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The odd Frechet inverse Weibull distribution with application

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Abstract

A new three parameters distribution called the odd Frechet inverse Weibull (OFIW) distribution is introduced. The reliability analysis of the new model is discussed. Several of its mathematical properties are studied. The maximum likelihood (ML) estimation are derived for OFIW parameters. The importance and flexibility of the OFIW is assessed using one real data set.

Keywords: Odd Frechet family, inverse Weibull distribution, order statistics, maximum likelihood.

2010 MSC: 60E05, 62E10, 62N05.

1. Introduction

Inverse Weibull (IW) distribution has wider application in the field of reliability and biological studies due to its failure rate. Keller and Kanath [10] introduced the IW distribution to study the shape of the density and the failure rate function. The IW distribution provides a good fit of several data sets in terms of times to breakdown of an insulating fluid, the subject leaded to the action of constant tension, see Nelson [15], Jiang et al. [9] presented Weibull and Weibull inverse mixture models. Jiang et al. [8] discussed the models involving two IW distributions. Khan et al. [13] studied the flexibility of the IW distribution.

The probability density function (pdf) and cumulative distribution function (cdf) of IW distribution

$$g\left(x{:}\alpha\right)=\frac{\beta\alpha}{x^{\beta+1}}e^{-\frac{\alpha}{x^{\beta}}},\quad x,\alpha,\beta>0, \tag{1.1}$$

and

$$G\left(x{:}\alpha\right)=e^{-\frac{\alpha}{x^{\beta}}},\quad x,\alpha\,,\beta>0. \tag{1.2}$$

IW distribution have recently been proposed in literature on statistical theory. de Gusmão et al. [3] pro-

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posed the generalized IW distribution and discussed several properties of this model with applications. Modified inverse Weibull distribution has been proposed by Khan and King [11], Shahbaz et al. [16] proposed the Kumaraswamy IW distribution. Hanook et al. [6] derived beta IW distribution, Khan et al. [12] studied characterizations of the transmuted IW distribution with an application to bladder cancer remission times data. Abbas et al. [1] introduced topp-Leone IW distribution. Elbatal et al. [4] studied the beta generalized IW geometric distribution.

Recently, Haq and Elgarhy [7] studied *odd Frechet generated (OF-G) family of distributions*. The cdf of OF-G is given by:

$$F(x{:}\ \theta,\ \xi)=\int_0^{\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]}\frac{\theta}{x^{\theta+1}}e^{-x^{-\theta}}dx=e^{-\left[\frac{1-G(x;\xi)}{G(x;\xi)}\right]^{\theta}},\qquad x\in\mathbb{R},\ \theta>0. \tag{1.3}$$

The corresponding pdf to (1.3) is given by

$$f(x:\ \theta,\ \xi)=\frac{\theta g(x;\xi)\left[1-G(x;\xi)\right]^{\theta-1}}{G(x;\xi)^{\theta+1}}e^{-\left[\frac{1-G(x;\xi)}{G(x;\xi)}\right]^{\theta}}, \tag{1.4}$$

where $g(x:\ \xi)$ considers a pdf of baseline distribution. Hereafter, a random variable X with density function (1.4) is denoted by $X\sim\text{OF-G}(\theta,\ \xi)$.

In this paper, we define a new lifetime model called the *odd Frechet inverse Weibull (OFIW)* distribution. The cdf of OFIW distribution with set of parameters $\varphi=(\alpha,\ \beta,\ \theta)$is obtained by substituting (1.2) in (1.3) as

$$F(x;\ \theta,\ \alpha,\beta)=e^{-\left[e^{\frac{\alpha}{x^{\beta}}}-1\right]^{\theta}},\quad x,\ \alpha,\theta,\ \beta>0. \tag{1.5}$$

The corresponding pdf to (1.5) is given by inserting (1.1) and (1.2) in (1.4) as

$$f(x;\ \theta,\ \alpha,\ \beta)=\frac{\beta\theta\alpha}{x^{\beta+1}}e^{\frac{\alpha}{x^{\beta}}}\left[e^{\frac{\alpha}{x^{\beta}}}-1\right]^{\theta-1}e^{-\left[e^{\frac{\alpha}{x^{\beta}}}-1\right]^{\theta}},\quad x,\ \alpha,\theta,\ \beta>0, \tag{1.6}$$

where α is a scale parameter and $\theta,\ \beta$ are two shape parameters.

The OFIW distribution is a very flexible model that includes some distributions when $\beta=1$ we get odd Frechet inverse exponential distribution and when $\beta=2$ we get odd Frechet inverse Rayleigh distribution. Figure 1 displays some plots of the OFIW pdf for some different values of parameters.

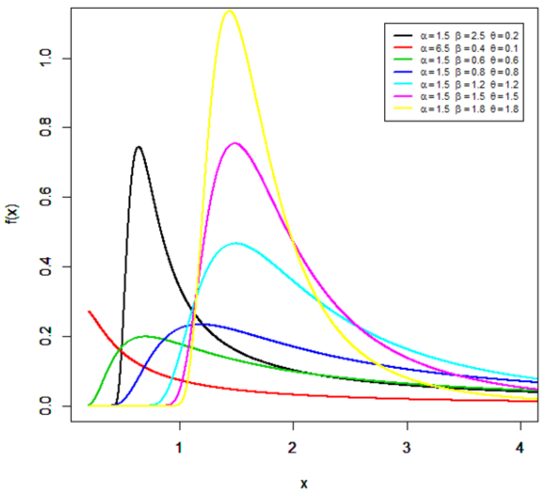

Figure 1: Plots of the pdf of the OFIW distribution for different values of parameters.

From Figure 1, we conclude that pdf of OFIW distribution can be uni-model and right skewed.

We aim that it will attract wider applications in engineering, medicine and other areas of research. This paper is organized as follows. In Section 2, reliability analysis is discussed. Section 3 studies the linear representation of the pdf for OFIW distribution. Statistical properties is studied in Section 4. The maximum likelihood method of estimation is applied to calculate the estimates of the OFIW parameters in Section 5. The analyses of one real data set is employed in Section 6. Concluding remarks are presented in Section 7.

2. The reliability analysis

The survival function (sf), hazard rate function (hrf), reversed hrf, and cumulative hrf of X are given, respectively, as

$$R(x;\theta,\alpha,\beta) = 1 - e^{-\left[e^{\frac{\alpha}{x^{\beta}}}-1\right]^{\theta}},$$

$$h(x;\theta,\alpha,\beta) = \frac{\frac{\beta\theta\alpha}{x^{\beta+1}} e^{\frac{\alpha}{x^{\beta}}}\left[e^{\frac{\alpha}{x^{\beta}}}-1\right]^{\theta-1} e^{-\left[e^{\frac{\alpha}{x^{\beta}}}-1\right]^{\theta}}}{1-e^{-\left[e^{\frac{\alpha}{x^{\beta}}}-1\right]^{\theta}}},$$

$$\tau(x;\theta,\alpha,\beta) = \frac{2\theta\alpha}{x^{3}} e^{\frac{\alpha}{x^{2}}}\left[e^{\frac{\alpha}{x^{2}}}-1\right]^{\theta-1},$$

and

$$H(x;\theta,\alpha,\beta) = -\ln\left(1-e^{-\left[e^{\frac{\alpha}{x^{2}}}-1\right]^{\theta}}\right).$$

Figure 2 displays some plots of the OFIW hrf for some different values of parameters.

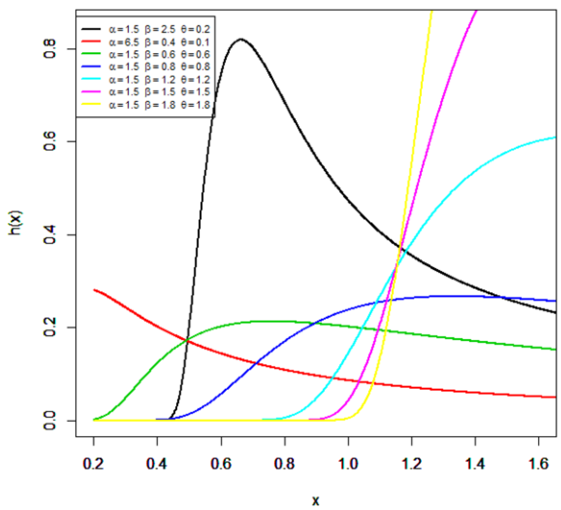

Figure 2: Plots of the hrf of the (OFIW) distribution for different values of parameters.

From Figure 2, we conclude that the hrf of OFIW distribution can be J-shaped and unimodal.

3. Useful expansion

In this section expansion of the pdf and cdf for OFIW distribution are calculated.

Haq and Elgarhy [7] expressed the equation (1.4) as

$$f(x) = \sum_{k=0}^{\infty} \eta_k g(x,\xi) G(x,\xi)^k, \tag{3.1}$$

where

$$\eta_k = \sum_{i,j=0}^{\infty} \frac{\theta(-1)^{i+k}}{i!} \binom{\theta(i+1)+j}{j} \binom{\theta(i+1)+j-1}{k}.$$

By inserting (1.6) in (3.1) we can rewrite the OFIW as a linear combination of IW distribution as

$$f(x) = \sum_{k=0}^{\infty} \frac{w_k}{x^{\beta+1}} e^{-\frac{\alpha(k+1)}{x^\beta}}, \tag{3.2}$$

where $w_k = \beta\alpha\eta_k$.

4. Statistical properties

In this section some statistical properties of the OFIW distribution are obtained.

4.1. *Quantile function*

The quantile function, say $Q(u) = F^{-1}(u)$ of X is given by

$$u = e^{-\left[e^{\frac{\alpha}{x^\beta}}-1\right]^{\theta}}.$$

After some simplifications, it reduces to the following form

$$Q(u) = \sqrt[\beta]{\frac{\alpha}{\ln\left(1+\left[\ln\left(\frac{1}{u}\right)\right]^{\frac{1}{\theta}}\right)}}, \tag{4.1}$$

where u is considered as a uniform random variable on the unit interval $(0, 1)$.

In particular, the median can be derived from (4.1) by setting $u = 0.5$. That is, the median (M) is given by

$$M = \sqrt[\beta]{\frac{\alpha}{\ln\left(1+[\ln(2)]^{\frac{1}{\theta}}\right)}}.$$

4.2. *Moments*

If X has the pdf (3.2), then its rthmoment is given from the following relation

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x;\varphi)dx. \tag{4.2}$$

Substituting (3.2) into (4.2) yields

$$\mu_r' = E(X^r) = \sum_{k=0}^{\infty} w_k \int_0^{\infty} x^{r-\beta-1} e^{-\alpha(k+1)x^{-\beta}} dx.$$

Let $y = x^{-\beta}$, then

$$\mu_r' = \sum_{k=0}^{\infty} \frac{w_k}{\beta} \int_0^{\infty} y^{\frac{-r}{\beta}} e^{-\alpha(k+1)y} dx.$$

Then μ_r' becomes

$$\mu_r' = \sum_{k=0}^{\infty} \frac{w_k \Gamma(1-\frac{r}{\beta})}{\beta\,[\alpha(k+1)]^{1-\frac{r}{\beta}}}, \quad r < \beta.$$

The moment generating function of OFIW distribution is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{r,k=0}^{\infty} \frac{t^r}{r!} \frac{w_k \Gamma(1-\frac{r}{\beta})}{\beta\,[\alpha(k+1)]^{1-\frac{r}{\beta}}}, \quad r < \beta.$$

4.3. Incomplete and conditional moments

The incomplete moments, say $\varphi_s(t)$, is given by

$$\varphi_s(t) = \int_0^t x^s f(x;\varphi)dx.$$

Using (3.2), then $\varphi_s(t)$ can be written as follows

$$\varphi_s(t) = \sum_{k=0}^{\infty} w_k \int_0^t x^{s-\beta-1} e^{-\alpha(k+1)x^{-\beta}} dx.$$

Then, using the lower incomplete gamma function, we obtain

$$\varphi_s(t) = \sum_{k=0}^{\infty} w_k \frac{\nu\left(1-\frac{s}{\beta}, \alpha(k+1)t^{-\beta}\right)}{\beta\,(\alpha(k+1))^{1-\frac{s}{\beta}}}, \quad s < \beta,$$

where $\nu(s,t) = \int_0^t x^{s-1} e^{-x} dx$ is the lower incomplete gamma function.

Further, the conditional moment say $\tau_s(t)$, is given by

$$\tau_s(t) = \int_t^{\infty} x^s f(x;\varphi)dx.$$

Hence, by using pdf (3.2), we can write

$$\tau_s(t) = \sum_{k=0}^{\infty} w_k \int_t^{\infty} x^{s-\beta-1} e^{-\alpha(k+1)x^{-\beta}} dx.$$

Then using the upper incomplete gamma function, we obtain

$$\tau_s(t) = \sum_{k=0}^{\infty} w_k \frac{\Gamma\left(1-\frac{s}{\beta}, \alpha(k+1)t^{-\beta}\right)}{\beta\,(\alpha(k+1))^{1-\frac{s}{\beta}}}, \quad s < \beta,$$

where $\Gamma(s,t) = \int_t^{\infty} x^{s-1} e^{-x} dx$ is the upper incomplete gamma function.

4.4. Order statistics

Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics of a random sample of size n following the OFIW distribution, with parameters α, β, and θ, then the pdf of the k^{th} order statistic, can be written as follows

$$f_{k:n}(x) = \frac{1}{B(k, n-k+1)} f(x)F(x)^{k-1}(1-F(x))^{n-k}, \tag{4.3}$$

where $B(.,.)$ is the beta function. By substituting (1.5) and (1.6) in (4.3), then

$$f_{k:n}(x) = \frac{\beta\theta\alpha}{B(k, n-k-1)} x^{-\beta-1} e^{\frac{\alpha}{x^{\beta}}} \left[e^{\frac{\alpha}{x^{\beta}}} - 1\right]^{\theta-1} e^{-k\left[e^{\frac{\alpha}{x^{\beta}}}-1\right]^{\theta}} \left(1 - e^{-\left[e^{\frac{\alpha}{x^{\beta}}}-1\right]^{\theta}}\right)^{n-k}. \tag{4.4}$$

When we put $k = 1$ in (4.4) we get the pdf of the smallest order statistics as

$$f_{1:n}(x) = n\beta\theta\alpha x^{-\beta-1} e^{\frac{\alpha}{x^\beta}} \left[e^{\frac{\alpha}{x^\beta}} - 1\right]^{\theta-1} e^{-\left[e^{\frac{\alpha}{x^\beta}}-1\right]^\theta} \left(1 - e^{-\left[e^{\frac{\alpha}{x^\beta}}-1\right]^\theta}\right)^{n-1},$$

when we put $k = n$ in (4.4) we get the pdf of the largest order statistics as

$$f_{k:n}(x) = n\beta\theta\alpha x^{-\beta-1} e^{\frac{\alpha}{x^\beta}} \left[e^{\frac{\alpha}{x^\beta}} - 1\right]^{\theta-1} e^{-n\left[e^{\frac{\alpha}{x^\beta}}-1\right]^\theta}.$$

4.5. *Inequality measures*

In this subsection, we will calculated Lorenz, Bonferroni, and Zenga curves for the OFIW distribution. The Lorenz, Bonferroni, and Zenga curves are obtained, respectively, as

$$L_F(x) = \frac{\int_0^t xf(x)dx}{E(X)} = \frac{\sum_{k=0}^{\infty} w_k \frac{\nu\left(\frac{1}{\beta}, \alpha(k+1)t^{-\beta}\right)}{\beta(\alpha(k+1))^{\frac{1}{\beta}}}}{\sum_{k=0}^{\infty} \frac{w_k \Gamma(\frac{1}{\beta})}{\beta[\alpha(k+1)]^{\frac{1}{\beta}}}},$$

$$B_F(x) = \frac{\int_0^t xf(x)dx}{E(X)F(x)} = \frac{L_F(x)}{F(x)} = \frac{\sum_{k=0}^{\infty} w_k \frac{\nu\left(\frac{1}{\beta}, \alpha(k+1)t^{-\beta}\right)}{\beta(\alpha(k+1))^{\frac{1}{\beta}}}}{\left(\sum_{k=0}^{\infty} \frac{w_k \Gamma(\frac{1}{\beta})}{\beta[\alpha(k+1)]^{\frac{1}{\beta}}}\right) e^{-\left(e^{\frac{\alpha}{x^\beta}}-1\right)^\theta}},$$

and

$$A_F(x) = 1 - \frac{\mu^-(x)}{\mu^+(x)},$$

where

$$\mu^-(x) = \frac{\int_0^t xf(x)dx}{E(X)} = \frac{\sum_{k=0}^{\infty} w_k \frac{\nu\left(\frac{1}{\beta}, \alpha(k+1)t^{-\beta}\right)}{\beta(\alpha(k+1))^{\frac{1}{\beta}}}}{\sum_{k=0}^{\infty} \frac{w_k \Gamma(\frac{1}{\beta})}{\beta[\alpha(k+1)]^{\frac{1}{\beta}}}},$$

and

$$\mu^+(x) = \frac{\int_t^\infty xf(x)dx}{1 - F(x)} = \frac{\sum_{k=0}^{\infty} w_k \frac{\Gamma\left(\frac{1}{\beta}, \alpha(k+1)t^{-\beta}\right)}{\beta(\alpha(k+1))^{\frac{1}{\beta}}}}{1 - e^{-\left(e^{\frac{\alpha}{x^\beta}}-1\right)^\theta}}.$$

5. Maximum likelihood estimation

The maximum likelihood estimates of the unknown parameters for the OFIW distribution are determined based on complete samples. Let $X_1, \ldots, X_n$ be observed values from the OFIW distribution with set of parameters $\varphi = (\alpha, \beta, \theta)^T$. The total log-likelihood function for the vector of parameters φ can be expressed as

$$\ln L(\varphi) = n \ln\theta + n\ln\beta + n\ln\alpha - (\beta+1)\sum_{i=1}^{n} \ln x_i$$

$$+\alpha\sum_{i=1}^{n}\frac{1}{x_i{}^{\beta}}+(\theta-1)\sum_{i=1}^{n}\ln\left(e^{\frac{\alpha}{x_i{}^{\beta}}}-1\right)-\sum_{i=1}^{n}\left(e^{\frac{\alpha}{x_i{}^{\beta}}}-1\right)^{\theta}.$$

The elements of the score function $U(\varphi)=(U_\alpha, U_\alpha, U_\theta)$ are given by

$$U_\alpha=\frac{n}{\alpha}+\sum_{i=1}^{n}\frac{1}{x_i{}^{\beta}}+(\theta-1)\sum_{i=1}^{n}\frac{\frac{1}{x_i{}^{\beta}}e^{\frac{\alpha}{x_i{}^{\beta}}}}{e^{\frac{\alpha}{x_i{}^{\beta}}}-1}-\theta\sum_{i=1}^{n}\frac{1}{x_i{}^{\beta}}e^{\frac{\alpha}{x_i{}^{\beta}}}\left(e^{\frac{\alpha}{x_i{}^{\beta}}}-1\right)^{\theta-1},$$

$$U_\beta=\frac{n}{\beta}-\sum_{i=1}^{n}\ln x_i-\alpha\beta\sum_{i=1}^{n}\frac{1}{x_i{}^{\beta+1}}-\alpha\beta(\theta-1)\sum_{i=1}^{n}\frac{\frac{1}{x_i{}^{\beta+1}}e^{\frac{\alpha}{x_i{}^{\beta}}}}{e^{\frac{\alpha}{x_i{}^{\beta}}}-1}+\alpha\beta\theta\sum_{i=1}^{n}\frac{1}{x_i{}^{\beta+1}}e^{\frac{\alpha}{x_i{}^{\beta}}}\left(e^{\frac{\alpha}{x_i{}^{\beta}}}-1\right)^{\theta-1},$$

and

$$U_\theta=\frac{n}{\theta}+\sum_{i=1}^{n}\ln\left(e^{\frac{\alpha}{x_i{}^{\beta}}}-1\right)-\sum_{i=1}^{n}\left(e^{\frac{\alpha}{x_i{}^{\beta}}}-1\right)^{\theta}\ln\left(e^{\frac{\alpha}{x_i{}^{\beta}}}-1\right).$$

Then the maximum likelihood estimators of the parameters α and θ are obtained by setting U_α, U_β, and U_θ to be zero and solving them. Clearly, it is difficult to solve them, therefore applying the Newton-Raphson's iteration method and using the computer packages such as Maple or R or other softwares.

6. Application

In this section, we provide an application to a real data set to assess the flexibility of the OFIW model. In order to compare the OFIW model with other fitted distributions we compare the fits of the OFIW distribution with the generalized Sujatha (GS) (Shanker et al. [19]), Sujatha (S) (Shanker [18]), Aradhana (A) (Shanker [17]), Lindley (L) (Lindley [14]), new modified Weibull (NMW) (Almalki and Yuan [2]), Weibull (W), and exponential distributions.

Table 1: MLEs -2ln L, AIC, CAIC, and BIC of the fitted distributions of data set.

Model	MLE	- 2log L	AIC	CAIC	BIC
OFIW	$\widehat{\theta}=0.208$ $\widehat{\alpha}=25.815$ $\widehat{\beta}=13.215$	38.253	44.253	45.753	42.156
NMW	$\widehat{\alpha}=0.1215$ $\widehat{\beta}=2.7837$ $\widehat{\gamma}=8.227\times10^{-5}$ $\widehat{\delta}=0.0003$ $\widehat{\theta}=2.7871$	41.173	51.173	55.459	47.678
GSD	$\widehat{\theta}=1.5712$ $\widehat{\alpha}=222.235$	45.97	49.96	50.67	51.96
S	$\widehat{\theta}1.1367$	57.49	59.49	59.71	60.49
A	$\widehat{\theta}1.1232$	56.37	58.37	58.59	59.36
L	$\widehat{\theta}=0.8161$	60.49	62.49	62.71	63.49
E	$\widehat{\theta}$ 0.5263	65.67	67.67	67.89	68.67
W	$\widehat{\alpha}=2.7870$ $\widehat{\beta}=2.1300$	41.1728	45.1728	47.1643	45.8787

The data set: (Gross and Clark [5]). The relief times of twenty patients receiving an analgesic is: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The ML estimates of the model parameters are provided in Table 1. In the same table, the analytical measures including minus double log-likelihood (-2log L), Akaike Information Criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion, and (BIC) are presented.

Table 1 lists the MLEs of the model parameters and the values of -2LogL, AIC, CAIC, BIC.

Table 1 compares the fits of the OFIW distribution with the NMW, GSD, S, A, L, E, and W distributions. The table shows that the OFIW model has the lowest values for -2LogL, AIC, CAIC, and BIC among all fitted distributions. So, it could be chosen as the best model.

7. Conclusion

In this paper, we propose a new three-parameter distribution named the odd Frechet inverse Weibull (OFIW) distribution. The pdf of OFIW can be expressed as a linear mixture of IW densities. We calculate explicit expressions for some of its statistical properties. We study maximum likelihood estimation. The proposed model provides better fits than some other competitive models using a real data set.

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