 5.2, we get the exact solution.
8. The proposed method is accurate and can be applied to more physical and engineering problems.

Acknowledgment

The authors would like to express their sincere appreciation to United Arab Emirates University for the financial support of Grant no. G00002758. Also, we would like to express our sincere thanks to the reviewers for their valuable comments.

References

[1] Q. M. Al-Mdallal, M. I. Syam, M. N. Anwar, *A collocation-shooting method for solving fractional boundary value problems*, Commun. Nonlinear Sci. Numer. Simul., **15** (2010), 3814–3822. 1
[2] H. Aminikhah, A. H. R. Sheikhani, H. Rezazadeha, *Approximate analytical solutions of distributed order fractional Riccati differential equation*, Ain Shams Eng. J., **2016** (2016), 8 pages. 1
[3] S. Balaji, *Legendre wavelet operational matrix method for solution of fractional order Riccati differential equation*, J. Egyptian Math. Soc., **23** (2015), 263–270. 1
[4] C. Bota, B. Caruntu, *Analytical approximate solutions for quadratic Riccati differential equation of fractional order using the Polynomial Least Squares Method*, Chaos Solitons Fractals, **102** (2017), 339–345. 1
[5] M. F. El-Sayed, M. I. Syam, *Electrohydrodynamic instability of a dielectric compressible liquid sheet streaming into an ambient stationary compressible gas*, Arc. Appl. Mech., **77** (2007), 613–626. 1
[6] M. F. El-Sayed, M. I. Syam, *Numerical study for the electrified instability of viscoelastic cylindrical dielectric fluid film surrounded by a conducting gas*, Phy. A: Stat. Mech. Appl., **377** (2007), 381–400. 1
[7] F. Ghomanjani, *A numerical technique for solving fractional optimal control problems and fractional Riccati differential equations*, J. Egyptian Math. Soc., **24** (2016), 638–643. 1

[8] M. Hamarsheh, A. I. Ismail, Z. Odibat, *An Analytic Solution for Fractional Order Riccati Equations by Using Optimal Homotopy Asymptotic Method*, Appl. Math. Sci., **10** (2016), 1131–1150. 1

[9] H. Jafari, H. Tajadodi, *He's Variational Iteration Method for Solving Fractional Riccati Differential Equation*, Int. J. Differ. Equ., **2010** (2010), 8 pages. 1

[10] M. M. Khader, *Numerical treatment for solving fractional Riccati differential equation*, J. Egyptian Math. Soc., **21** (2013), 32–37. 1

[11] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70. 1

[12] N. A. Khan, A. Ara, N. A. Khan, *Fractional-order Riccati differential equation: Analytical approximation and numerical results*, Adv. Difference Equ., **2013** (2013), 16 pages. 1

[13] N. A. Khana, A. Ara, M. Jamil, *An efficient approach for solving the Riccati equation with fractional orders*, Comput. Math. Appl., **61** (2011), 2683–2689. 1

[14] J. D. Lambert, *Numerical Methods for Ordinary Differential Systems*, John Wiley & Sons, Chichester, (1991). 2.1, 2.2, 2.3, 2.4, 2.5

[15] Y. L. Li, *Solving a nonlinear fractional differential equation using Chebyshev wavelets*, Commun. Nonlinear Sci. Numer. Simul., **15** (2010), 2284–2292. 5, 5.1, 2, 6

[16] X. Y. Li, B. Y. Wu, R. T. Wang, *Reproducing kernel method for fractional Riccati differential equations*, Abstr. Appl. Anal., **2014** (2014), 6 pages. 5, 5.1, 2, 6

[17] Bin Lu , *Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations*, Phys. Lett. A, **376** (2012), 2045–2048. 1

[18] S. Momani, N. Shawagfeh, *Decomposition method for solving fractional Riccati differential equations*, Appl. Math. Comput., **182** (2006), 1083–1092. 1

[19] Z. Odibat, S. Momani, *Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order*, Chaos Solitons Fractals, **36** (2008), 167–174. 5, 5.1, 2, 6

[20] M. Z. Raja, M. Manzar, R. Samarc. *An efficient computational intelligence approach for solving fractional order Riccati equations using ANN and SQP*, Appl. Math. Model., **39** (2015), 3075–3093. 1

[21] M. G. Sakar , A. Akgül, D. Baleanu, *On solutions of fractional Riccati differential equations*, Adv. Difference Equ., **2017** (2017), 10 pages. 5, 5.1, 2, 6

[22] M. I. Syam, *Cubic spline interpolation predictors over implicitly defined curves*, J. Comput. Appl. Math., **157** (2003), 283–295. 1

[23] M. I syam, *The modified Broyden-variational method for solving nonlinear elliptic differential equations*, Chaos Solitions Fractals, **32** (2007), 392–404.

[24] M. I. Syam, B. S. Attili, *Numerical solution of singularly perturbed fifth order two point boundary value problem*, Appl. Math. Comput., **170** (2005), 1085–1094.

[25] M. I. Syam, H. I. Siyyam, *An efficient technique for finding the eigenvalues of fourth-order Sturm-Liouville problems*, Chaos Solitons Fractals, **39** (2009), 659–665.

[26] M. I. Syam, H. I. Siyyam, *Numerical differentiation of implicitly defined curves*, J. Comput. Appl. Math., **108** (1999), 131–144. 1