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Ekeland's variational principle in complete quasi-G-metric spaces

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Abstract

In this paper, by concept of Γ-function which is define on q-G-m (quasi-G-metric) space, we establish a generalized Ekeland's variational principle in the setting of lower semicontinuous from above. As application we prove generalized flower petal theorem in q-G-m.

Keywords: Γ-Function, q-G-m space, generalized EVP, lower semicontinuous from above function, generalized Caristi's (common) fixed point theorem, nonconvex minimax theorem, generalized flower petal theorem.

2010 MSC: 54H25, 54C60.

1. Introduction

EVP was first studied in 1972. Many equivalents have been found by scholars over the years for primitive EVP[10, 11], see [2–7, 17, 18, 20, 22]. Interesting applications in various fields of applied mathematics are found. A number of generalized of these results have been reviewed by other researchers [1–4, 8, 12–16, 23–30].

2. Ekeland's variational principle

In this paper, $\theta : (-\infty, \infty) \longrightarrow (0, \infty)$ is a nondecreasing function, a function $g : U \longrightarrow (-\infty, \infty)$ is said to be lower semicontinuous from above (shortly Lsca) at r_0, when for each sequence $\{r_n\}$ in U such that $r_n \longrightarrow r_0$ and $g(r_1) \geqslant g(r_2) \geqslant \cdots \geqslant g(r_n) \geqslant \cdots$, we have $g(r_0) \leqslant \lim_{n\to\infty} g(r_n)$. The function g is said to be Lsca on U, when g is Lsca at every point of U, g is proper when $h \not\equiv \infty$.

Theorem 2.1 ([9, Ekeland theorem])**.** *Let U be a complete metric space with meter d, $g : U \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, semicontinuous, and bounded below function. Then there exists $v \in U$ such that $g(v) \leqslant g(u)$, $d(u, v) \leqslant 1$, and $g(w) > g(v) - \epsilon d(v, w)$ for all $v \neq w$.*

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Definition 2.2 ([19]). Assume that U is a nonempty set and mapping

$$G : U \times U \times U \longrightarrow [0, \infty)$$

is satisfying the following conditions:

(i) $G(r, s, t) = 0$ if $r = s = t$;
(ii) $G(r, r, s) > 0$ for all $r, s \in U$, where $r \neq s$;
(iii) $G(r, r, t) \leqslant G(r, s, t)$ for all $r, s, t \in U$ with $r \neq t$;
(iv) $G(r, s, t) = G(p\{r, s, t\})$ such that p is a permutation of r, s, t;
(v) $G(r, s, t) \leqslant G(r, \alpha, \alpha) + G(\alpha, s, t)$ for all r, s, t, α in U.

Then G is said to be G-metric and pair (U, G) is said to be G-metric space.

Definition 2.3 ([19]). Let (U, G) be a G-metric space. A sequence $\{r_n\}$ in U is said to be

(a) G-Cauchy sequence if for all $\epsilon > 0$, there exists $q_0 \in \mathbb{N}$ such that for every $p, q, l \in \mathbb{N}$ and $p, q, l \geqslant q_0$ then $G(r_q, r_m, r_l) < \epsilon$;
(b) G-convergent to $r \in U$ if for all $\epsilon > 0$, there exists natural number q_0 such that for all $p, q \geqslant q_0$, then $G(r_q, r_p, r) < \epsilon$.

Proposition 2.4 ([19]). *Assume that (U, G) is a G-metric space, then the following statements are equivalent:*

(a) *$\{r_n\}$ is a G-caushy sequence;*
(b) *for each $\epsilon > 0$, there exists natural number q_0 such that for all $p, q \geqslant q_0$, then $G(r_q, r_p, r_p) < \epsilon$.*

Definition 2.5. A function $\sigma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is subaddtive when $\sigma(r + s) \leqslant \sigma(r) + \sigma(s)$, and $\sigma(\epsilon r) = \epsilon\sigma(r)$ for every $\epsilon > 0$.

Definition 2.6. Let U be a nonempty set. A function

$$G : U \times U \times U \longrightarrow [0, \infty)$$

is said to be quasi-G-metric (q-G-m) if the following conditions be satisfied

1. $G(r, s, t) = 0$ if $r = s = t$;
2. $G(r, r, s) > 0$ for all $r, s \in U$, $\quad r \neq s$;
3. $G(r, r, t) \leqslant G(r, s, t)$ for all $r, s, t \in U$, $\quad t \neq s$;
4. $G(r, s, t) \leqslant G(r, \varepsilon, \varepsilon) + G(\varepsilon, s, t)$ for all $r, s, t, \varepsilon \in U$.

(U, G) is said to be q-G-m space when U is a nonempty set and G is a q-G-m. The concept of Cauchy sequence, convergence, and complete space are defined as G-metric space.

Definition 2.7. Let (U, G) be a q-G-m space. A function $\Gamma : U \times U \times U \longrightarrow [0, \infty)$ is said to be Γ-function when

(1) $\Gamma(r, s, t) \leqslant \Gamma(r, \varepsilon, \varepsilon) + \Gamma(\varepsilon, s, t)$ for all $r, s, t, \varepsilon \in U$;
(2) if $r \in U$, $\{s_n\}_{n \in \mathbb{N}}$ be a sequence in U which is convergent to s in U and $\Gamma(r, s_n, s_n) \leqslant M$, then $\Gamma(r, s, s) \leqslant M$;
(3) for every $\epsilon > 0$, there exists $\delta > 0$ such that $\Gamma(r, \varepsilon, \varepsilon) \leqslant \delta$ and $\Gamma(\varepsilon, s, t) \leqslant \delta$ imply $G(r, s, t) \leqslant \epsilon$.

Example 2.8 ([21]). Let (U, d) be a metric space and $G : U^3 \longrightarrow [0, \infty)$ defined by $G(r, s, t) = \max\{d(r, s), d(r, t), d(s, t)\}$ for all $r, s, t \in U$. Then $\Gamma = G$ is a Γ-function on U.

Example 2.9. Assume that

$$G : U^3 \longrightarrow [0, \infty), \qquad G(r, s, t) = \frac{1}{3}(\mid t - r \mid + \mid r - s \mid)$$

is a function, then G is a q-G-m but isn't G metric.

Proof. q-G-m is obvious. We show that $G(r,s,t) \neq G\{p(r,s,t)\}$ (p is a permutation of r,s,t). Since

$$G(3,5,2) = \frac{1}{3}(\mid 2-3 \mid + \mid 3-5 \mid) = 1, \qquad G(2,3,5) = \frac{1}{3}(\mid 3-2 \mid + \mid 5-2 \mid) = \frac{4}{3},$$

then G is not a G-metric. □

Example 2.10. Let $G(r,s,t)$ be the same as in the previous example. Then $\Gamma = G$ is a Γ-function.

Proof. (a) and (b) are obvious. Let $\epsilon > 0$ be given, put $\delta = \frac{\epsilon}{2}$ if $\Gamma(r,\varepsilon,\varepsilon) = \frac{1}{3}(\mid t-\varepsilon \mid + \mid \varepsilon - s \mid) < \frac{\epsilon}{2}$, then

$$G(r,s,t) = \frac{1}{3}(\mid t-s \mid + \mid r-s \mid) \leqslant \frac{1}{3}(\mid t-\varepsilon \mid + \mid \varepsilon - r \mid + \mid r-\varepsilon \mid + \mid \varepsilon - s \mid) < \epsilon.$$

So, (c) is established. □

Lemma 2.11 ([21]). *Assume that (U,G) is a G-metric space and Γ is a Γ-function on U. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in U, $\{\rho_n\}$ and $\{\phi_n\}$ be in $[0,\infty]$, which are convergent to zero. Let $u,v,w,\varepsilon \in U$, then*

(1) *if $\Gamma(v,u_n,u_n) \leqslant \rho_n$ and $\Gamma(u_n,v,w) \leqslant \phi_n$ for all $n \in \mathbb{N}$, then $G(v,v,w) < \epsilon$ and hence $w = v$;*
(2) *if $\Gamma(u_n,u_n,u_n) \leqslant \rho_n$ and $\Gamma(u_n,u_m,w) \leqslant \phi_n$ for every $m > n$, then $G(v_n,v_m,w)$ is convergent to zero and hence $v_n \to w$;*
(3) *if $\Gamma(u_n,u_m,u_l) \leqslant \rho_n$ for all $m,n,l \in \mathbb{N}$ with $n \leqslant m \leqslant l$, then $\{u_n\}$ is a G-Cauchy sequence;*
(4) *if $\Gamma(u_n,\varepsilon,\varepsilon) \leqslant \rho_n$ for all $n \in \mathbb{N}$, then $\{u_n\}$ is a G-Cauchy sequence.*

Lemma 2.12. *Let Γ be a Γ-function on $U \times U \times U$. If sequence $\{r_n\}$ be in U that $\limsup_{n\longrightarrow\infty}\{\Gamma(r_n,r_m,r_l), n \leqslant m \leqslant l\} = 0$, then $\{r_n\}$ will be a G-Cauchy sequence in U.*

Proof. Assume $\rho_n = \sup\{\Gamma(r_n,r_m,r_l)\}$, then $\lim_{n\to\infty} \rho_n = 0$. By Lemma 2.11 (3), $\{r_n\}$ is a G-Cauchy sequence. □

Lemma 2.13. *Let $g : U \longrightarrow [-\infty,\infty]$ be a function and Γ be a Γ-function on $U \times U \times U$. The set $P(r)$ is defined by*

$$P(r) = \{s \in U; \quad s \neq r, \quad \Gamma(r,s,s) \leqslant \theta(g(r))(g(r)-g(s))\}\cdot$$

If $P(r)$ be nonempty, then for every $s \in P(r)$, we will have

$$P(s) \subseteq P(r) \ \text{ and } \ g(s) \leqslant g(r).$$

Proof. Let $s \in P(r)$. So $s \neq r$ and $\Gamma(r,s,s) \leqslant \theta(g(r))(g(r)-g(s))$. Since $\Gamma(r,s,s) \geqslant 0$ and θ is nondecreasing and positive function, then $g(r) \geqslant g(s)$. If $P(s) = \emptyset$ then $P(s) \subseteq P(r)$. Therefore $t \neq s$ and $\Gamma(s,t,t) \leqslant \theta(g(s))(g(s)-g(t))$ as above $g(s) \geqslant g(t)$. Since Γ be a Γ-function, then

$$\Gamma(r,t,t) \leqslant \Gamma(r,s,s) + \Gamma(s,t,t) \leqslant \theta(g(r))(g(r)-g(t)).$$

We claim that $t \neq r$. Assume that $t = r$ so $\Gamma(r,t,t) = 0$. On the other hand

$$\Gamma(r,s,s) \leqslant \theta(g(r))(g(r)-g(s)) \leqslant \theta(g(r))(g(r)-g(t)) = 0 \Longrightarrow \Gamma(r,s,s) = 0,$$

then $\Gamma(r,s,s) = 0$. For every $\epsilon > 0$, we have $\Gamma(r,t,t) = 0 < \epsilon$ and $\Gamma(t,s,s) = 0 < \epsilon$ then by definition Γ-function, we have $G(t,s,s) < \epsilon$, so $G(t,s,s) = 0$ and $t = s$. This is a contradiction, therefore $t \in P(r)$ and $P(s) \subseteq P(r)$. □

Proposition 2.14. *Assume that (U,G) is a complete q-G-m space and $g : U \longrightarrow [-\infty,\infty]$ is a proper and bounded below function, Γ is a Γ-function on $U \times U \times U$. Let*

$$P(r) = \{s \in U; s \neq r, \Gamma(r,s,s) \leqslant \theta(g(r))(g(r)-g(s))\}\cdot$$

Let $\{r_n\}$ be a sequence in U such that $P(r_n)$ be nonempty and for all $n \in \mathbb{N}$, $r_{n+1} \in P(r_n)$. Then, there exists

$r_0 \in U$ *such that* $r_n \longrightarrow r_0$ *and* $r_0 \in \bigcap_{n=1}^{\infty} P(r_n)$. *Also if for every* $n \in \mathbb{N}$, *we have* $g(r_{n+1}) \leqslant \inf_{t \in P(r_n)} g(t) + \frac{1}{n}$, *then* $\bigcap_{n=1}^{\infty} P(r_n)$ *will only has one member.*

Proof. At first we prove that $\{r_n\}$ is a Cauchy sequence by Lemma 2.13, $g(r_n) \geqslant g(r_{n+1})$ for all $n \in \mathbb{N}$. Therefore $\{g(r_n)\}$ is nonincreasing. On the other hand g is bounded below then $\lim_{n\to\infty} g(r_n) = u$, and $g(r_n) \geqslant u$ for all $n \in \mathbb{N}$. We claim that

$$\lim_{n\to\infty} \sup\{\Gamma(r_n, r_m, r_m) : m > n\} = 0\cdot$$

We have

$$\begin{aligned}\Gamma(r_n, r_m, r_m) &\leqslant \Gamma(r_n, r_{n+1}, r_{n+1}) + \Gamma(r_{n+1}, r_m, r_m)\\ &\leqslant \Gamma(r_n, r_{n+1}, r_{n+1}) + \Gamma(r_{n+1}, r_{n+2}, r_{n+2}) + \cdots + \Gamma(r_{m-1}, r_m, r_m),\end{aligned}$$

then

$$\Gamma(r_n, r_m, r_m) \leqslant \sum_{j=n}^{m-1} \Gamma(r_n, r_m, r_m) \leqslant \theta(g(r))(g(r_n) - u)$$

for all $m, n \in \mathbb{N}$ with $m > n$.

Put $\rho_n = \theta(g(r))(g(r_n) - u)$, then $\sup\{\Gamma(r_n, r_m, r_m) : m > n\} \leqslant \rho_n\cdot$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} g(r_n) = u$, we result

$$\lim_{n\longrightarrow\infty} \sup\{\Gamma(r_n, r_m, r_m) : m > n\} = 0$$

and $\lim_{n\to\infty} \rho_n = 0\cdot$. By Lemma 2.12, $\{u_n\}$ is a G-Cauchy sequence. Then, there exists $r_0 \in U$ such that $r_n \to u_0$. We show that $r_0 \in \bigcap_{n=1}^{\infty} P(r_n)$. Since g is Lsca, then $g(r_0) \leqslant \lim_{n\to\infty} g(r_n) = u \leqslant g(r_k)$.

Let $n \in \mathbb{N}$, we have

$$\Gamma(r_n, r_m, r_m) \leqslant \sum_{j=n}^{m-1} \Gamma(r_j, r_{j+1}, r_{j+1}) \leqslant \theta(g(r_n))(g(r_n) - g(r_0))$$

for all $m \in \mathbb{N}$ with $m > n$. By Definition 2.7 (2), we have

$$\Gamma(r_n, r_0, r_0) \leqslant \theta(g(r_n)(g(r_n) - g(r_0))$$

for all $n \in \mathbb{N}$. Also $r_0 \neq r$ for all $n \in \mathbb{N}$, suppose it is not, then there exists $j \in \mathbb{N}$ such that $r_0 = r_j$. Since

$$\Gamma(r_j, r_{j+1}, r_{j+1}) \leqslant \theta(g(r_j))(g(r_j) - g(r_{j+1})) \leqslant \theta(g(r_j))(g(r_j) - g(r_0)) = 0,$$

then we have $\Gamma(r_j, r_{j+1}, r_{j+1}) = 0$ and in the same way

$$\Gamma(r_{j+1}, r_{j+2}, r_{j+2}) = 0.$$

Now assume $\epsilon > 0$, $\Gamma(r_j, r_{j+1}, r_{j+1}) = 0 < \delta$, and $\Gamma(r_{j+1}, r_{j+2}, r_{j+2}) = 0 < \delta$. Therefor by Definition 2.7 (3) we get it $G(r_j, r_{j+2}, r_{j+2}) < \epsilon$. Then $r_j = r_{j+2}$ that is a contradiction because of $r_j \neq r_{j+2}$. Since $r_{j+1} \in P(r_j)$, then $P(r_{j+1}) \subseteq P(r_j)$ and $r_{j+2} \in P(r_{j+1})$. So $r_{j+2} \in P(r_j)$. We suppose $r_{j+2} \neq r_j$ for all $n \in \mathbb{N}$. We have $r_0 \in \bigcap_{n=1}^{\infty} P(r_n)$, then $\bigcap_{n=1}^{\infty} P(r_n) \neq \emptyset$. Let $g(r_{n+1}) \leqslant \inf_{t \in P(r_n)} g(t) + \frac{1}{n}$ for all $r_0 \neq r_n$. We show that $\bigcap_{n=1}^{\infty} P(r_n) = \{r_0\}$. Assume that $w \in \bigcap_{n=1}^{\infty} P(r_n)$, then

$$\Gamma(r_n, w, w) \leqslant \theta(g(r_n))(g(r_n) - g(w)) \leqslant \theta(g(r_1))(g(r_n) - \inf_{t \in P(r_n)} g(t)) \leqslant \theta(g(r_1))(g(r_n) - g(r_{n+1}) + \frac{1}{n}).$$

Let

$$\varphi_n = \theta(g(r_1))(g(r_n) - g(r_{n+1}) + \frac{1}{n})$$

for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \varphi_n = 0$, we get it $\lim_{n\to\infty} \Gamma(r_n, w, w) = 0$. On the other hand $\{r_m\}$ is a G-Cauchy sequence. Then $\lim_{n\to\infty} \Gamma(r_m, r_m, r_n) = 0$ and we get it $r_n \to \infty$, by uniqueness $w = r_0$. Then $\bigcap_{n=1}^{\infty} P(r_n) = \{r_0\}$. □

Theorem 2.15 (Generalized Ekeland's variational principle)**.** *Assume that* (U, G) *is a complete q-G-m space and* $g : U \longrightarrow (-\infty, \infty]$ *be a proper, bounded below and Lsca function.* Γ *is a* Γ*-function on* $U \times U \times U$*, then there exists* $r \in U$ *such that*

$$\Gamma(v, r, r) > \theta(g(r))(g(r) - g(v))$$

for all $r \in U$ *with* $v \neq r$.

Proof. Suppose it isn't true. Then for every $r \in U$, there exists $s \in U$, $s \neq r$ such that $\Gamma(r, s, s) \leqslant \theta(g(r))(g(r) - g(s))$. That is $P(r) \neq \emptyset$. We define the sequence $\{r_n\}$ as follows. Put $r_1 = \varepsilon$, we choose $r_2 \in P(r_1)$ such that $g(r_2) \leqslant \inf_{r\in P(r_1)} g(r) + 1$. In the same way suppose that $r_n \in U$ is given. We choose $r_{n+1} \in P(r_n)$ such that $g(r_{n+1}) \leqslant \inf_{r\in P(r_n)} g(r) + \frac{1}{n}$. By proposition 2.14, there exists $r_0 \in U$ such that

$$\bigcap_{n=1}^{\infty} P(r_n) = \{r_0\}.$$

By lemma 2.13, we have $P(r_0) \subseteq \bigcap_{n=1}^{\infty} P(r_n) = \{r_0\}$ then $P(r_0) = \{r_0\}$. This is a contradiction. Therefore there exists $v \in U$ such that

$$\Gamma(v, r, r) > \theta(g(v))(g(v) - g(r)).$$

□

Theorem 2.16 (Generalized Caristi's common fixed point theorem for a family of multivalued maps)**.** *Assume that* (U, G) *is a complete q-G-m space and* $g : U \longrightarrow (-\infty, \infty]$ *be a proper, bounded below and Lsca function.* Γ *is a* Γ*-function on* $U \times U \times U$*. Let J be any index set and for each* $j \in J$*, suppose* $T_j : U \to 2^U$ *is multivalued map such that for each* $r \in U$*, there is* $s = s(r, j) \in T_j(r)$ *with*

$$\Gamma(r, s, s) \leqslant \theta\big(g(r)\big)\big(g(r) - g(s)\big). \tag{2.1}$$

Then there is $w \in U$ *such that* $w \in \bigcap_{j\in J} T_j(w)$*, and* $\Gamma(w, w, w) = o$.

Proof. By Theorem 2.15, there exists $w \in U$ such that $\Gamma(w, r, r) > \theta\big(g(w)\big)\big(g(w) - g(r)\big)$ for all $r \in U$ with $r \neq w$. Now we show that $w \in \bigcap_{j\in J} T_j(w)$ and $\Gamma(w, w, w) = 0$. According to the assumption, there exists $r(t, j) \in T_j(w)$ such that $\Gamma(w, t, t) \leqslant \theta\big(g(t)\big)\big(g(w) - g(t(w, j))\big)$. We show that $t(w, j) = w$ for all $j \in J$. On the contrary, let $t(w, j_0) \neq w$ for some $j_0 \in J$, then

$$\Gamma(w, t, t) \leqslant \theta\big(g(w)\big)\big(g(w) - g(t)\big) < \Gamma(w, t, t),$$

which is a contradiction. Therefore $w = t(w, j) \in T_j(w)$ for all $j \in T$.

Since $\Gamma(w, w, w) \leqslant \theta\big(g(w)\big)\big(g(w) - g(w)\big) = 0$, we obtain $\Gamma(w, w, w) = 0$. □

Remark 2.17. We conclude that Theorem 2.16 concludes Theorem 2.15.

On the contrary, for each $r \in U$, there exists $s \in U$ with $s \neq r$ such that

$$\Gamma(r, s, s) \leqslant \theta\big(g(r)\big)\big(g(r) - g(s)\big).$$

Put $T : U \longrightarrow 2^U \backslash \{\emptyset\}$ by

$$T(r) = \{s \in U : s \neq r, \Gamma(r, s, s) \leqslant \theta\big(g(r)\big)\big(g(r) - fg(s)\big)\}.$$

By Theorem 2.16, T has a fixed point $w \in U$, this means, $w \in T(w)$. This is a contradiction, because $w \notin T(w)$.

Theorem 2.18 (Nonconvex maximal element theorem for a family of multivalued maps)**.** *Assume that* (U, G) *is a complete q-G-m space and* $g : U \longrightarrow (-\infty, \infty]$ *be a proper, bounded below and Lsca function.* Γ *is a* Γ*-function on* $U \times U \times U$*, and J be any index set. For each* $j \in J$*, let* $T_j : U \longrightarrow 2^U$ *be a multivalued map. Suppose that for each* $(r, j) \in U \times J$ *with* $T_j(r) \neq \emptyset$*, there exists* $s = s(r, j) \in U$ *with* $s \neq r$ *such that* (2.1) *holds. Then there exists* $w \in U$ *such that* $T_j(W) = \emptyset$ *for each* $j \in J$.

Proof. By Theorem 2.15, there exists $w \in U$, such that $\Gamma(w, r, r) > \theta\big(g(w)\big)\big(g(w) - f(r)\big)$ for all $r \in U$ with $r \neq w$. We prove that $T_j(w) = \emptyset$ for each $j \in J$. Indeed, if $T_{j_0}(w) \neq \emptyset$, for some $j_0 \in J$, according to the assumption, there exists $t = t(w, j_0) \in U$ with $t \neq w$ such that $\theta(w, t, t) \leqslant \theta\big(g(w)\big)\big(g(w) - g(t)\big)$. Also $\Gamma(w, t, t) > \theta\big(g(w)\big)\big(g(w) - g(t)\big)$, which is a contradiction. □

Remark 2.19. We conclude that Theorem 2.18 concludes Theorem 2.15.

On the contrary, thus for each $r \in U$, there exists $s \in U$ with $s \neq r$ such that

$$\Gamma(r, s, s) \leqslant \theta\big(g(r)\big)\big(g(r) - g(s)\big).$$

For each $r \in U$, we define $T(r) = \{s \in U : s \neq r, (r, s, s) \leqslant \theta\big(g(r)\big)\big(g(r) - g(s)\big)$. Then $T(r) \neq \emptyset$ for all $r \in U$. But by Theorem 2.18, there exists $w \in U$ such that $T(w) = \emptyset$, which is a contradiction.

3. Nonconvex optimization and minimax theorems

Theorem 3.1 (Generalized Takahashi's nonconvex minimization theorem)**.** *Assume that* (U, G) *is a complete q-G-m space and* $g : U \longrightarrow (-\infty, \infty]$ *be a proper, bounded below and Lsca function.* Γ *is a* Γ*-function on* $U \times U \times U$*. Suppose that for any* $r \in U$ *with* $g(r) > \inf_{w \in U} fg(w)$ *there exists* $s \in U$ *with* $s \neq r$ *such that* (2.1) *holds. Then there exists* $w \in U$ *such that* $g(w) = \inf_{t \in U} g(t)$.

Proof. By Theorem 2.15, there exists $w \in U$ such that $\Gamma(w, r, r) > \theta\big(g(w)\big)\big(g(w) - g(r)\big)$ for all $r \in U$, $r \neq w$. Now we prove that $g(w) = \inf_{t \in U} g(t)$.

On the contrary, then $g(w) > \inf_{t \in U} g(t)$. According to the assumption, there exists $s = s(w) \in U$, with $s \neq w$ such that $\Gamma(w, s, s) \leqslant \theta\big(g(w)\big)\big(g(w) - g(s)\big)$. Then we have $\Gamma(w, s, s) \leqslant \theta\big(g(w)\big)\big(g(w) - g(s)\big) < \Gamma(w, s, s)$, which is a contradiction. □

Remark 3.2. Using Theorem 3.1, we can conclude Theorem 2.15.

On the contrary, then for each $r \in U$, there exists $s \in U$ with $s \neq r$ such that $\Gamma(r, s, s) \leqslant \theta\big(g(r)\big)\big(g(r) - g(s)\big)$. By Theorem 3.1, there exists $w \in U$ such that $g(w) = \inf_{t \in U} g(t)$. According to the assumption, there exists $z \in U$ with $z \neq r$, such that $\Gamma(w, z, z) \leqslant \theta\big(g(w)\big)\big(g(w) - g(z)\big) \leqslant 0$. Then $\Gamma(w, z, z) = 0$ and $g(w) = g(z) = \inf_{t \in U} g(t)$. There exists $t \in U$ with $t \neq z$ such that $\Gamma(z, t, t) \leqslant \theta\big(g(z)\big)\big(g(z) - g(t)\big) \leqslant 0$. Then we have $\Gamma(z, t, t) = 0$ and $g(w) = g(z) = g(t) = \inf_{r \in U} g(r)$. Since $\Gamma(w, t, t) \leqslant \Gamma(w, z, z) + \Gamma(z, t, t)$, then $\Gamma(w, t, t) = 0$. For $\epsilon > 0$ we have $\Gamma(w, z, z) = 0 < \delta$, $\Gamma(z, t, t) = 0 < \delta$ then $G(w, t, t) < \epsilon$, that is, $w = t$. Also for $\epsilon > 0$ we have $\Gamma(z, w, w) = 0 < \delta$, $\Gamma(w, t, t) = 0 < \delta$, then $G(z, t, t) < \epsilon$ that is, $z = t$, which is a contradiction.

Theorem 3.3 (Nonconvex minimax theorem)**.** *Assume that* (U, G) *is a complete q-G-m space and* Γ *is a* Γ*-function on* $U \times U \times U$*. Let* $F : U \times U \to (-\infty, \infty]$ *be a proper lsca and bounded below function in the first argument. Suppose that for each* $r \in U$ *with* $\{x \in U : F(r, x) > \inf_{a \in U} F(a, x)\} \neq \emptyset$*, there exists* $s = s(r) \in U$ *with* $s \neq r$ *such that*

$$\Gamma(r, s, s) \leqslant \theta\big(F(r, w)\big)\big(F(r, w) - F(s, w)\big) \tag{3.1}$$

for all $w \in \{x \in U : F(r, x) > \inf_{a \in U} F(a, x)\}$*. Then* $\inf_{r \in U} \sup_{s \in U} F(u, s) = \sup_{s \in U} \inf_{r \in U} F(r, s)$.

Proof. By Theorem 3.1, for every $s \in U$, there exists $r(s) \in U$ such that $F(r(s), s) = \inf_{r\in U} F(r, s)$. Then $\sup_{s\in U} F(r(s), s) = \sup_{s\in U} \inf_{r\in U} F(r, s)$.

By displacement of $r(s)$ with an arbitrary $r \in U$ and then getting inf, we obtain $\inf_{r\in U} \sup_{s\in U} F(r, s) = \sup_{s\in U} \inf_{r\in U} F(r, s)$. □

Theorem 3.4 (Nonconvex equilibrium theorem)**.** *Assume that* (U, G) *is a complete q-G-m space and* Γ *is a* Γ*-function on* $U \times U \times U$. *Let* F *and* θ *be the same as in Theorem 3.3. Let, for each* $r \in U$ *with* $\{x \in U : F(r, x) < 0\} \neq \emptyset$, *there exists* $s = s(r) \in U$ *with* $s \neq r$ *such that* (3.1) *holds for all* $t \in U$. *Then there exists* $y \in U$ *such that* $F(y, s) \geqslant 0$ *for all* $s \in U$.

Proof. From Theorem 2.15 for each $t \in U$, there exists $y(t) \in U$ such that $\Gamma(y(t), r, r) > \theta\big(F(y(t), t)\big)\big(F(y(t), t) - F(r, t)\big)$ for all $r \in U$ with $r \neq y(t)$. We show that there exists $y \in U$ such that $F(y, s) \geqslant 0$ for all $s \in U$. On the contrary, for each $r \in U$ there exists $s \in U$ such that $F(r, s) < 0$. Then for each $r \in U$, $\{x \in U : F(r, x) < 0\} \neq \emptyset$. According to the assumption, there exists $s = s(y(t))$, $y \neq y(t)$ such that $\Gamma(y(t), s, s) \leqslant \theta\big(F(y(t), t)\big)\big(F(y(t), t) - F(s, t)\big)$, which is a contradiction. □

Example 3.5. Let $U = [0, 1]$ and $G(r, s, t) = \max\{|r - s|, |r - t|, |s - t|\}$. Then (X, G) is a complete q-G-m space. Suppose that a, b be positive real numbers with $a \geqslant b$. Suppose $H : U \times U \longrightarrow R$ with $H(r, s) = \frac{a}{2}r - \frac{b}{3}s$. Therefore, function $r \longrightarrow H(r, s)$ is proper, lower semicontinuous and bounded below, and $H(1, s) \geqslant 0$ for every $s \in U$. Also $H(r, s) \geqslant 0$ for every $r \in [\frac{b}{a}, 1]$ and for every $s \in U$. In fact, for every $r \in [0, \frac{b}{a}]$, $H(r, s) = ar - bs < 0$ when $s \in [\frac{a}{b}r, 1]$. Then set $\{x \in U : H(r, x) < 0\} \neq \emptyset$ for every $r \in [0, \frac{b}{a}]$. Let $r, s \in U$, $r \geqslant s$, we have $r - s = \frac{2}{a}\{(\frac{a}{2}r - \frac{b}{3}x) - (\frac{a}{2}s - \frac{b}{3}x)\}$ for every $x \in U$. Let $\theta : [0, \infty) \longrightarrow [0, \infty)$ with $\theta(t) = \frac{2}{a}$ be defined. Therefore $G(r, s, s) \leqslant \theta\big(H(r, x)\big)\big(H(r, x) - H(s, x)\big)$ for every $r \geqslant s$, and $r, s, x \in U$. By Theorem 3.4 there exists $y \in U$ such that $H(y, s) \geqslant 0$ for every $s \in U$.

4. Applications

Definition 4.1. Let (U, G) be a q-G-m space and $a, b \in U$. Suppose that $\lambda : U \to (0, \infty)$ be a function and Γ be a Γ-function on U. Define

$$\Gamma_\epsilon(a, b, \lambda) = \{r \in U : \epsilon\Gamma(a, r, r) \leqslant \lambda(a)\big(\Gamma(b, a, a) - \Gamma(b, r, r)\big)\}$$

such that $\epsilon \in (0, \infty)$ and $a, b \in U$.

Lemma 4.2. *Assume that* (U, G) *is a complete q-G-m space and* $g : U \longrightarrow (-\infty, \infty]$ *be a proper, bounded below and Lsca function and* Γ *is a* Γ*-function on* $U \times U \times U$. *Let* $\epsilon > 0$. *Suppose that there exists* $x \in U$ *such that* $g(x) < \infty$ *and* $\Gamma(x, x, x) = 0$. *Then there exists* $t \in U$ *such that*

(i) $\epsilon\Gamma(x, t, t) \leqslant \theta\big(g(x)\big)\big(g(x) - g(t)\big)$;
(ii) $\Gamma(t, r, r) > \theta\big(g(t)\big)\big(g(t) - g(r)\big)$ *for all* $r \in U$ *with* $r \neq t$.

Proof. Let $x \in U$, $g(x) < +\infty$ and $\Gamma(x, x, x) = 0$. Put

$$S = \{r \in U : \epsilon\Gamma(x, r, r) \leqslant \theta\big(g(x)\big)\big(g(x) - g(r)\big).$$

Therefore (S, G) is a nonempty complete q-G-m space. By Theorem 2.15, there exists $t \in S$ such that $\epsilon\Gamma(t, , r, r) > \theta\big(g(t)\big)\big(g(t) - g(r)\big)$ for all $r \in S$ with $r \neq t$. For any $r \in U \backslash S$, since $\epsilon[\Gamma(x, t, t) + \Gamma(t, r, r)] \geqslant \epsilon\Gamma(x, r, r) > \theta\big(g(x)\big)\big(g(x) - g(r)\big) \geqslant \epsilon\Gamma(x, t, t) + \theta\big(g(t)\big)\big(g(t) - g(r)\big)$, therefore $\epsilon\Gamma(t, r, r) > \theta\big(g(t)\big)\big(g(t) - g(r)\big)$ for all $r \in U \backslash S$. Then $\epsilon\Gamma(t, r, r) > \theta\big(g(t)\big)\big(g(t) - g(t)\big)$ for all $r \in U$ with $r \neq t$. □

Theorem 4.3 (Generalized flower petal theorem)**.** *Suppose that P be a proper complete subset of a q-G-m space U and* $a \in P$. *Let* Γ *be a* Γ*-function on U with* $\Gamma(a, a, a) = 0$. *Let* $b \in U \backslash P$, $\Gamma(b, P, P) = \inf_{r\in P} \Gamma(b, r, r) \geqslant u$ *and* $\Gamma(b, a, a) = s > 0$ *and there exists a function* λ *from U into* $(0, \infty)$ *satisfying* $\lambda(r) = \theta\big(\Gamma(b, r, r)\big)$ *for some nondecreasing function* θ *from* $(-\infty, \infty]$ *into* $(0, \infty)$. *Then for each* $\epsilon > 0$, *there exists* $t \in P \bigcap \Gamma_\epsilon(a, b, \lambda)$*such that* $\Gamma_\epsilon(t, b, \lambda) \bigcap (P \backslash \{t\}) = \emptyset$ *and* $(a, t, t) \leqslant \epsilon^{-1}\lambda(a)(s - r)$.

Proof. (P, G) is a complete q-G-m space. Consider $g : P \longrightarrow (-\infty, \infty], g(r) = \Gamma(b, r, r)$. Since $g(a) =$

$\Gamma(b, a, a) = s < \infty$ and $\Gamma(b, P, P) = \inf_{r \in P} \Gamma(b, r, r) \geqslant u$ then g is a proper lower semicontinuous and bounded below function. By Lemma 4.2, there exists $t \in P$ such that

(i) $\epsilon\Gamma(a, t, t) \leqslant \lambda(a)\big(g(a) - g(t)\big)$;

(ii) $\epsilon\Gamma(t, r, r) > \lambda(t)\big(g(t) - g(t)\big)$ for all $r \in P$ with $r \neq t$.

Applying (i), we have $t \in P \bigcap \Gamma_\epsilon(a, b, \lambda)$. Also, applying (i) again, we have $\Gamma(a, t, t) \leqslant \epsilon^{-1}\lambda(a)\big(\Gamma(b, a, a) - \Gamma(b, t, t)\big) \leqslant \epsilon^{-1}\lambda(a)(s - r)$. By (ii), we obtain $\epsilon(t, r, r) > \lambda(t)\big(\Gamma(b, t, t) - \Gamma(b, r, r)\big)$ for all $r \in P$ with $r \neq t$. Therefore $u \notin \Gamma_\epsilon(t, b, \lambda)$ for all $r \in P \backslash \{t\}$ or $\Gamma_\epsilon(t, b, \lambda) \bigcap (P \backslash \{t\}) = \emptyset$. □

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